Stretched exponentials and beyond

Computer Algebra for Functional Equations in Combinatorics and Physics

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Asymptotic counting

Landau notation

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$, $b_n > 0$ be two sequences.

- $a_n = \mathcal{O}(b_n)$ if $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$
- $a_n = \Theta(b_n)$ if $0 < \liminf_{n\to\infty} \frac{|a_n|}{b_n}$ and $\limsup_{n\to\infty} \frac{|a_n|}{b_n} < \infty$
- $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{|a_n|}{b_n} = 1$

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Examples:

Stirling's formula

- $n! = \mathcal{O}(n^n)$
- $n! = \Theta\left(n^{n+1/2}\, e^{-n}\right)$
- $n! \sim \sqrt{2\pi n} n^n e^{-n}$

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Binomial coeffs

- $\binom{2n}{n} = \mathcal{O}(4^n)$
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Double factorials

- $(2n-1)!! = \mathcal{O}(n! 2^n)$
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What is a stretched exponential?

General question

How does a sequence $(a_n)_{n\geq 0}$ behave for large n?

- Often we observe

$$C \cdot R^n \cdot n^\alpha,$$

for constants $C, R, \alpha \in \mathbb{R}$.

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- Much more seldom we observe (or are able to prove)
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with a *stretched exponential* μ^{n^σ} with $\mu > 0$ and $\sigma \in (0,1)$.

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Some deeper reasons why they are “seldom”

- Generating function cannot be algebraic
- It can be *D*-finite (satisfy a linear differential equation with polynomial coefficients), but only only with an *irregular singularity*, e.g., $\exp(\frac{z}{1-z})$

Appearances of stretched exponentials

Known exactly:

- Number theory (integer partitions):
$$\sim (4\sqrt{3})^{-1} e^{\pi(2n/3)^{1/2}} n^{-1}$$
- Theoretical physics (pushed Dyck paths [Beaton, McKay 14], [Guttmann 15]):
$$\sim C_1 4^n e^{-3\left(\frac{\pi \log 2}{2}\right)^{2/3} n^{1/3}} n^{-5/6}$$
- Phylogenetics (phylogenetic tree-child networks [Fuchs, Yu, Zhang 20]):
$$\Theta\left(n^{2n}(12e^{-2})^n e^{a_1(3n)^{1/3}} n^{-2/3}\right)$$

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Conjectured:

- Permutations avoiding 1324 [Conway, Guttmann, Zinn-Justin 18]:

$$\approx \mu^n e^{-cn^{1/2}}$$

- Pushed self avoiding walks [Beaton, Guttmann, Jensen, Lawler 15]:

$$\approx \mu^n e^{-cn^{3/7}}$$

- and recently more and more appear in group theory, queuing theory, . . .

Stretched exponentials in DAG counting

Biology: d-combining tree-child networks

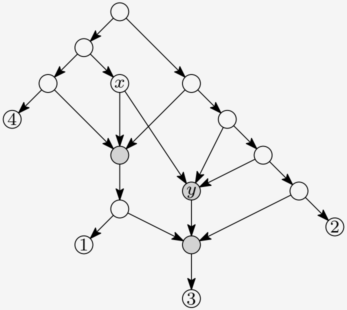
Definition

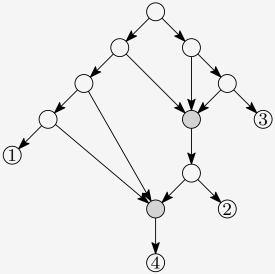
A *d-ary rooted phylogenetic network* is a DAG with nodes of the type:

- *unique root*: indegree 0, outdegree 2
- *leaf*: indegree 1, outdegree 0
- *tree node*: indegree 1, outdegree 2
- *reticulation node*: indegree d, outdegree 1

Furthermore, the n leaves are labeled bijectively by $\{1, \ldots, n\}$.

Tree-child: every non-leaf node has at least one child that is not a reticulation.





Asymptotics of d-combining tree-child networks

A *stretched exponential* μ^{n^σ} appears!

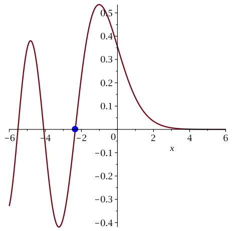
Theorem [Chang, Fuchs, Liu, W, Yu 2023]

The number $\mathrm{TC}_n^{(d)}$ of d-combining tree-child networks with n leaves satisfies

$$\mathrm{TC}_n^{(d)} = \Theta\left((n!)^d \, \gamma(d)^n \, e^{3a_1\beta(d)n^{1/3}} \, n^{\alpha(d)} \right) \qquad \text{for } n \to \infty,$$

with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x)$ and

$$\alpha(d) = -\frac{d(3d-1)}{2(d+1)}, \qquad \beta(d) = \left(\frac{d-1}{d+1}\right)^{2/3}, \qquad \gamma(d) = 4\frac{(d+1)^{d-1}}{(d-1)!}.$$



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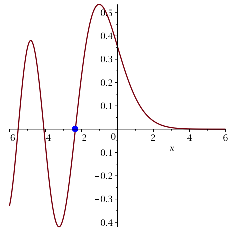
Questions we will answer next

- How to prove this?
- Why is there a stretched exponential?
- Why does the Airy function appear?
  → Previously, e.g., in random maps [Banderier, Flajolet, Schaeffer, Soria 2001] and Brownian excursion area [Flajolet, Louchard 2001]



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     Let $\mathrm{TC}_{n,k}^{(d)}$ be TC networks with n leaves and k reticulation nodes, then

$$\mathrm{TC}_n^{(d)} \sim c_d \mathrm{TC}_{n,n-1}^{(d)}$$

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2. Two parameter recurrence relation

$$e_{n,m} = \mu_{n,m}\, e_{n-1,m+1} + \nu_{n,m}\, e_{n-1,m-1}$$

$n \geq 3$ and $m \geq 0$, $e_{n,-1} = e_{2,n} = 0$ except for $e_{2,0} = 1$,

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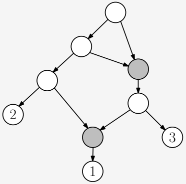
$$\mu_{n,m} = 1 + \frac{2(d-1)}{(d+1)n + (d-1)m - 2(d+1)} \qquad \text{and} \qquad \nu_{n,m} = \prod_{i=2}^{d} \left(1 - \frac{2(m+i)}{(d+1)(n+m)}\right).$$

We are interested in $e_{2n,0}$, as $\mathrm{TC}_n^{(d)} = \Theta\left((n!)^d \left(\frac{\gamma(d)}{4}\right)^n n^{1-d}\, e_{2n,0}\right)$.

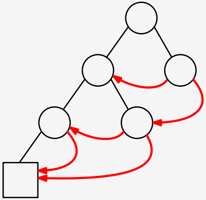
More objects with bivariate recurrences giving stretched exponentials

6	10	14	15	17	18
3	5	9	12	13	16
2	1	7	4	11	8

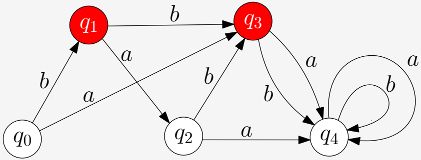
Young tableaux with walls



TC networks



Compressed trees



Minimal automata

BAADBACFCBEDECDFEF

Constrained words

Computer Science: Compacted trees

Definition

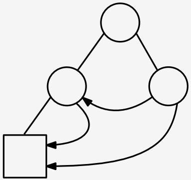
A compacted k-ary tree is a DAG with nodes of the type:

- *unique root*: outdegree k
- *unique sink*: outdegree 0
- *internal nodes*: outdegree k

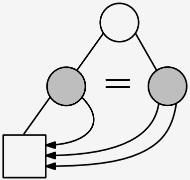
Furthermore,

(O) the **children are ordered** and
(U) all **fringe subgraphs are unique**.

A relaxed k-ary tree is a compacted k-ary tree without condition (U).



Compacted binary tree



Relaxed binary tree

Why are they interesting?

- Applications:
  - **XML-Compression** [Bousquet-Mélou, Lohrey, Maneth, Noeth 2015]
  - **Data storage** [Meinel, Theobald 1998], [Knuth 1968]
  - **Compilers** [Aho, Sethi, Ullman 1986]
  - **LISP** [Goto 1974]
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Reverse question

How many compacted trees of (compacted) size n exist?

