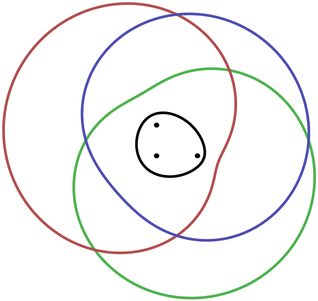
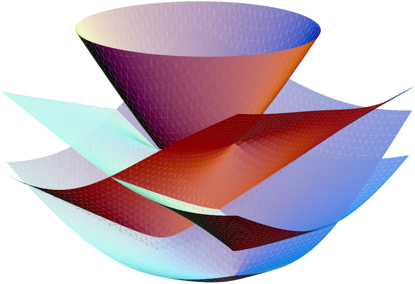
The Quadratic Equation Revisited

Bernd Sturmfels
MPI Leipzig





An Invitation to Non-Linear Algebra

General Audience Lecture

IHP Paris, October 11, 2023

Back in Ninth Grade

A quadratic equation has the form

$$ax^2 + bx + c = 0.$$

The letter x is the unknown.

The three quantities a, b, c are parameters. In applications, they are measurements from an experiment. They change many times.

How do we solve this equation?

Back in Ninth Grade

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The teacher presents a general formula.
The students memorize that formula.

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Why do we solve this equation?

No clue.
The curriculum requires it.

Math class is totally boring....

The Formula

A general quadratic equation $ax^2 + bx + c = 0$ has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The **discriminant** is the expression

$$D = b^2 - 4ac$$

There is a case distinction concerning the nature of the solution:

$$D > 0 \quad \textit{oder} \quad D = 0 \quad \textit{oder} \quad D < 0.$$

There are almost always two complex solutions.

The number of real solutions is

`Two` *or* `one` *or* `zero.`

Completing the Square

Derivation: The following equations are all equivalent:

$$
\begin{aligned}
ax^2 + bx + c &= 0 \\
ax^2 + bx &= -c \\
ax^2 + bx + \tfrac{b^2}{4a} &= \tfrac{b^2}{4a} - c \\
a\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a} \\
\left(x + \tfrac{b}{2a}\right)^2 &= \tfrac{b^2-4ac}{4a^2} \\
x + \tfrac{b}{2a} &= \tfrac{\pm\sqrt{b^2-4ac}}{2a} \\
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\end{aligned}
$$

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\end{aligned}$$

What to do with polynomials in x of higher degree?

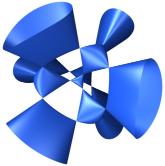
Numerical solutions, symbolic representations of the roots, ...

What to do with polynomials in several unknowns?

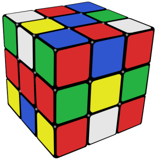
Gröbner bases, numerical algebraic geometry, ...

Math is Not Boring

Mathematics is the language in which God has written the universe. Galileo Galilei



Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different. Johann Wolfgang von Goethe



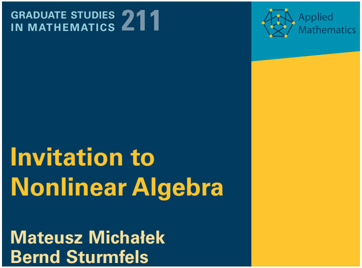
Mathematics, rightly viewed, possesses not only truth, but supreme beauty. Bertrand Russell

Cayley's Cubic Surface

[Arthur Cayley, 1821-1895]

Logo of the Nonlinear Algebra Group at the

Max-Planck Institute for Mathematics in the Sciences



$$x^2 + y^2 + z^2 - 2xyz - 1 \;=\; 0$$

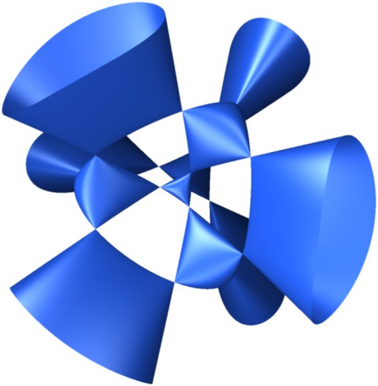
Optimization, Statistics,...

Kummer's Quartic Surface

The equation

$$x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 - x^2 - y^2 - z^2 + 1 \;=\; 0$$

describes a surface of degree four in $\mathbb{R}^3$ with 16 singular points:



[Ernst Eduard Kummer, 1810-1893]

Kummer surfaces have applications in cryptography.

