Beyond Painlevé: The need for computational tools to reveal hidden structure.

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   - the eigenvalue distribution of the Wishart matrix $H^{\dagger}H$ where H is a Ginibre random matrix

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   - singular value distribution of a product of two Ginibre random matrices

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   - Rank 3 Isomonodromic Deformations with one irregular singularity

- Ginibre matrix $H \in \mathbb{C}^{m\times n}$ with i.i.d entries $H_{i,j} \sim \mathrm{N}[0,\frac{1}{\sqrt{2}}] + i\mathrm{N}[0,\frac{1}{\sqrt{2}}]$
- Shape parameter

$$M = \max(m,n), \quad N = \min(m,n), \quad \alpha = M - N$$

- Complex Wishart matrix

$$W = \begin{cases} H^{\dagger}H & m > n, \\ HH^{\dagger} & n > m, \end{cases}.$$

- Eigenvalues of W are $\{x_1,\ldots x_N\}$ with $x_j \in [0,\infty)$
- Joint Eigenvalue PDF

$$p(x_1,\ldots x_N) = \frac{1}{C_N}\prod_{l=1}^{N} x_l^{\alpha} e^{-x_l} \prod_{1\le j<k\le N}(x_k - x_j)^2$$

Normalisation

$$\int_0^{\infty} dx_1 \ldots \int_0^{\infty} dx_N p(x_1,\ldots x_N) = 1$$

Let $\{p_n(x)\}_{n=0,1,\dots}$ the orthonormal polynomials with respect to the weight function $w(x)$,

$$K(x,y) \equiv [w(x)w(y)]^{1/2} \sum_{n=0}^{N-1} p_n(x)p_n(y)$$

With a_n denoting the coefficient of x^n in $p_n(x)$ the Christoffel-Darboux summation gives

$$K(x,y) = \frac{a_{N-1}}{a_N}[w(x)w(y)]^{1/2} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}$$

The classical weights of the Askey Tableaux are

$$w(x) = \begin{cases} e^{-x^2}, & \text{Gaussian} \\ x^a e^{-x}, \;\; x>0, & \text{Laguerre} \\ (1-x)^a(1+x)^b, \;\; -1<x<1, & \text{Jacobi}, \end{cases}$$

Gap probability as a Fredholm determinant

$$E(0;I) = \det(\mathbb{I} - \mathbb{K}|_I) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_I dx_1 \dots \int_I dx_n \; \det[K(x_j,x_k)]_{j,k=1}^{n}$$

where $\mathbb{K}$ is the integral operator with kernel $K(x,y)$.

Chapter 32 Digital Library of Mathematical Functions, http://dlmf.nist.gov/32

Type	Differential Equation	Parameters	Classical Functions
P-I	$\frac{d^2y}{dx^2} = 6y^2 + x$	–	–
P-II	$\frac{d^2y}{dx^2} = 2y^3 + xy + \nu$	ν	Airy
P-III	$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$	α, β $\gamma = 1, \delta = -1$	Bessel
P-IV	$\frac{d^2y}{dx^2} = \frac{1}{2y}\left(\frac{dy}{dx}\right)^2 + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}$	α, β	Hermite-Weber
P-V	$\frac{d^2y}{dx^2} = \left\{\frac{1}{2y} + \frac{1}{y-1}\right\}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\left\{\alpha y + \frac{\beta}{y}\right\} + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}$	$\alpha, \beta, \gamma,$ $\delta = -1/2$	Confluent Hypergeometric
P-VI	$\frac{d^2y}{dx^2} = \frac{1}{2}\left\{\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right\}\left(\frac{dy}{dx}\right)^2 - \left\{\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right\}\frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left\{\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x(x-1)}{(y-x)^2}\right\}$	$\alpha, \beta, \gamma, \delta$	Gauß Hypergeometric

[Tracy & Widom 1994]