Asymptotics of relaxed k-ary trees

A *stretched exponential* μ^{n^σ} appears!

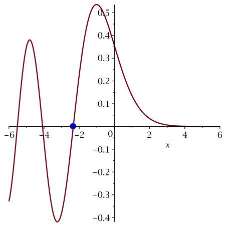
Theorem [Ghosh Dastidar, W 2024+]

The number r_n of relaxed k-ary trees with n internal nodes satisfies

$$r_n = \Theta\left((n!)^{k-1}\,\gamma(k)^n\, e^{3a_1\beta(k)n^{1/3}}\, n^{\alpha(k)}\right),$$

with $a_1 \approx -2.338$: largest root of the Airy function $\mathrm{Ai}(x)$ and

$$\alpha(k) = \frac{7k-8}{6}, \qquad \beta(k) = \left(\frac{k(k-1)}{2}\right)^{1/3}, \qquad \gamma(k) = \frac{k^k}{(k-1)^{k-1}}.$$



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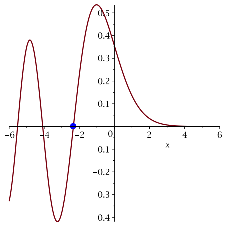
Proof strategy

1. Bijective Comb.: Bijection to decorated Dyck paths
2. Enumerative Comb.: Two-parameter recurrence
3. Calculus + ODEs: Heuristic analysis of recurrence
4. Computer algebra: Inductive proof of asymptotically tight bounds



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Asymptotics in the binary case

Theorem [Elvey Price, Fang, W 2021]

The number of relaxed and compacted **binary** trees satisfy for $n \to \infty$

$$r_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right) \qquad \text{and} \qquad c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$

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Conjecture

Experimentally we find

$$r_n \sim \gamma_r n! 4^n e^{3a_1 n^{1/3}} n \qquad \text{and} \qquad c_n \sim \gamma_c n! 4^n e^{3a_1 n^{1/3}} n^{3/4},$$

where

$$\gamma_r \approx 166.95208957 \qquad \text{and} \qquad \gamma_c \approx 173.12670485.$$

Bijection to decorated paths

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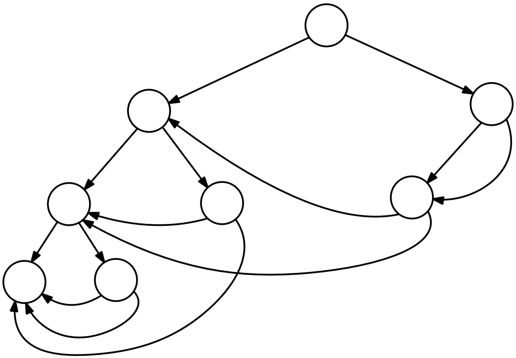

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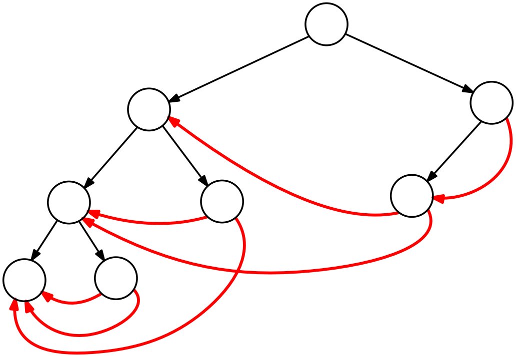

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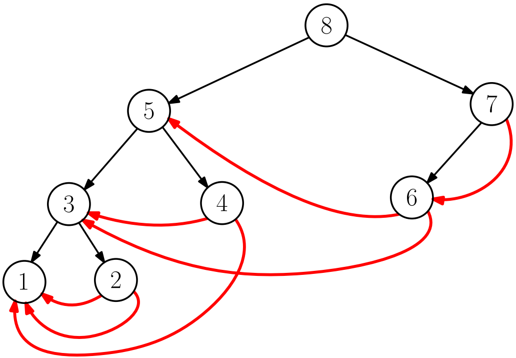

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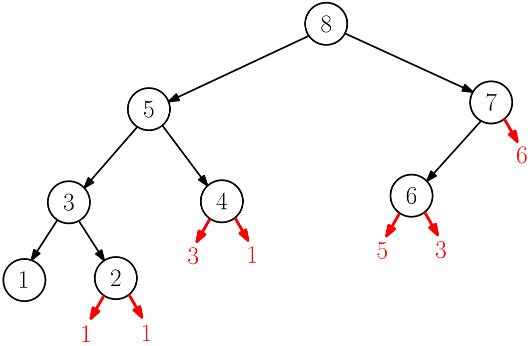

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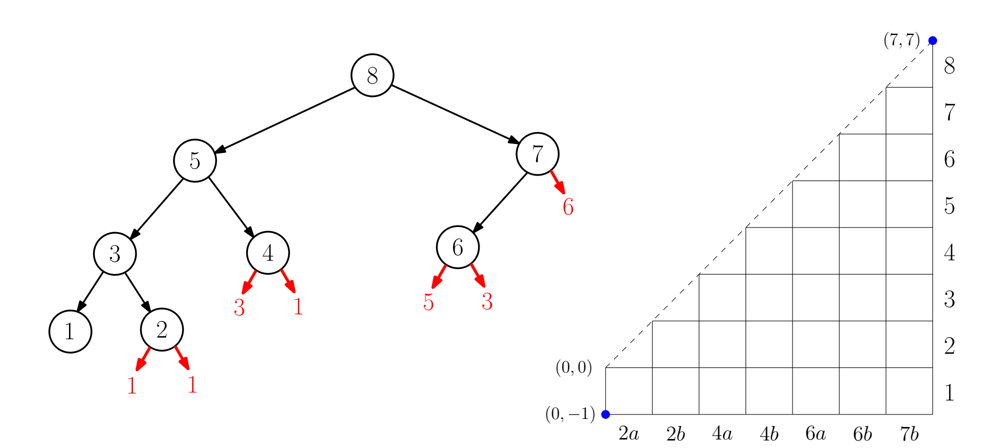

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   - going up: add up step
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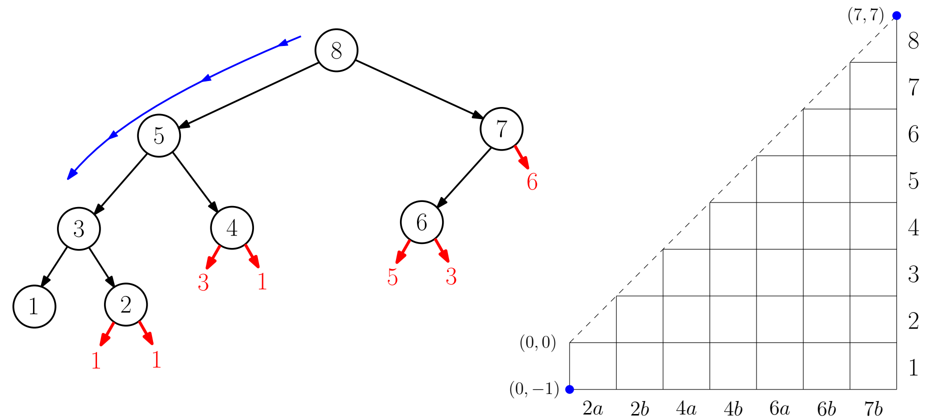