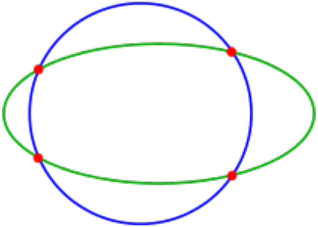
Varieties

The set of solutions to a system of polynomial equations in n variables is called a *variety* in $\mathbb{R}^n$. *"supreme beauty"*

Example: Quadratic curves in the plane ($n = 2$):

$$a_1 \cdot x^2 + a_2 \cdot xy + a_3 \cdot y^2 + a_4 \cdot x + a_5 \cdot y + a_6 \quad = \quad 0$$

Two quadratic equations in x und y ...



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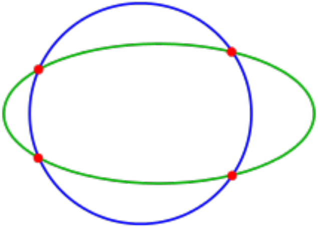

... have almost always four complex solutions. [Bézout 1764]

The **discriminant** is a polynomial in the coefficients.
It specifies the case distinction: 0,1,2,3 or 4 real solutions.

Discriminant ... has 3210 terms

```
256*a1^4*a3^2*u3^2*u6^4-128*a1^4*a3^2*u3*u5^2*u6^3+16*a1^4*a3^2*u5^4
*u6^2-256*a1^4*a3*a5*u3^2*u5*u6^3+128*a1^4*a3*a5*u3*u5^3*u6^2-16*a1^
4*a3*a5*u5^5*u6-512*a1^4*a3*a6*u3^3*u6^3+512*a1^4*a3*a6*u3^2*u5^2*u6
^2-160*a1^4*a3*a6*u3*u5^4*u6+16*a1^4*a3*a6*u5^6+256*a1^4*a5^2*u3^3*u
6^3-128*a1^4*a5^2*u3^2*u5^2*u6^2+16*a1^4*a5^2*u3*u5^4*u6-256*a1^4*a5
*a6*u3^3*u5*u6^2+128*a1^4*a5*a6*u3^2*u5^3*u6-16*a1^4*a5*a6*u3*u5^5+2
56*a1^4*a6^2*u3^4*u6^2-128*a1^4*a6^2*u3^3*u5^2*u6+16*a1^4*a6^2*u3^2*
u5^4-128*a1^3*a2^2*a3*u3^2*u6^4+64*a1^3*a2^2*a3*u3*u5^2*u6^3-8*a1^3*
a2^2*a3*u5^4*u6^2+64*a1^3*a2^2*a5*u3^2*u5*u6^3-32*a1^3*a2^2*a5*u3*u5
^3*u6^2+4*a1^3*a2^2*a5*u5^5*u6+128*a1^3*a2^2*a6*u3^3*u6^3-128*a1^3*a
2^2*a6*u3^2*u5^2*u6^2+40*a1^3*a2^2*a6*u3*u5^4*u6-4*a1^3*a2^2*a6*u5^6
-256*a1^3*a2*a3^2*u2*u3*u6^4+64*a1^3*a2*a3^2*u2*u5^2*u6^3+128*a1^3*a
2*a3^2*u3*u4*u5*u6^3-32*a1^3*a2*a3^2*u4*u5^3*u6^2+128*a1^3*a2*a3*a4*
u3^2*u5*u6^3-64*a1^3*a2*a3*a4*u3*u5^3*u6^2+8*a1^3*a2*a3*a4*u5^5*u6+2
56*a1^3*a2*a3*a5*u2*u3*u5*u6^3-64*a1^3*a2*a3*a5*u2*u5^3*u6^2+128*a1^
3*a2*a3*a5*u3^2*u4*u6^3-192*a1^3*a2*a3*a5*u3*u4*u5^2*u6^2+40*a1^3*a2
*a3*a5*u4*u5^4*u6+768*a1^3*a2*a3*a6*u2*u3^2*u6^3-512*a1^3*a2*a3*a6*u
2*u3*u5^2*u6^2+80*a1^3*a2*a3*a6*u2*u5^4*u6-512*a1^3*a2*a3*a6*u3^2*u4
*u5*u6^2+320*a1^3*a2*a3*a6*u3*u4*u5^3*u6-48*a1^3*a2*a3*a6*u4*u5^5-25
6*a1^3*a2*a4*a5*u3^3*u6^3+128*a1^3*a2*a4*a5*u3^2*u5^2*u6^2-16*a1^3*a
2*a4*a5*u3*u5^4*u6+128*a1^3*a2*a4*a6*u3^3*u5*u6^2-64*a1^3*a2*a4*a6*u
3^2*u5^3*u6+8*a1^3*a2*a4*a6*u3*u5^5-384*a1^3*a2*a5^2*u2*u3^2*u6^3+12
8*a1^3*a2*a5^2*u2*u3*u5^2*u6^2-8*a1^3*a2*a5^2*u2*u5^4*u6+128*a1^3*a2
*a5^2*u3^2*u4*u5*u6^2-32*a1^3*a2*a5^2*u3*u4*u5^3*u6+384*a1^3*a2*a5*a
6*u2*u3^2*u5*u6^2-128*a1^3*a2*a5*a6*u2*u3*u5^3*u6+8*a1^3*a2*a5*a6*u2
*u5^5+128*a1^3*a2*a5*a6*u3^3*u4*u6^2-192*a1^3*a2*a5*a6*u3^2*u4*u5^2*
u6+40*a1^3*a2*a5*a6*u3*u4*u5^4-512*a1^3*a2*a6^2*u2*u3^3*u6^2+192*a1^
3*a2*a6^2*u2*u3^2*u5^2*u6-16*a1^3*a2*a6^2*u2*u3*u5^4+128*a1^3*a2*a6^
2*u3^3*u4*u5*u6-32*a1^3*a2*a6^2*u3^2*u4*u5^3-512*a1^3*a3^3*u1*u3*u6^
4+128*a1^3*a3^3*u1*u5^2*u6^3+256*a1^3*a3^3*u2^2*u6^4-256*a1^3*a3^3*u
2*u4*u5*u6^3+128*a1^3*a3^3*u3*u4^2*u6^3+32*a1^3*a3^3*u4^2*u5^2*u6^2+
128*a1^3*a3^2*a4*u2*u3*u5*u6^3-32*a1^3*a3^2*a4*u2*u5^3*u6^2-512*a1^3
*a3^2*a4*u3^2*u4*u6^3+192*a1^3*a3^2*a4*u3*u4*u5^2*u6^2-16*a1^3*a3^2*
a4*u4*u5^4*u6+768*a1^3*a3^2*a5*u1*u3*u5*u6^3-192*a1^3*a3^2*a5*u1*u5^
3*u6^2-384*a1^3*a3^2*a5*u2^2*u5*u6^3+128*a1^3*a3^2*a5*u2*u3*u4*u6^3+
352*a1^3*a3^2*a5*u2*u4*u5^2*u6^2-256*a1^3*a3^2*a5*u3*u4^2*u5*u6^2-32
*a1^3*a3^2*a5*u4^2*u5^3*u6+512*a1^3*a3^2*a6*u1*u3^2*u6^3-640*a1^3*a3
```

