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<u>The $\mathfrak{J}$ team</u> (joint with Armin Straub)

$A^{(s)}(n) = \sum_{k=0}^{n} \binom{n}{k}^s$ generalized Franel numbers

$A^{(1)}(n) = 2^n$ $\quad A^{(1)}(n+1) - 2A^{(1)}(n) = 0$

$A^{(2)}(n) = \binom{2n}{n}$ $\quad (n+1)A^{(2)}(n+1) - 2(2n+1)A^{(2)}(n) = 0$

Franel 1894 $\quad (n+1)^2 A^{(3)}(n+1) - (7n^2+7n+2)A^{(3)}(n) - 8n^2 A^{(3)}(n-1) = 0$

Franel 1895 $\quad (n+1)^3 A^{(4)}(n+1) - 2(2n+1)(3n^2+3n+1)A^{(4)}(n)$

$\quad - 4n(4n-1)(4n+1)A^{(4)}(n-1) = 0$

Predicted a general shape of the recursion, in particular that the order is $\left[\frac{s+1}{2}\right]$

1987 Perlstadt: $s = 5, 6$

1989 McIntosh: $s = 5, 6, \ldots, 10$

Nowadays (≥ 1990) such things are done using the algorithm of creative telescoping invented by Zeilberger

What it does?

It takes a hypergeometric term $a(n,k)$, e.g. $= \binom{n}{k}^s$, and returns the difference operator $P(n,N) \in \mathbb{Z}[n][N]$, where $N: n \mapsto n+1$, and another hypergeometric term $B(n,k)$ such that

telescoper for $a(n,k)$

$$P(n,N)\,a(n,k) = B(n,k+1) - B(n,k)$$

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Sum both sides over $k \in \mathbb{Z}$ (assuming, for example, that the support in k of $a(n,k)$ & $b(n,k)$ for any fixed n is finite):

$$P(n,N) \sum_{k\in\mathbb{Z}} a(n,k) = 0$$

A caveat: the operator $P(n,N)$ may be not of minimal order (in N), reducible.

For example, when $\sum_{k\in\mathbb{Z}} a(n,k)$ admits a different representation, $\sum_{k\in\mathbb{Z}} \tilde{a}(n,k)$ say.

Then $\tilde{P}(n,N)$ ~~and $\tilde{b}(n,k)$~~ can be different from $P(n,N)$ while $\tilde{b}(n,k)$ is certainly different.

For example, run on $A^{(3)}(n) = \sum_{k=0}^{n} \underbrace{\binom{n}{k}^2\binom{2k}{n}}_{\tilde{a}(n,k)}$ to get $\tilde{P} = P$ but different $\tilde{b}(n,k)$.

Conclusion: $P(n,N)$ and its minimality may depend on the choice of $a(n,k)$.

Our result with Armin Straub: (2022)

Theorem: With the choice $a(n,k) = \binom{n}{k}^s$, the order of minimal recursion for $A^{(s)}(n)$ is at least $\lfloor \frac{s+1}{2} \rfloor$.

A bit of history:

1997 M. Stoll showed the upper bound $\lfloor \frac{s+1}{2} \rfloor$ for the recursion for $A^{(s)}(n)$.

He uses estimates from Linear Algebra, but not creative telescoping (but his estimates also depend on representation with $a(n,k) = \binom{n}{k}^s$!)

2007: Yuan, Lu & Schmidt uses an ingeneous congruence argument to give the lower bound ≥ 2 for every $A^{(s)}(n)$ with $s \geq 3$

Nowadays: for each fixed s, to show that the operator is of minimal order one uses available algorithms from computer algebra

By hand: up to $s = 20$.

Our result is for general s.

There is a different aspect of such recursions of order ≥ 2: Apéry limits.

If one chooses two linearly independent solutions of such a recursion, $A(n)$ and $B(n)$ say, then a question of interest is the limit

$$\lim_{n\to\infty} \frac{B(n)}{A(n)}$$

Apéry (1978) used the recursion

$$(n+1)^3 A(n+1) - (2n+1)(17n^2+17n+5)A(n) + n^3 A(n-1) = 0$$

for $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ and another solution $B(n)$, $B(0)=0$, $B(1)=6$

to show that $\lim_{n\to\infty} \frac{B(n)}{A(n)} = \zeta(3)$.

In a recent paper, Chamberland & Straub conjectured that one can choose indep. solutions $\underbrace{A_j^{(s)}(n)}_{j=0,1,\lfloor\frac{s+1}{2}\rfloor-1}$ of the recursion for $A^{(s)}(n)$ such that $A_0^{(s)}(n) = A^{(s)}(n)$ and

$$\lim \frac{A_j^{(s)}(n)}{A_0^{(s)}(n)}$$ is a nonzero rational multiple of π^{2j}

for $j=0,1,\dots,\lfloor\frac{s+1}{2}\rfloor-1$.

We confirm this observation by constructing explicitly such ~~a basis for~~ $\left[\frac{s+1}{2}\right]$ independent solutions (based on $a(n,k) = \binom{n}{k}^s$) and in fact use Apéry limits to prove that order in the main theorem is $\geq \left[\frac{s+1}{2}\right]$.

The idea of constructing the independent solutions is quite simple:

$$P(n,N)\, a(n,k) = B(n,k+1) - B(n,k)$$

can be used not only for integral k!

Using it for $a(n,k-t)$ where $t \in \mathbb{R}$ (say $|t|<1$) we obtain that

$$A^{(s)}(n,t) = \left(\frac{\pi t}{\sin \pi t}\right)^s \sum_{k=0}^{n} \binom{n}{k-t}^s$$

satisfies $\quad P(n,N)\, A^{(s)}(n,t) = \underline{O}(t^s)$ as $t \to 0$

In other words, all coeffs in the t-expansion of $A^{(s)}(n,t)$ for t^j with $j < s$ satisfy the same recursion!

And $A^{(s)}(n,t)$ is an even function, one can check the non-zero coeffs of t^{2j} are actually positive.

The ζ team

Finally, $$\lim_{n\to\infty} \frac{A^{(s)}(n,t)}{A^{(s)}(n)} = \left(\frac{\pi t}{\sin \pi t}\right)^s$$

$$= \left(\sum_{j\geq 1}^{\infty} \left(2 - \frac{1}{2^{2j-2}}\right) \zeta(2j)\, t^{2j}\right)^s$$

$$= \left(\sum_{j=1}^{\infty} \left(\frac{1}{2^{2j-1}} - 1\right) \frac{B_{2j}}{(2j)!} (2\pi i t)^{2j}\right)^s$$

$$\in \mathbb{Q}[[\pi^2 t^2]]$$

The estimate for the order:

Assume that the solutions $A_j^{(s)}(n)$ so constructed are linearly dependent $\Rightarrow$ lin dependent over $\mathbb{Q}$ because $A_j^{(s)}(n) \in \mathbb{Q}$ $\forall j, n$

Write $$0 = \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor - 1} \lambda_j A_j^{(s)}(n) \qquad \lambda_j \in \mathbb{Q}$$
for all $n \geq n_0$.

Divide by $A_0^{(s)}(n)$ and take the limit as $n \to \infty$

$$0 = \sum_j \lambda_j \varphi_j \pi^{2j} \qquad \text{where } \varphi_j \in \mathbb{Q}^\times$$

but the powers of π^2 are lin. independent over $\mathbb{Q}$