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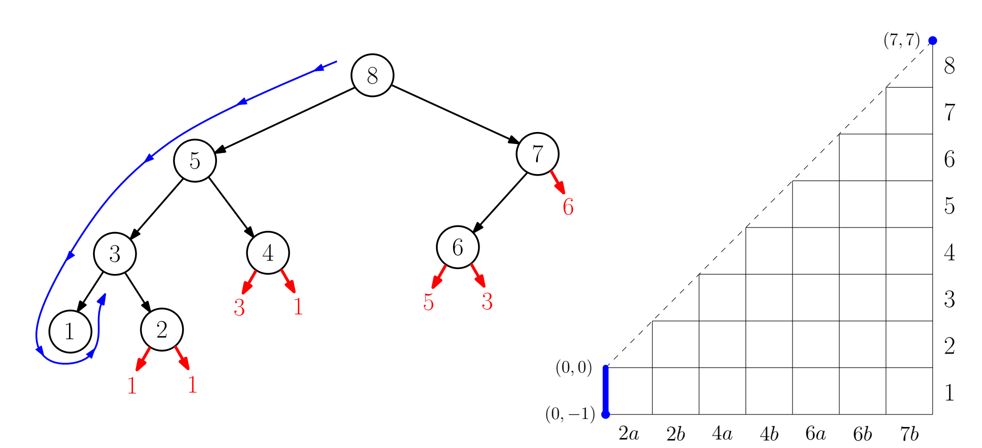

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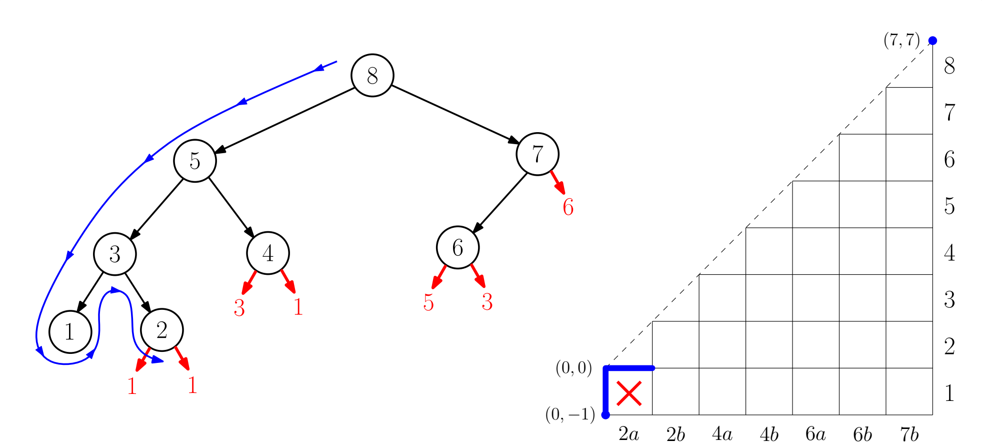

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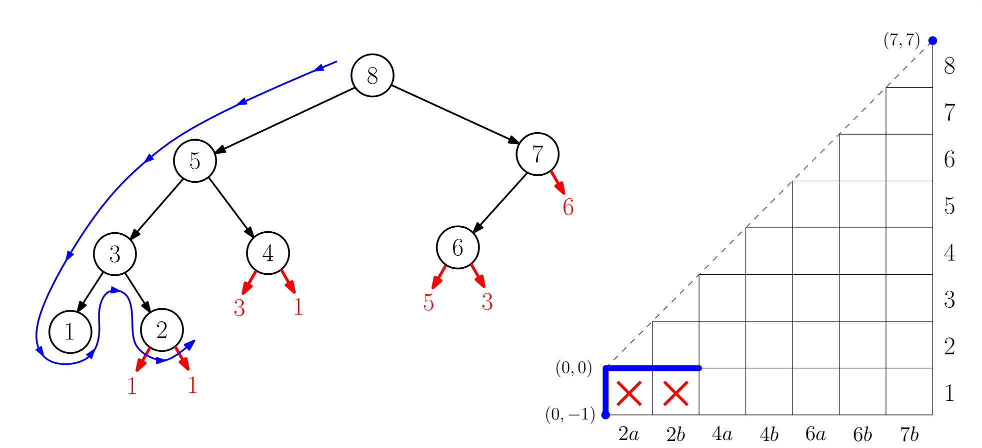

1. **Spanning tree** distinguishes internal edges and pointers
2. Label nodes and pointers in **post-order**
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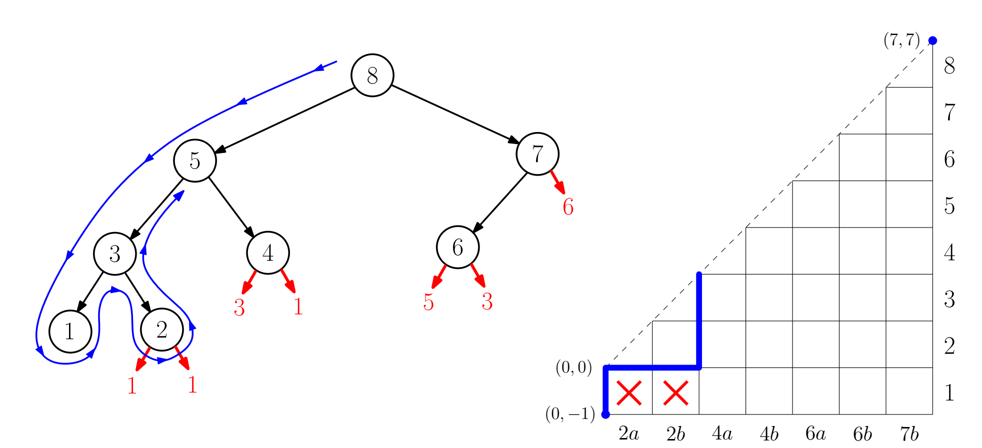

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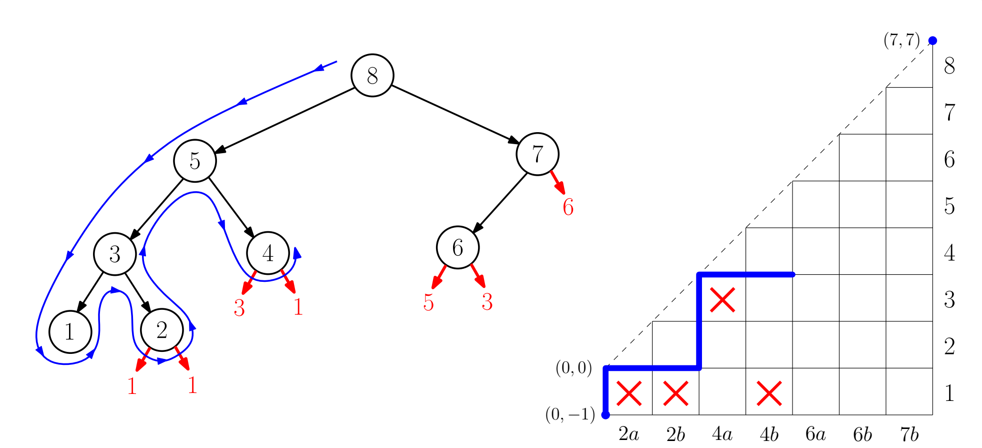

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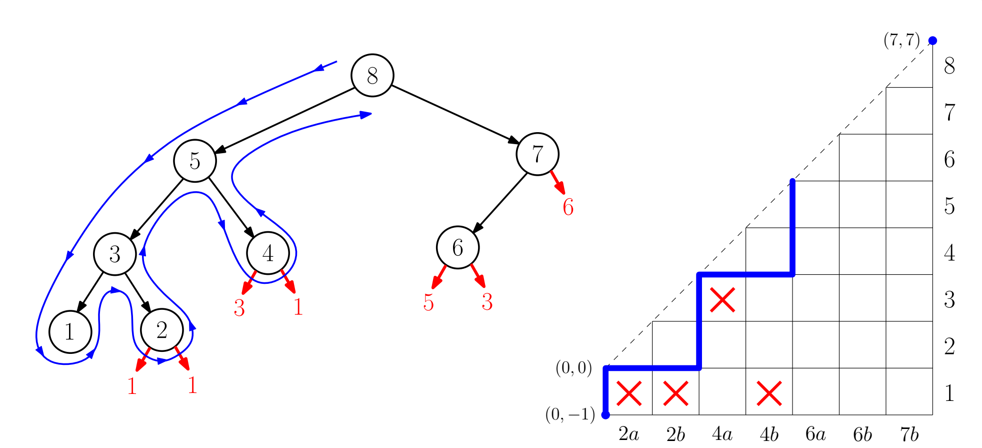