Many Conics


3264
CONICS
IN A
SECOND
Paul Breiding
Bernd Sturmfels
Sascha Timme
In 1848 Jakob Steiner asked
«How many conics are tangent to five conics?»
In 2019 we ask
«Which conics are tangent to your five conics?»
Curious to know the answer?
Find out at:
juliahomotopycontinuation.org/do-it-yourself/
MAX-PLANCK-GESELLSCHAFT

Watching too much soccer on TV leads to hair loss?

296 people were asked about their hair length and how many hours per week they watch soccer on TV. The data:

$$U = \begin{array}{r} \\ \leq 2 \text{ hours} \\ 2\text{–}6 \text{ hours} \\ \geq 6 \text{ hours} \end{array} \begin{array}{ccc} \text{full hair} & \text{medium} & \text{little hair} \\ \end{array}$$

	full hair	medium	little hair
≤ 2 hours	51	45	33
2–6 hours	28	30	29
≥ 6 hours	15	27	38

Is there a correlation between watching soccer and hair loss?

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Is there a correlation between watching soccer and hair loss?

<u>Extra info</u>: Our study involved 126 men and 170 women:

$$U = \begin{pmatrix} 3 & 9 & 15 \\ 4 & 12 & 20 \\ 7 & 21 & 35 \end{pmatrix} + \begin{pmatrix} 48 & 36 & 18 \\ 24 & 18 & 9 \\ 8 & 6 & 3 \end{pmatrix}$$

We cannot reject the **null hypothesis**:

Hair length is conditionally independent of soccer on TV given gender.

Algebraic Statistics

Philosophy: Statistical models are **varieties**.