Ensemble	$w(x) =$	OPS	$I =$	P_X	Parameters
GUE	e^{-x^2}	Hermite $H_N(x)$	(s, ∞)	IV	$\alpha = 2N-1,\ \beta = 0$
LUE	$x^a e^{-x}$	associated Laguerre $L_N^a(x)$	$(0, s)$	V	$\alpha = \frac{1}{2}$ $\beta = -\frac{1}{2}a^2$ $\gamma = 2N+a$ $\delta = -\frac{1}{2}$
JUE	$(1-x)^a(1+x)^b$	Jacobi $J_N^{a,b}(x)$	$(-1, s)$	VI	$\alpha = \frac{1}{2}$ $\beta = -\frac{1}{2}a^2$ $\gamma = \frac{1}{2}b^2$ $\delta = \frac{1}{2}[1-(2N+a+b)^2]$

The resolvent function $\eta_0(s)$ satisfies a specialised σ-form equation for Painlevé III′

$$s^2(\eta_0'')^2 - e_1^2(\eta_0')^2 + 4(\eta_0')^2\left(s\eta_0' - \eta_0 + s + e_2\right) - 4\eta_0\eta_0' = 0,$$

subject to the boundary conditions at $s = 0$ [FW2002]

$$\eta_0(s) \underset{s\to 0}{\sim} -\frac{s^{\nu_1-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)\Gamma(\nu_1-\nu_0+1)}.$$

The resolvent is given in terms of Okamoto's function $h(s; v_1, v_2)$ (see Prop. 4.1 of [FW2002], or Eq. (0.7) of [Okamoto 1987])

$$\eta_0(s) = h(s) - \frac{s}{2} - \frac{1}{4}(\nu_1-\nu_0)^2,$$

for the special case $v_1 = v_2 = \pm(\nu_1 - \nu_0)$ [Okamoto 1987].

In the 2×2 matrix formulation of the isomonodromic problem the $\Psi(x, s)$ system is

$$\frac{\partial \Psi}{\partial z} = \left\{ sE + \frac{C - A^{(2)}}{z} + \frac{A^{(2)}}{z-1} \right\} \Psi,$$
$$\frac{\partial \Psi}{\partial s} = \left\{ s^{-1} E z + s^{-1} C \right\} \Psi.$$

Definitions

$$E := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} -\eta_0 & -1 \\ \xi_0 - \eta_1 & \xi_1 \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} x_0 y_0 & x_0 y_1 \\ x_1 y_0 & x_1 y_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 & y_1 \end{pmatrix},$$

- regular singularities: 0 and 1, an irregular singularity ∞,
- resonant or ramified case arises because E is nilpotent with eigenvalues $0, 0$
- the eigenvalues of $C - A^{(1)}$ are $-\nu_0, -\nu_1$,
- $A^{(2)}$ is a rank 1 matrix whose eigenvalues are $0, 0$.

The Schlesinger equations are now ($' = d/ds$)

$$sA^{(2)\prime} = \left[C + sE, A^{(2)} \right], \qquad C' = \left[E, A^{(2)} \right].$$

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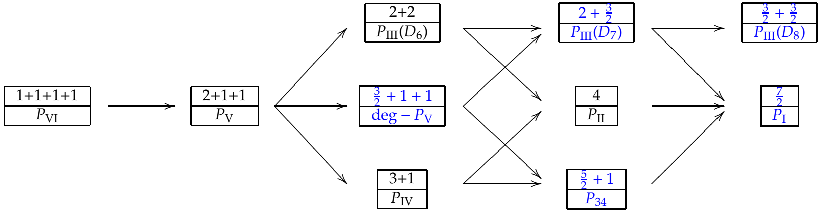
The system has a singularity pattern $\frac{3}{2}+1+1$with Riemann-Papperitz symbol

$$\left\{ \begin{array}{cccc} 0 & 1 & \multicolumn{2}{c}{\infty(\frac{1}{2})} \\ -\nu_0 & 0 & i\sqrt{s} & -\frac{1}{2} \\ -\nu_1 & 0 & -i\sqrt{s} & \nu_0 + \nu_1 \end{array} \right\}.$$

The degeneration scheme of the Painlevé equations interpreted through their isomonodromic deformation problems.