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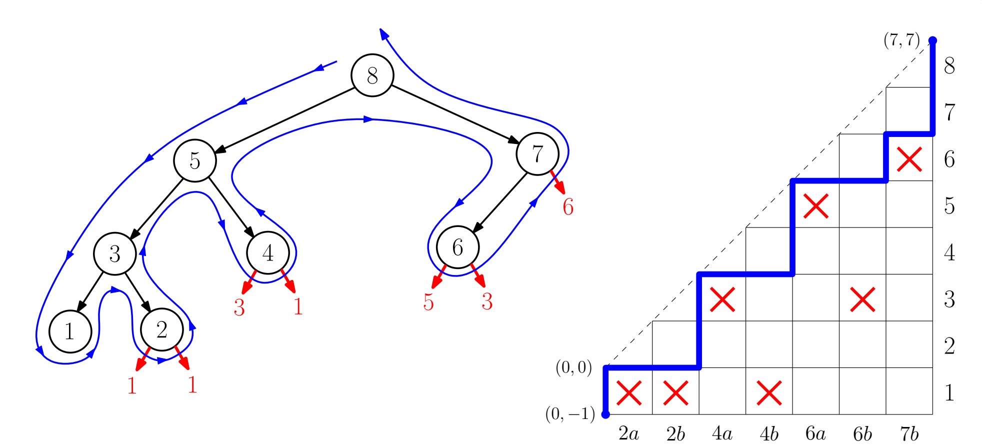

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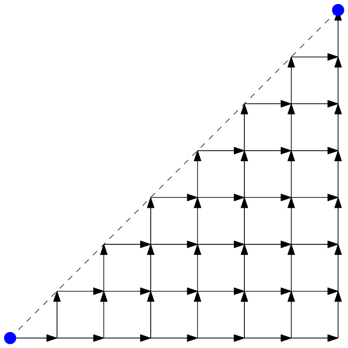

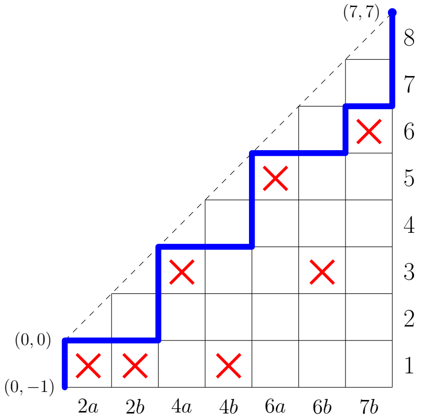

- Path starts at $(0, -1)$ and ends at (n, n)
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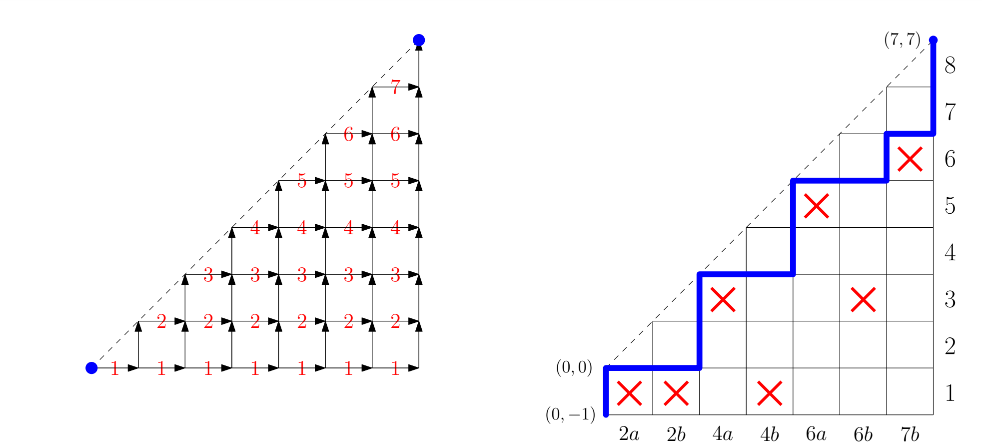
Decorated paths



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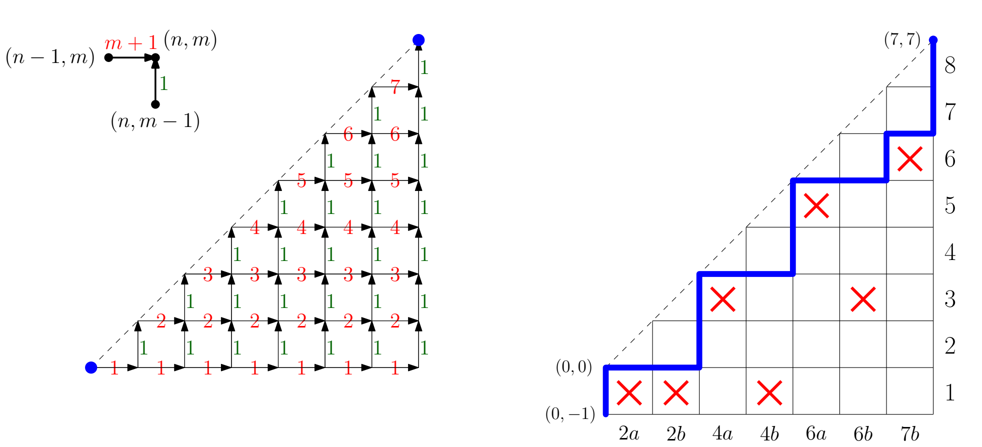

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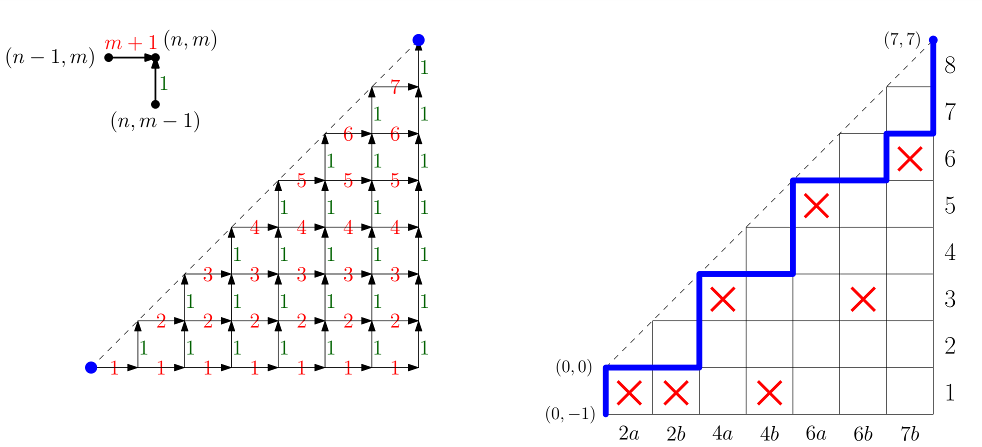
Recurrence for decorated paths



Recurrence: Let $a_{n,m}$ be the number of paths ending at (n,m)

$$a_{n,m} = a_{n,m-1} + (m+1)a_{n-1,m}, \qquad \text{for } n \geq m$$

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Number of relaxed trees is $r_n = a_{n,n}$

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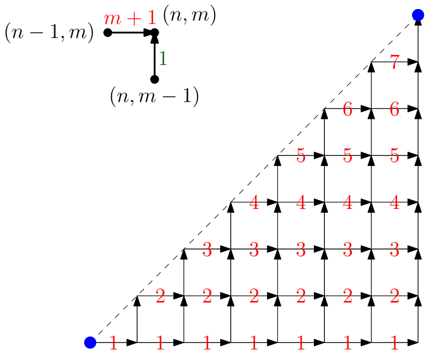

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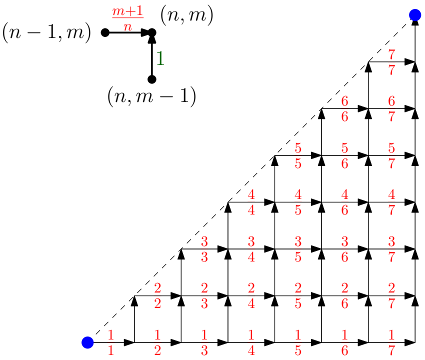

Recurrence: Let $\tilde{a}_{n,m}$ be the number of paths ending at (n,m) with weights divided by column number