Conditional independence of two ternary random variables:

This is the cubic hypersurface in $\mathbb{R}^9$ defined by

$$\det \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \quad = \quad 0.$$

Given any data matrix (u_{ij}), one seeks to maximize the likelihood function $p_{11}^{u_{11}} p_{12}^{u_{12}} \cdots p_{33}^{u_{33}}$ over all points in this model.

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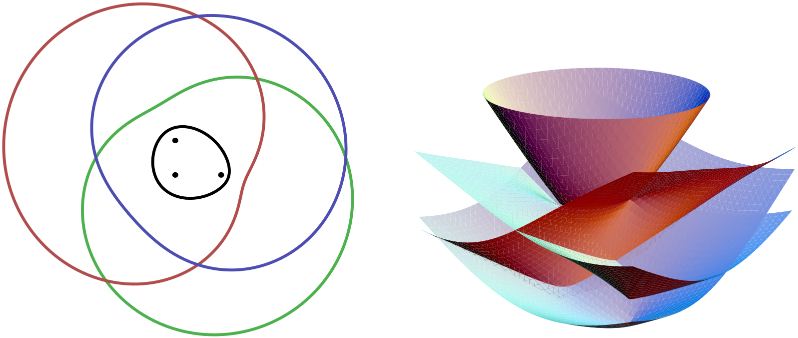
This leads to a system of polynomial equations. It has almost always 10 complex solutions. The **discriminant** is a polynomial in the data $u_{11}, u_{12}, \ldots, u_{33}$. It specifies the case distinction.

[J. Hauenstein, J. Rodriguez, B. St: *Maximum likelihood for matrices with rank constraints*, J. Alg. Stat (2014)]
[J. Rodriguez, X. Tang: *Data discriminants of likelihood equations*, ISSAC 2015]

Optimization

Here is a **3-ellipse**: *"supreme beauty"*



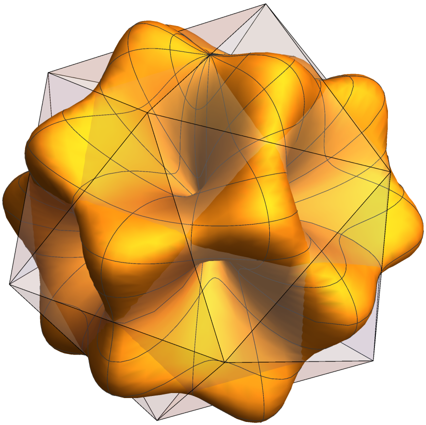
This **variety** is an algebraic curve of degree 8.
If we vary the radius then we obtain a surface of degree 8.

Discriminant? Case Distinction?

Symmetric Tensors of format 3×3×3×3×3×3

A *ternary sextic* has up to 20 local maxima on the sphere,
and up to 62 critical points (eigenvectors).

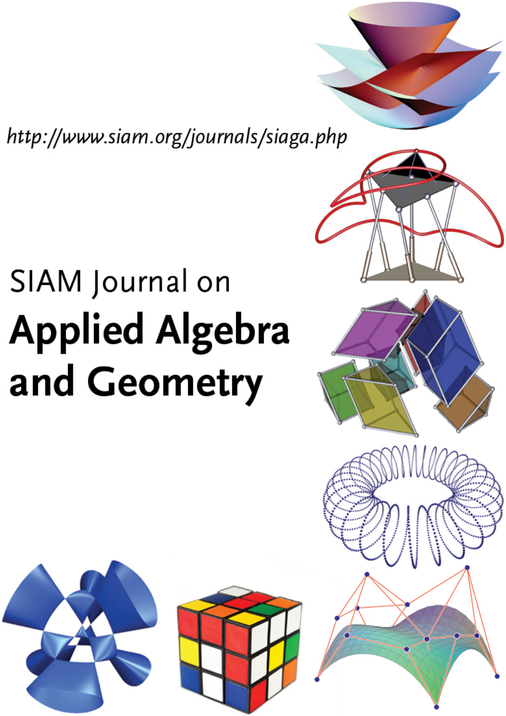
Example: Morse complex is the icosahedron: f-vector (12, 30, 20)



The **eigendiscriminant** has degree 150 in the 28 coefficients.