[Kapaev & Hubert 1999], [Kapaev 2002] and [Ohyama and Okumura 2006]



Notes:

- unramified and ramified cases are given in black and blue entries respectively,
- singularity confluence transitions are given by black arrows,
- drop in the Poincaré index transitions (in this case always 1/2) are given by red arrows,
- deg − $P_{\rm V}$ system is equivalent to the $P_{\rm III}(D_6)$ system,
- P_{34} is equivalent to $P_{\rm II}$.

Hamiltonian $\{s; H\}$ with two sets of co-ordinates and momenta $\{x_0, x_1; y_0, y_1\}$ and $\{\xi_0, \xi_1; \eta_0, \eta_1\}$

$$H = -\eta_0 x_0 y_0 + (\xi_0 - \eta_1 + s) x_0 y_1 - x_1 y_0 + \xi_1 x_1 y_1,$$

Hamiltonian equations of motion

$$s x_j' = \frac{\partial}{\partial y_j} H, \quad s y_j' = -\frac{\partial}{\partial x_j} H, \quad j = 0, 1$$

$$\eta_j' = \frac{\partial}{\partial \xi_j} H, \quad \xi_j' = -\frac{\partial}{\partial \eta_j} H, \quad j = 0, 1$$

Coupled quasi-linear ODEs ($' = d/ds$) with respect to s

$$\begin{aligned} s x_0' &= -\eta_0 x_0 - x_1, & s y_0' &= -\xi_0 y_1 - s y_1 + \eta_0 y_0 + \eta_1 y_1, \\ s x_1' &= -\eta_1 x_0 + s x_0 + \xi_0 x_0 + \xi_1 x_1, & s y_1' &= -\xi_1 y_1 + y_0, \\ \xi_0' &= x_0 y_0, & \eta_0' &= x_0 y_1, \\ \xi_1' &= x_0 y_1, & \eta_1' &= x_1 y_1 \end{aligned}$$

Gap probability is

$$\det(\mathbb{1} - \mathbb{K}_1) = \exp\left(\int_0^s \frac{dt}{t} \eta_0(t) \right)$$

Define elementary symmetric functions $e_j, j = 1, 2$ of v_0, v_1.
Trace relations

$$\xi_1 = \eta_0 - e_1, \quad \mathrm{Tr}C = -e_1;$$

Orthogonality relation

$$\mathrm{Tr}A^{(2)} = x_0 y_0 + x_1 y_1 = 0;$$

Further integrals of motion

$$\eta_1 + \xi_0 = e_2;$$

$$s x_0 y_1 = -\eta_0 \xi_1 + \eta_0 + \xi_0 - \eta_1 - e_2;$$

η_0 identified as a Hamiltonian, the identity

$$\eta_0 x_0 y_0 + (\eta_1 - \xi_0 - s) x_0 y_1 + x_1 y_0 - \xi_1 x_1 y_1 + \eta_0 = 0.$$

Folding relations between the two sets (x_0, y_0) and (x_1, y_1)
Assume $x_0 \neq 0$. Then x_1, y_1 are related to x_0, y_0 by

$$x_1 = -s^{e_1} y_0, \quad y_1 = s^{e_1} x_0.$$

$J = (0, s)$, The Hamiltonian is now

$$H = -\eta_0 x_0 y_0 - \eta_1 x_0 y_1 + (\xi_0 - \eta_2 - s) x_0 y_2 - x_1 y_0 - x_2 y_1 + \xi_1 x_1 y_2 + \xi_2 x_2 y_2,$$

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$$H = -\eta_0 x_0 y_0 - \eta_1 x_0 y_1 + (\xi_0 - \eta_2 - s) x_0 y_2 - x_1 y_0 - x_2 y_1 + \xi_1 x_1 y_2 + \xi_2 x_2 y_2,$$