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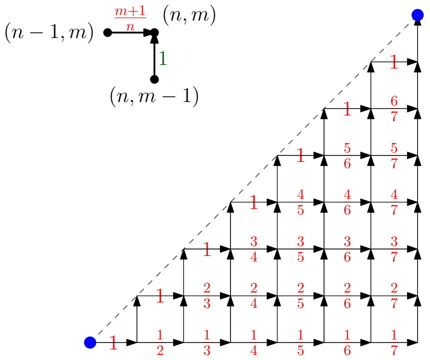

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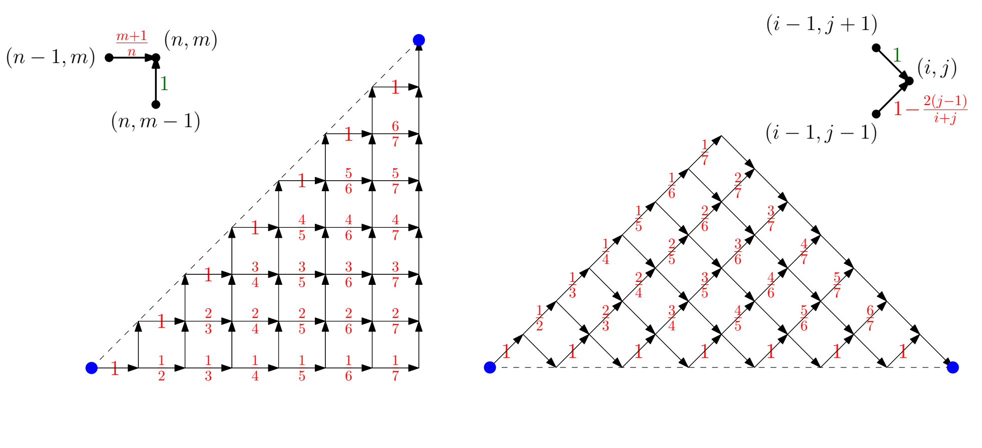

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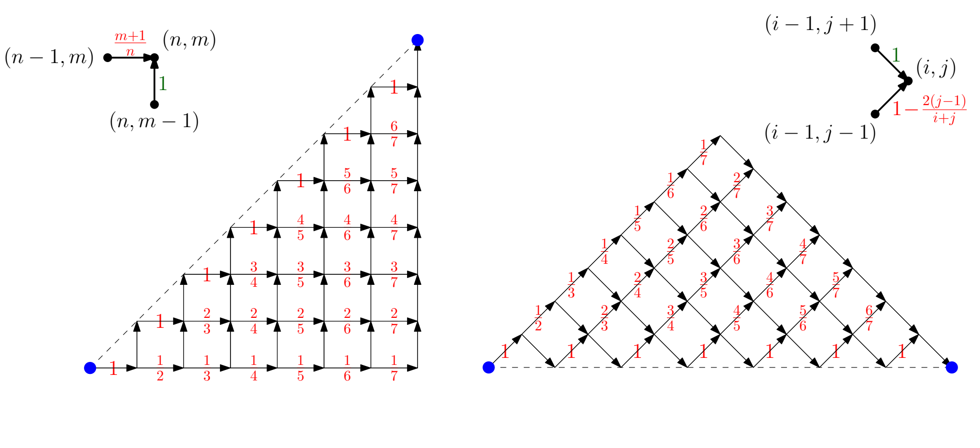


Recurrence: Let $d_{i,j}$ be the number of decorated paths ending at (i, j) shown on the right

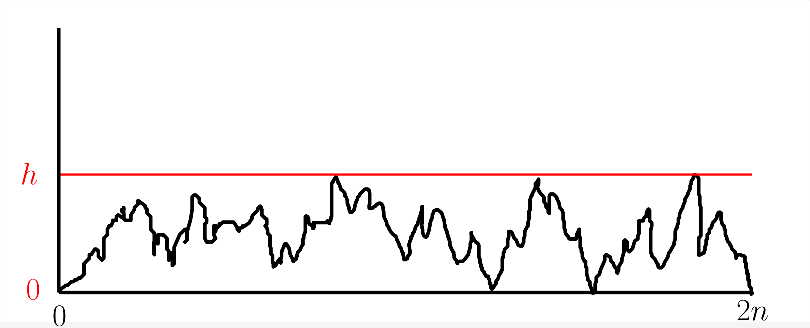
$$d_{i,j} = d_{i-1,j+1} + \left(1 - \frac{2(j-1)}{i+j}\right) d_{i-1,j-1}, \qquad \text{for } i > 0,\ j \geq 0$$

$$d_{0,0} = 1.$$

Number of relaxed trees is $r_n = n!\, d_{2n,0}$

Intuition stretched exponential: Pushed Dyck paths

Dyck paths of length $2n$ where paths of height h get weight 2^{-h}



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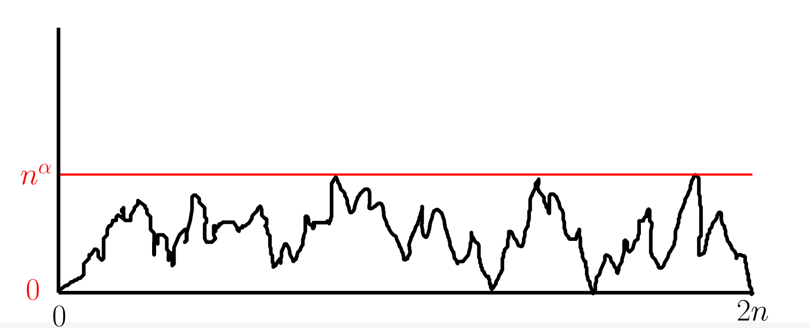

Consider paths with max height $h = n^\alpha$ (for $0 < \alpha \leq 1/2$):

$$\textbf{Number of paths} \approx 4^n e^{-c_1 n^{1-2\alpha}}, \qquad \textbf{Weight} = 2^{-n^\alpha} = e^{-\log(2) n^\alpha}.$$

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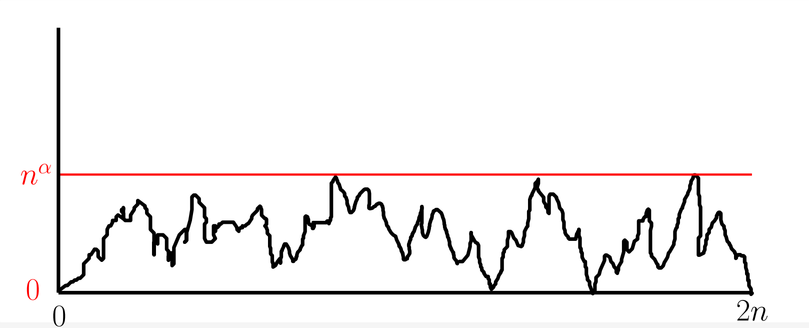

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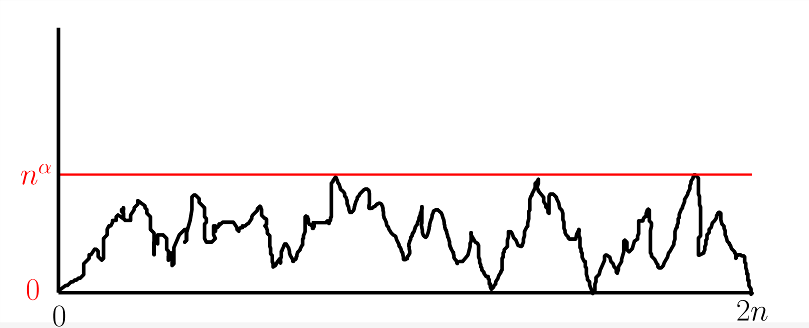

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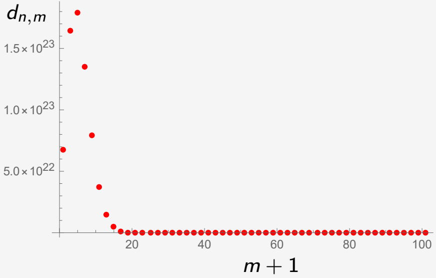
Our case: weights decrease similarly with height so we expect similar behavior

Heuristic analysis of recurrence

Heuristics: What happens for large (fixed) n?

$$d_{n,m} = d_{n-1,m+1} + \left(1 - \frac{2(m-1)}{n+m}\right) d_{n-1,m-1}$$



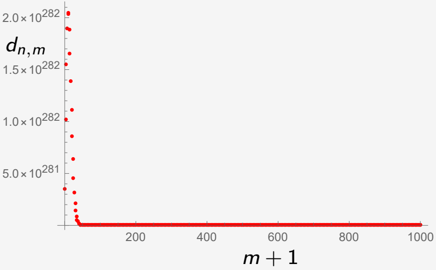


Figure: Plots of $d_{n,m}$ against $m+1$. **Left:** $n = 100$, **Right:** $n = 1000$.