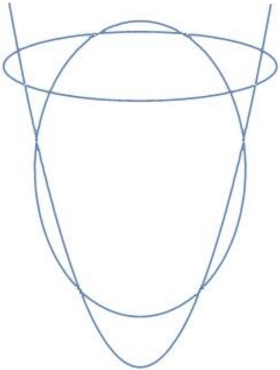
Journal

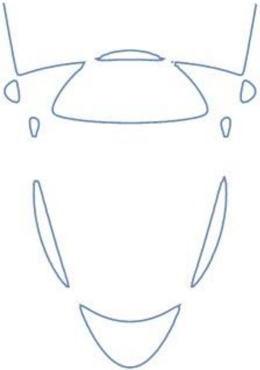
http://www.siam.org/journals/siaga.php



So many varieties, so little time

How to draw all possible **curves of degree** 6 **in the plane**?

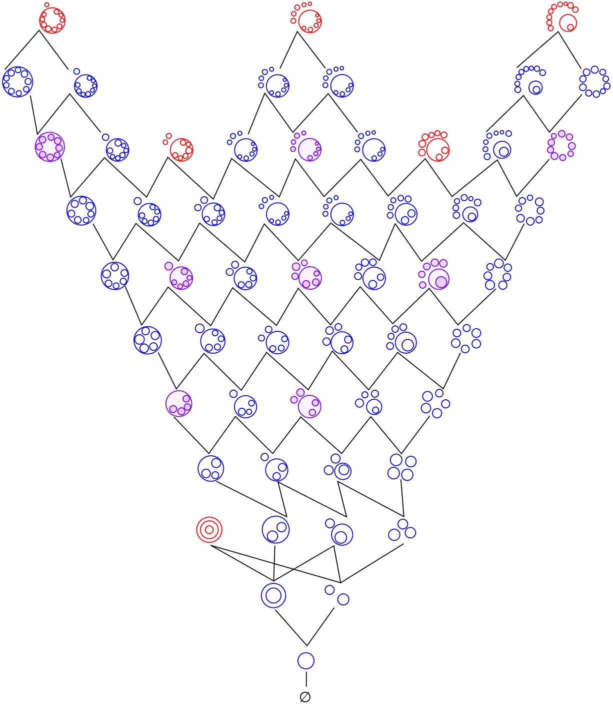




A big **discriminant** furnishes the case distinction.

[N. Kaihnsa et al: Sixty-four Curves of Degree Six Experimental Mathematics 2019]

Curves



Projective Plane

A *line* in the plane $\mathbb{P}^2_{\mathbb{R}}$ is the zero set of a linear form

$$f \;=\; c_1x + c_2y + c_3z.$$

Quiz: What do you get by removing a line from the plane $\mathbb{P}^2_{\mathbb{R}}$?

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Quiz: What do you get by removing a line from the plane $\mathbb{P}^2_{\mathbb{R}}$?

A *conic* in the plane $\mathbb{P}^2_{\mathbb{R}}$ is the zero set of a quadratic form

$$f \;=\; c_1 x^2 + c_2 xy + c_3 xz + c_4 y^2 + c_5 yz + c_6 z^2.$$

A conic is either an oval or empty, depending on the discriminant

$$\det \begin{pmatrix} 2c_1 & c_2 & c_3 \\ c_2 & 2c_4 & c_5 \\ c_3 & c_5 & 2c_6 \end{pmatrix}$$

Quiz: What do you get by removing a conic from the plane $\mathbb{P}^2_{\mathbb{R}}$?

A Big Discriminant

A sextic in $\mathbb{P}^2_{\mathbb{R}}$ is the zero set of

$$f \;=\; c_1x^6 + c_2x^5y + c_3x^5z + c_4x^4y^2 + c_5x^4yz + \cdots + c_{28}z^6$$

The **discriminant** of f is a polynomial Δ of degree 75 in the 28 coefficients $c_1, c_2, c_3, \ldots, c_{28}$.

Can we give a formula?

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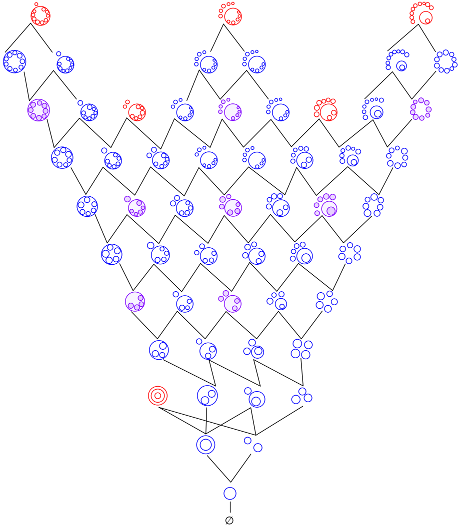
Hilbert's 16th Problem (1900):
Classify all algebraic curves of degree six in the plane $\mathbb{P}^2_\mathbb{R}$.