Yields the following system of coupled ODEs

$$\begin{aligned} s x_0' &= -\eta_0 x_0 - x_1, & s y_0' &= -\xi_0 y_2 + s y_2 + \eta_0 y_0 + \eta_1 y_1 + \eta_2 y_2, \\ s x_1' &= -\eta_1 x_0 - x_2, & s y_1' &= -\xi_1 y_2 + y_0, \\ s x_2' &= -\eta_2 x_0 - s x_0 + \xi_0 x_0 + \xi_1 x_1 + \xi_2 x_2, & s y_2' &= -\xi_2 y_2 + y_1, \\ \xi_0' &= -x_0 y_0, & \eta_0' &= -x_0 y_2, \\ \xi_1' &= -x_0 y_1, & \eta_1' &= -x_1 y_2, \\ \xi_2' &= -x_0 y_2, & \eta_2' &= -x_2 y_2. \end{aligned}$$

3×3 Lax Pairs

$$\frac{\partial}{\partial z}\Psi = \left\{ sE + \frac{C - A^{(2)}}{z} + \frac{A^{(2)}}{z-1} \right\}\Psi,$$

and

$$\frac{\partial}{\partial s}\Psi = \left\{ s^{-1}Ez + s^{-1}C \right\}\Psi.$$

Now the definitions are

$$E := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad C := \begin{pmatrix} -\eta_0 & -1 & 0 \\ -\eta_1 & 0 & -1 \\ \xi_0 - \eta_2 & \xi_1 & \xi_2 \end{pmatrix},$$

and

$$A^{(2)} = \begin{pmatrix} x_0 y_0 & x_0 y_1 & x_0 y_2 \\ x_1 y_0 & x_1 y_1 & x_1 y_2 \\ x_2 y_0 & x_2 y_1 & x_2 y_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix}.$$

Note:

- E is nilpotent with eigenvalues $0, 0, 0$ in Jordan blocks of size 2 & 1, i.e. the resonant or ramified case,
- the eigenvalues of $C - A^{(2)}$ are $-\nu_0, -\nu_1, -\nu_2$,
- $A^{(2)}$ is a rank 1 matrix and its eigenvalues are $0, 0, 0$.

The Schlesinger equations take the standard form

$$sA^{(2)\prime} = \left[C + sE, A^{(2)} \right], \qquad C' = \left[E, A^{(2)} \right].$$

System has the singularity pattern $\frac{4}{3}$+1+1 with Riemann-Papperitz symbol

$$\left\{ \begin{array}{cccc} 0 & 1 & & \infty(\frac{1}{3}) \\ -\nu_0 & 0 & s^{1/3} & -\frac{2}{3} \\ -\nu_1 & 0 & \omega s^{1/3} & -1 \\ -\nu_2 & 0 & \omega^2 s^{1/3} & -\frac{1}{3} + \nu_0 + \nu_1 + \nu_2 \end{array} \right\}, \qquad \omega^3 = 1.$$

Let us assume the generic condition $\nu_2 - \nu_1 \neq \mathbb{Z}$ holds.

Energy conservation

$$\eta_0 x_0 y_0 + \eta_1 x_0 y_1 + (-\xi_0 + \eta_2 + s) x_0 y_2 + x_1 y_0 + x_2 y_1 - \xi_1 x_1 y_2 - \xi_2 x_2 y_2 + \eta_0 = 0;$$

Linear Trace relations

$$\xi_2 = \eta_0 - e_1, \quad \mathrm{Tr} C = -e_1, \qquad s x_0 y_2 = \eta_0 \xi_2 + \eta_1 - \xi_1 + e_2 - \eta_0$$