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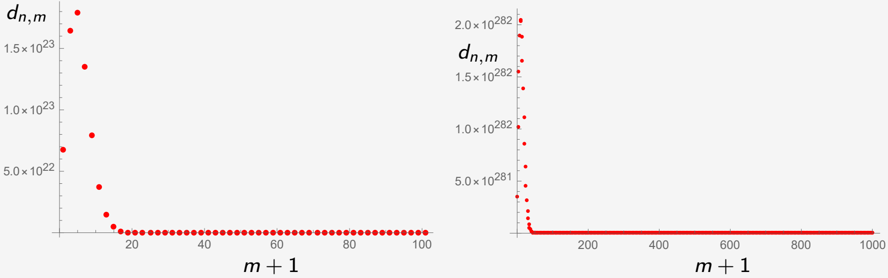

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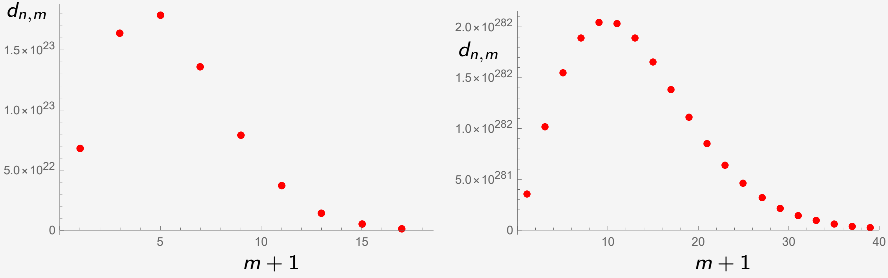

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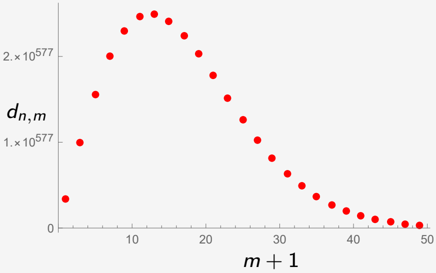

Figure: **Left:** Plot of $d_{n,m}$ against $m+1$ for $n = 2000$. **Right:** Limiting function $f(x)$.

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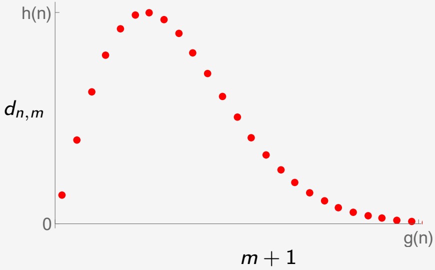

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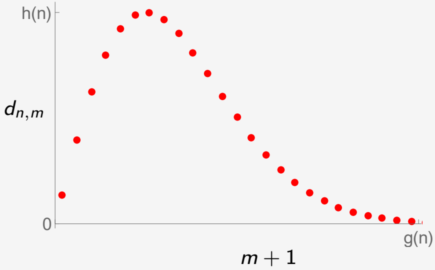

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$$\textbf{Ansatz:} \quad d_{n,m} \approx h(n) f\left(\frac{m+1}{g(n)}\right)$$

Does this ansatz work in the unweighted or unconstrained model?

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$$h(n) \approx \frac{c}{n} 4^n, \qquad g(n) = \sqrt{n}, \qquad f(x) = xe^{-x^2}.$$

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3. Relaxed binary trees $\mu_{n,m} = 1$ and $\nu_{n,m} = 1 - \frac{2(m-1)}{n+m}$ with $m \geq 0$:
$\Rightarrow$ Based on the relation with pushed Dyck paths, we guess $g(n) = \sqrt[3]{n}$.

What are $h(n)$ and $f(x)$?

Heuristic analysis of weighted paths of relaxed binary trees

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$$\frac{h(n)}{h(n-1)} \approx 2 + \frac{f''(x) - 2xf(x)}{f(x)} n^{-2/3} + O(n^{-1})$$

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  Solution

  $$f''(x) = (2x+c) f(x) \qquad \Rightarrow \qquad f(x) = \mathrm{Ai}(2^{-2/3}(2x+c))$$

  where c is a constant and Ai is the Airy function.

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- **Boundary condition:** $d_{n,-1} = 0$ and $d_{n,m} \geq 0$.

  Then $f(0) = 0$ implies $c = 2^{2/3} a_1$, where $a_1 \approx -2.338$ satisfies $\mathrm{Ai}(a_1) = 0$.

Inductive proof

Proof method

Find **explicit sequences** $X_{n,m}$ and $Y_{n,m}$ with the **same asymptotic form**, such that

$$X_{n,m} \leq d_{n,m} \leq Y_{n,m},$$

for all m and all n large enough.

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for all m and all n large enough.

How to find them?

1. Use heuristics
2. Adapt until $X_{n,m}$ and $Y_{n,m}$ satisfy the recurrence of $d_{n,m}$ with the equalities replaced by inequalities:
$$= \quad \longrightarrow \quad \leq \text{ and } \geq$$
3. Prove $X_{n,m} \leq d_{n,m} \leq Y_{n,m}$ by induction.

Relaxed trees: Proof idea – lower bound

Main idea

Suppose $(X_{n,m})_{n\geq m\geq 0}$ and $(s_n)_{n\geq 1}$ satisfy

$$X_{n,m}s_n \leq X_{n-1,m+1} + \left(1 - \frac{2(m+1)}{n+m}\right) X_{n-1,m-1}, \tag{1}$$

for all sufficiently large n and all integers $m \in [0, n]$.

Define $(h_n)_{n\geq 0}$ by $h_0 = 1$ and $h_n = s_n h_{n-1}$; then prove that

$$X_{n,m}h_n \leq b_0 d_{n,m}$$

for some constant b_0 by induction:

Relaxed trees: Proof idea – lower bound

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for some constant b_0 by induction:

Relaxed trees: Proof idea – lower bound

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where $(\sigma_i, \tau_j \in \mathbb{R})$

$$s_n := \sigma_0 + \frac{\sigma_1}{n^{1/3}} + \frac{\sigma_2}{n^{2/3}} + \frac{\sigma_3}{n} + \frac{\sigma_4}{n^{7/6}},$$

$$X_{n,m} := \left(1 + \frac{\tau_2 m^2 + \tau_1 m}{n}\right) \mathrm{Ai}\left(a_1 + \frac{2^{1/3}(m+1)}{n^{1/3}}\right).$$

2. Expand $\mathrm{Ai}(z)$ in a neighborhood of

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$$P_{n,m} = p_{n,m}\mathrm{Ai}(\alpha) + p'_{n,m}\mathrm{Ai}'(\alpha),$$

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Lower bound – Colorful Polygons

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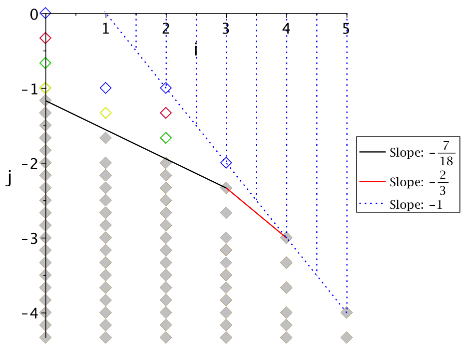