Theorem (Rokhlin-Nikulin Classification, 1980)

The complement of the discriminant hypersurface in $\mathbb{P}^{27}_\mathbb{R}$ has 64 connected components. The 64 rigid isotopy types are grouped into 56 topological types, with number of ovals ranging from 0 to 11.

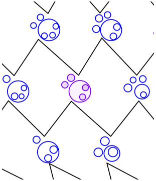
Blue, Red, Purple

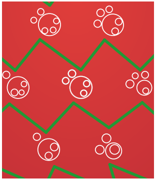


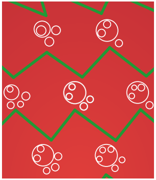
Corollary

Of the 56 *topological types of smooth plane sextics,* 42 *types are non-dividing, six are dividing, and eight can be dividing or non-dividing. This accounts for all* 64 *rigid isotopy types in* $\mathbb{P}^{27}_{\mathbb{R}} \backslash \Delta$.

Transitions

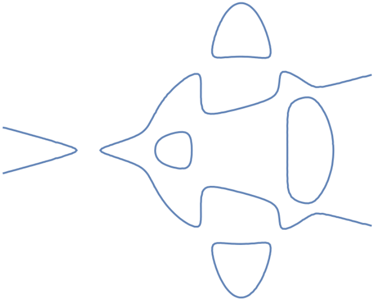


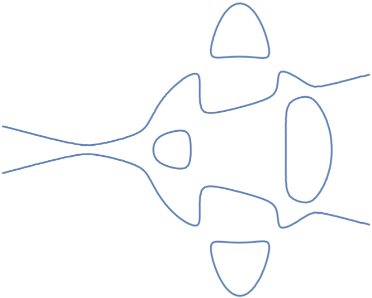




Theorem

For curves of even degree, every **discriminantal transition** *between rigid isotopy types is one of the following: shrinking an ovals, fusing two ovals, and turning an oval inside out.*





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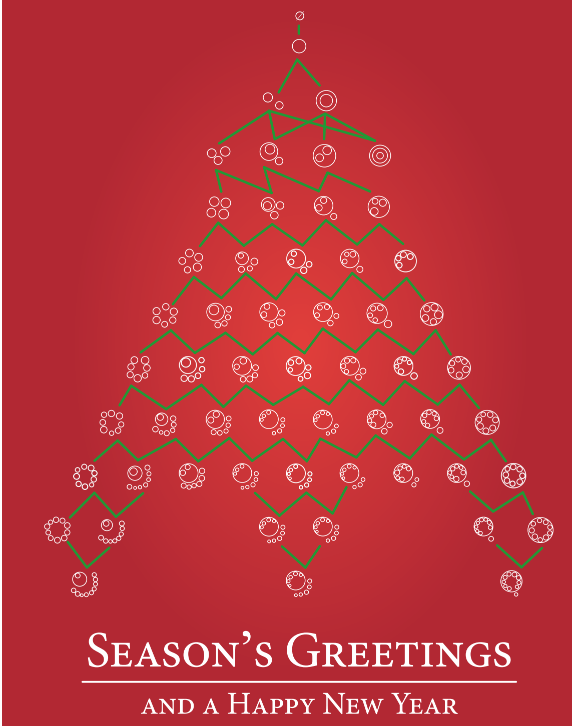

Conclusion

The quadratic equation $ax^2 + bx + c = 0$ has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The **discriminant** is the expression $D = b^2 - 4ac$.

It characterizes the case distinction for the nature of the solutions:

$$D > 0 \quad or \quad D = 0 \quad or \quad D < 0.$$

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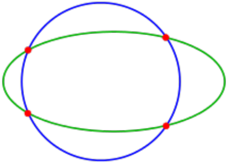

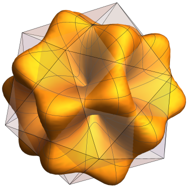

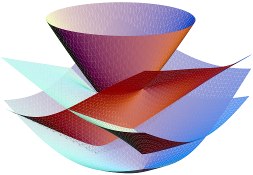

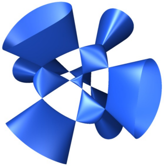

Discriminants are everywhere. They are very important...

... and beautiful. Not just in 9th grade.

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Check out my new book with Kathlén Kohn and Paul Breiding on

Metric Algebraic Geometry