Orthogonality relation

$$\mathrm{Tr} A^{(2)} = x_0 y_0 + x_1 y_1 + x_2 y_2 = 0.$$

In addition the latter relation can be integrated once again to give

$$-3e_3 + e_2(e_1 + \eta_0 - 1) - \eta_0(e_1 - \eta_0 + 1)(e_1 + \eta_0 - 2) + (2e_1 - 1)\eta_1 + (1 - e_1)\xi_1 \\ - s x_0 y_1 + s x_0 y_2(-2\eta_0 + \xi_2 + 2) + s x_1 y_2 - 3(\eta_2 + \xi_0) = 0,$$

and furthermore can be split into the two independent integrals

$$3e_3 + e_2(-2e_1 + \eta_0 - 4) + \eta_0(e_1 - \eta_0 + 1)(2e_1 - \eta_0 + 2) + (-e_1 - 1)\eta_1 + (2e_1 - 3\eta_0 + 4)\xi_1 \\ + 2s x_0 y_1 + s x_0 y_2(2e_1 - \eta_0 + 2) + s x_1 y_2 + 3\xi_0 = 0,$$

$$e_2(e_1 + \eta_0 - 1) - \eta_0(e_1 - \eta_0 + 1)(e_1 + \eta_0 - 2) + (-e_1 + 3\eta_0 - 4)\eta_1 + (1 - e_1)\xi_1 \\ - s x_0 y_1 - s x_0 y_2(e_1 + \eta_0 - 2) - 2s x_1 y_2 + 3\eta_2 = 0.$$

Final integral of the motion is

$$e_3 + \xi_0 - \eta_2 - \eta_0 \xi_1 - \xi_2 \eta_1 - x_2 y_0 + \eta_0 x_2 y_1 - \xi_2 x_1 y_0 + \eta_0 \xi_2 x_1 y_1 \\ - \xi_1 x_0 y_0 + (\xi_0 - \eta_2 - \xi_2 \eta_1) x_0 y_1 + \xi_1 \eta_1 x_0 y_2 + (\xi_0 - \eta_2 - \eta_0 \xi_1) x_1 y_2 + \eta_1 x_2 y_2 = 0.$$

In addition to H there is the radical F by

$$F^2 := 4e_1^2\eta_0'^2 - 12e_2\eta_0'^2 + 12\eta_0\eta_0'^2 - 36s\eta_0'^3 + 9s^2\eta_0''^2 - 12s\eta_0'(\eta_0'' + s\eta_0^{(3)}).$$

But! The quantity F^2 is a perfect square and the radical F is

$$F = -3x_0y_1 - 3x_1y_2 - e_1x_0y_2.$$

The sign is chosen here so that $F > 0$ for the appropriate solution to the boundary conditions.

The resolvent function $\eta_0(s)$ satisfies a scalar ordinary differential equation with degrees $2, 3, 4, 8$ in $\eta_0^{(4)}, \eta_0^{(3)}, \eta_0^{(2)}, \eta_0^{(1)}$