- blue terms: $\sigma_0 = 2$
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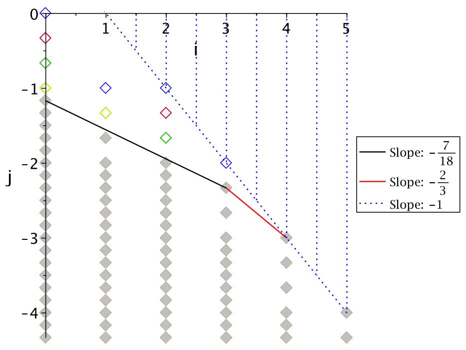

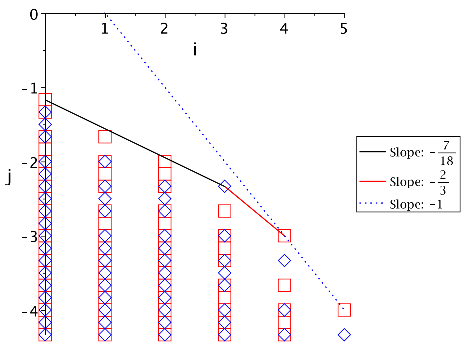

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Case analysis on non-zero coefficients!

$$\Rightarrow \boldsymbol{P_{n,m} \geq 0} \quad \text{for} \quad \boldsymbol{m \in [0, n^{2/3-\varepsilon})}$$

Results and further perturbations

Theorem

The number r_n (c_n) of relaxed (compacted) binary trees, b_n of minimal DFAs recognizing a finite binary language, and y_n of $3 \times n$ Young tableaux with walls in the bottom row satisfy for $n \to \infty$

$$r_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right),$$ [Elvey Price, Fang, W 2021]

$$c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$ [Elvey Price, Fang, W 2021]

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Associated recurrence relations ($n \geq m \geq 0$):

$$r_n = a_{n,n}, \quad \text{where} \quad a_{n,m} = a_{n,m-1} + (m+1)a_{n-1,m}$$

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$$b_n = b_{n,n}, \quad \text{where} \quad b_{n,m} = 2b_{n,m-1} + (m+1)b_{n-1,m} - mb_{n-2,m-1}$$

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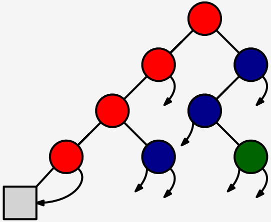
Compacted binary trees of bounded right height

Bounded right height

The **right height** of a binary tree is the maximal number of right children on any path from the root to a leaf (not going through pointers).



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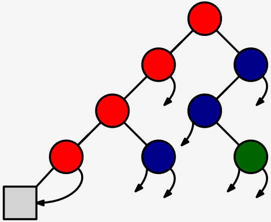

Theorem [Genitrini, Gittenberger, Kauers, W 2020]

The number $r_{k,n}$ ($c_{k,n}$) of relaxed (compacted) trees with right height at most k satisfies for $n \to \infty$

$$r_{k,n} \sim \gamma_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2}},$$

$$c_{k,n} \sim \kappa_k n! \left(4\cos\left(\frac{\pi}{k+3}\right)^2\right)^n n^{-\frac{k}{2}-\frac{1}{k+3}-\left(\frac{1}{4}-\frac{1}{k+3}\right)\cos\left(\frac{\pi}{k+3}\right)^{-2}},$$

where $\gamma_k, \kappa_k \in \mathbb{R} \setminus \{0\}$ are independent of n.

Main idea: Exponential generating functions

- Problem: super-exponential growth $r_{k,n} = \Theta(n!)$ *but* unlabeled structures!
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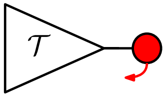
$T(z) \mapsto zT(z)$

Append a new node with a pointer to the class $\mathcal{T}$.



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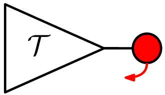

Proof:

$$t_k = k![z^k]zT(z) = \quad k \quad \cdot \quad t_{k-1}$$

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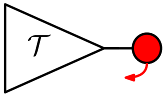


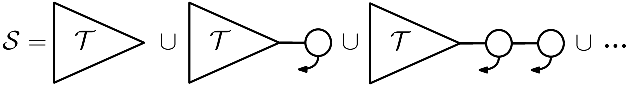
Proof:

$$t_k = k![z^k]zT(z) = \underbrace{k}_{k \text{ possible pointers}} \cdot \underbrace{t_{k-1}}_{k-1 \text{ internal nodes}}$$ □

Further constructions

$S : T(z) \mapsto \frac{1}{1-z} T(z)$

Append a (possibly empty) sequence at the root.



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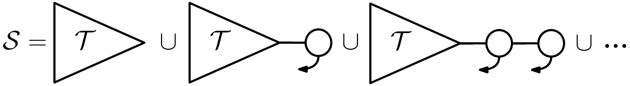

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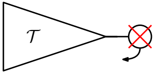

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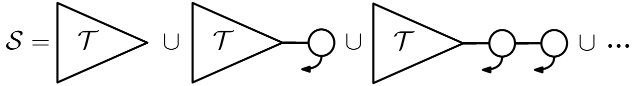

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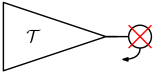
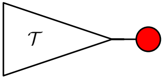


$I : T(z) \mapsto \int T(z)$

Add top node without pointers.



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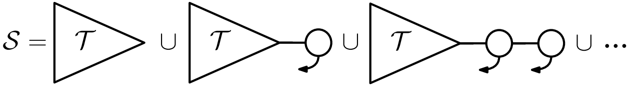

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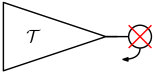

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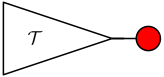
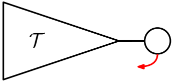
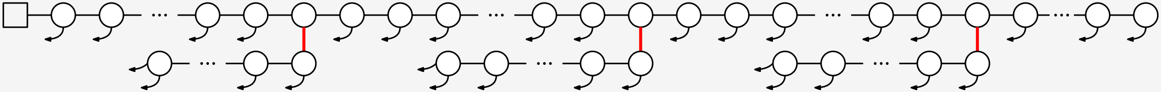


$P : T(z) \mapsto z \frac{d}{dz} T(z)$

Add a new pointer to the top node.



Bounded right height ≤ 1: $R_1(z)$



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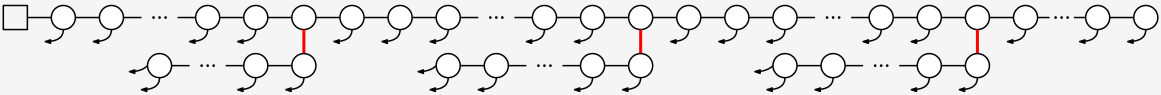

Symbolic construction

$$(1-2z)\,R_1'(z) - R_1(z) = 0,$$
$$R_1(0) = 1,$$

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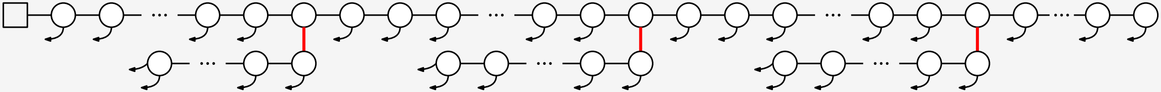

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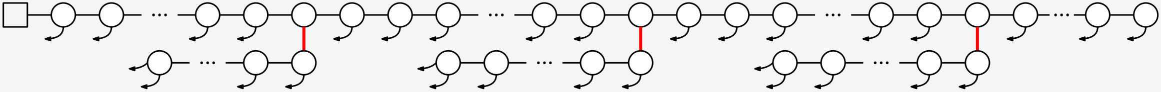

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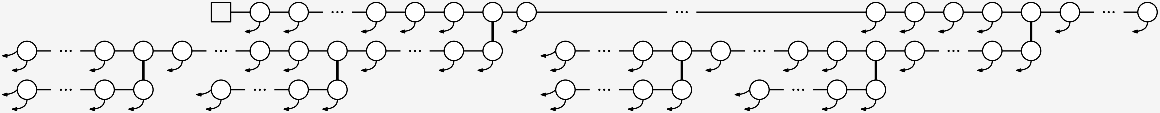
and the coefficients

$$r_{1,n} = \frac{n!}{2^n}\binom{2n}{n} = (2n-1)\cdot(2n-3)\cdots 3\cdot 1.$$

[W 2019, "A bijection of plane increasing trees with relaxed binary trees of right height at most one"].