$$
\begin{aligned}
&27s^6 \left(\eta_0^{(4)}\right)^2 \eta_0'^2 \\
&\quad + 27s^4 \left[-F\eta_0'' + 3s^2\eta_0''^3 + 6s\eta_0'^3\eta_0'' + 2\eta_0'^2\left(\eta_0'' + 3s\eta_0^{(3)}\right)\right. \\
&\quad \left. -5s\eta_0'\eta_0''\left(\eta_0'' + s\eta_0^{(3)}\right) + 4\eta_0'^4\right]\eta_0^{(4)} \\
&\quad + 81s^6\left(\eta_0^{(3)}\right)^3\eta_0' \\
&\quad + \left[-27e_1^2s^4\eta_0'^2 + 81e_2s^4\eta_0'^2 + 18Fs^4 - 54s^6\eta_0''^2 - 162s^5\eta_0'\eta_0''\right. \\
&\quad \left. +567s^5\eta_0'^3 - 81s^4\eta_0\eta_0'^2 + 243s^4\eta_0'^2\right]\left(\eta_0^{(3)}\right)^2 \\
&\quad - 3s^2\left[F\left(15s\eta_0'' - 2\eta_0'\left(e_1^2 - 3\left(e_2 + 3s\eta_0' - 7\eta_0\right)\right)\right)\right. \\
&\quad +9s^2\eta_0'\eta_0''^2\left(-2e_1^2 + 6e_2 + 54s\eta_0' - 6\eta_0 + 11\right) + 4\eta_0'\left(9s\left(e_1^2 - 3e_2 + 3\eta_0 - 3\right)\eta_0'^3\right. \\
&\quad \left. -\left(27\left(e_3 + s\right) + 2e_1^3 - 9e_2e_1\right)\eta_0' - 108s^2\eta_0'^4 + 27\eta_0\right) \\
&\quad \left. -18s\eta_0'^2\eta_0''\left(-e_1^2 + 3e_2 + 25s\eta_0' - 3\eta_0 + 3\right) - 45s^3\eta_0''^3\right]\eta_0^{(3)} \\
&\quad + 27s^4\left(-e_1^2 + 3e_2 + 27s\eta_0' - 3\eta_0 + 1\right)\eta_0''^4 \\
&\quad - 54s^3\eta_0'\left(-e_1^2 + 3e_2 + 24s\eta_0' - 3\eta_0 + 1\right)\eta_0''^3 \\
&\quad - 9s^2\left[F\left(-e_1^2 + 3e_2 + 18s\eta_0' + 6\eta_0 + 1\right) - 3s\left(4e_1^2 - 12e_2 + 12\eta_0 + 17\right)\eta_0'^3\right.
\end{aligned}
$$

Recall the generic condition $\nu_2 - \nu_1 \neq \mathbb{Z}$ holds.

Solutions are defined as $s \to 0^+$

$$\eta_0(s) \sim -\frac{\Gamma(\nu_2-\nu_1)\, s^{\nu_1-\nu_0+1}}{\Gamma(\nu_1-\nu_0+2)\,\Gamma(\nu_1-\nu_0+1)\,\Gamma(\nu_2-\nu_0+1)} - \frac{\Gamma(\nu_1-\nu_2)\, s^{\nu_2-\nu_0+1}}{\Gamma(\nu_1-\nu_0+1)\,\Gamma(\nu_2-\nu_0+2)\,\Gamma(\nu_2-\nu_0+1)},$$

Want the asymptotic behaviour as $s \to +\infty$, $|\arg(s)| < \frac{3}{4}\pi$

In this regime the solution for a general resolvent function η_0 permits the asymptotic expansion

$$\eta_0(s) = -\frac{3}{2^{4/3}} s^{2/3} + \mathrm{O}(s^{1/3}, 1/\log(s)).$$

For large s the gap probability $E_2 := E_{\nu_1,\nu_2}$ has the asymptotic form, i.e. the tails of the gap probability

$$E_{\nu_1,\nu_2}(0;(0,s)) = e^{-\frac{9}{2^{7/3}} s^{2/3} + O(s^{1/3})}.$$

Four families and their master cases:
Garnier systems, Fuji-Suzuki systems, Sasano systems and matrix Painlevé systems - [Kawakami, Nakamura & Sakai 2013, 2017, 2018]

1+1+1+1+1
11, 11, 11, 11, 11 $H^{1+1+1+1+1}_{\text{Garnier}}$

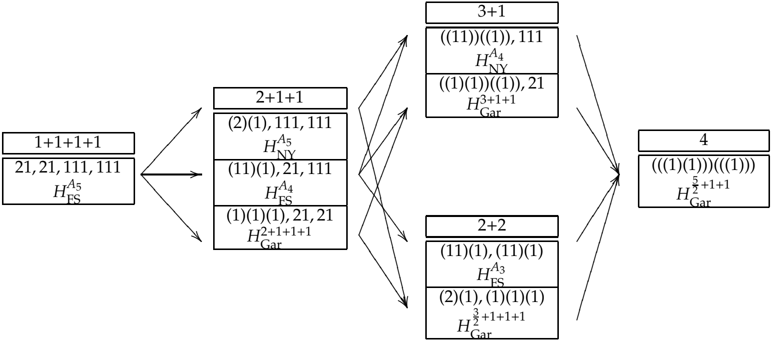
1+1+1+1
21, 21, 111, 111 $H^{A_5}_{\text{Fuji-Suzuki}}$

1+1+1+1
31, 22, 22, 1111 $H^{D_6}_{\text{Sasano}}$

1+1+1+1
22, 22, 22, 211 $H^{\text{Matrix}}_{\text{VI}}$

Identification? Higher order analogues of Painlevé III

Unramified Fuji-Suzuki family which have 3×3 Lax pairs



In addition to the nine shown above another seven ramified cases are given [Kawakami 2015].

However of those only one is a possibility, namely the onen with the singularity pattern $\frac{4}{3}+1+1$ and spectral type $(1)_3, 21, 111$ and has a Riemann-Papperitz symbol

$$\left\{ \begin{array}{cccc} 0 & 1 & & \infty(\frac{1}{3}) \\ 0 & 0 & t^{1/3} & \theta^{\infty}_1/3-\frac{2}{3} \\ \theta^0_1 & 0 & \omega t^{1/3} & \theta^{\infty}_1/3-\frac{2}{3} \\ \theta^0_2 & \theta^1 & \omega^2 t^{1/3} & \theta^{\infty}_1/3-\frac{2}{3} \end{array} \right\}$$

For $\nu_1 = -1/2, \nu_2 = 0$ we compute that the initial terms are

$$\eta_0(s) = -2\frac{\sqrt{s}}{\sqrt{\pi}} - 2(4-\pi)\frac{s}{\pi} - \frac{32}{3}(3-\pi)\frac{s^{3/2}}{\pi^{3/2}} - \frac{16}{9}(72-32\pi+3\pi^2)\frac{s^2}{\pi^2}$$
$$- \frac{64}{45}(360-200\pi+27\pi^2)\frac{s^{5/2}}{\pi^{5/2}} - \frac{512}{675}(2700-1800\pi+347\pi^2-15\pi^3)\frac{s^3}{\pi^3} + \mathrm{O}(s^{7/2}).$$

We have extended this series to high order, and have computed as well the series expansion of F as implied by to high order.
We find the remarkable but not understood relation

$$6 - 2F = \eta_0'.$$

Substituting for F and squaring we find the even more remarkable, and similarly not understood, result that the resolvent function satisfies the much simpler third-order non-linear ODE

$$12s^2\eta_0'{\eta_0}^{(3)} - 9s^2\eta_0''^2 + 12s\eta_0'\eta_0'' - \frac{3}{4}\eta_0'\left[\eta_0'(-48s\eta_0' + 16\eta_0 + 1) + 4\right] + 9 = 0.$$

[Pantone+W]

- $\nu_0 = 0, \nu_1 = -1/2, \nu_2 = 0$:

$$2F = 6 - \eta'$$

$$s^2(\eta'')^2 - \frac{1}{4}(\eta')^2 + 4(\eta')^2 (s\eta' - \eta) + \eta' - 1 = 0,$$

- $\nu_0 = 0, \nu_1 = 0, \nu_2 = +1/2$:

$$2F = 6 + \eta'$$

$$s^2(\eta'')^2 - \frac{1}{4}(\eta')^2 + 4(\eta')^2 (s\eta' - \eta) - \eta' - 1 = 0,$$

- $\nu_0 = 0, \nu_1 = 0, \nu_2 = +1/3$:

$$3F = 9 + \eta'$$

$$36s^2\eta'\eta^{(3)} - 27s^2\eta''^2 + 36s\eta'\eta'' + \eta'\left[36\eta'(3s\eta' - \eta) - \eta' + 6\right] + 27 = 0,$$

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