Bounded right height ≤ 2: $R_2(z)$



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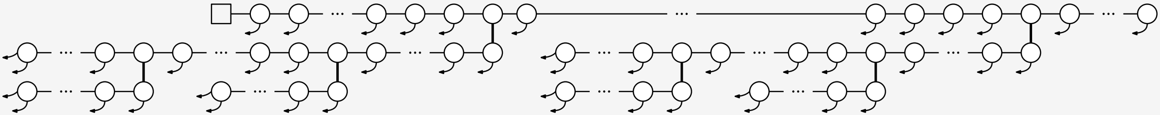

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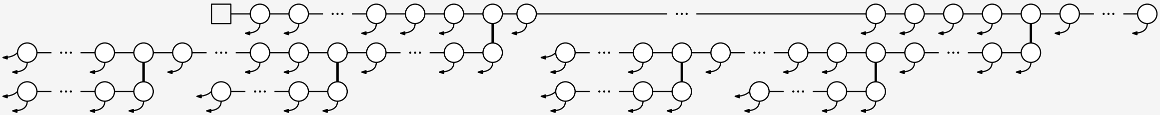

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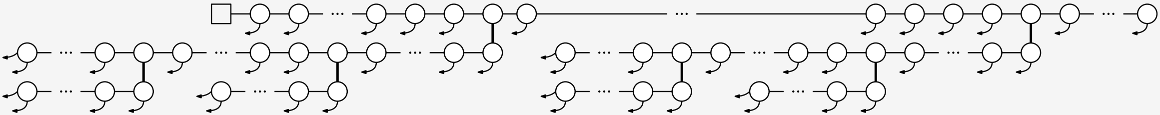

Symbolic construction

$$\left(1 - 3z + z^2\right) R_2''(z) + (2z - 3)\, R_2'(z) = 0,$$
$$R_2(0) = 1, \; R_2'(0) = 1,$$

then we get the closed form

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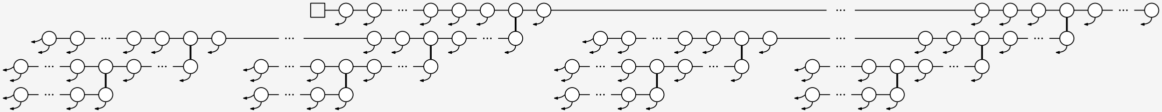
and the coefficients

$$r_{2,n} = \frac{(n-1)!}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{2n} - \left(\frac{1-\sqrt{5}}{2} \right)^{2n} \right).$$

Bounded right height ≤ 3: $R_3(z)$



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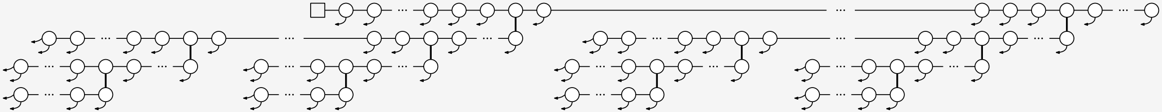

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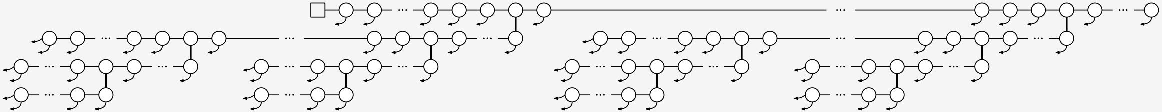

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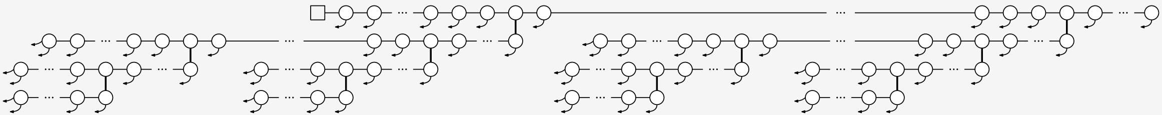

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and the asymptotics of the coefficients

$$r_{3,n} = n![z^n]R_3(z) = \frac{n!}{\sqrt{6}\,(2-\sqrt{3})^{1/\sqrt{3}}}\frac{3^n}{n^{3/2}\sqrt{\pi}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Differential operators

Theorem

Let $D = \frac{d}{dz}$ *and* $(L_k)_{k\geq 0}$ *be a family of differential operators given by*

$$L_0 = (1-z),$$
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$$L_k = L_{k-1}\cdot D - L_{k-2}\cdot D^2 \cdot z, \qquad k \geq 2.$$

Then the exponential generating function $R_k(z)$ *for relaxed trees with right height* $\leq k$ *satisfies*

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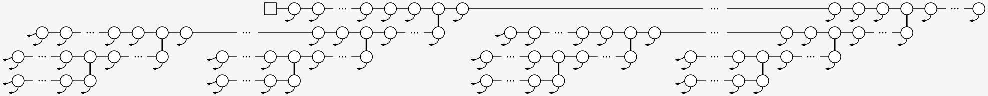
Proof of asymptotics of compacted trees of bounded right height



1. Let $\ell_{k,i} \in \mathbb{C}[z]$ be such that
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Find recurrences for $\ell_{k,i}(z)$ using **Guess'n'Prove techniques**.

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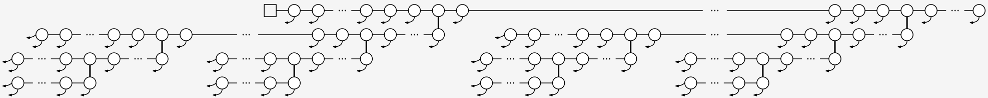

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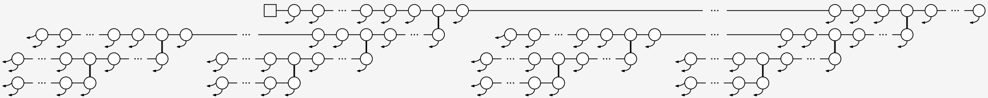

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   2. Subexponential growth:
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      - Use the indicial polynomial derived from the $\ell_{k,i}(z)$.
      - Find a basis of solutions for differential equation: Only one is singular at ρ_k!

Conclusion

Theorem [Genitrini, Gittenberger, Kauers, W 2020]

The number of relaxed and compacted binary trees with **right height at most k** satisfy for $n \to \infty$

$$r_{k,n} \sim \gamma_k n!\, 4^n \cos\left(\frac{\pi}{k+3}\right)^{2n} n^{-\frac{k}{2}} \qquad \text{and} \qquad c_{k,n} \sim \kappa_k n!\, 4^n \cos\left(\frac{\pi}{k+3}\right)^{2n} n^{-\frac{k}{2}-\frac{1}{k+3}-\frac{k-1}{4(k+3)\cos\left(\frac{\pi}{k+3}\right)^2}}.$$

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$$r_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n\right) \quad \text{and} \quad c_n = \Theta\left(n!\, 4^n e^{3a_1 n^{1/3}} n^{3/4}\right),$$

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- Universality of $e^{c\, a_1\, n^{1/3}}$
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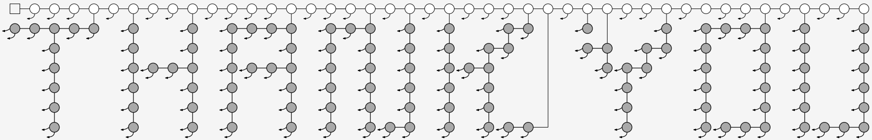

Backup

Comparing compacted and relaxed trees

Asymptotics of compacted and relaxed trees

$$c_{k,n} \sim \kappa_k n! r_k^n n^{\alpha_k} \qquad \text{and} \qquad r_{k,n} \sim \gamma_k n! r_k^n n^{\beta_k}.$$

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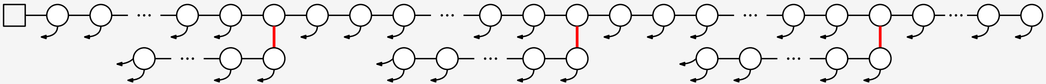
k	r_k	$r_k \approx$	$\kappa_k \approx$	α_k	$\alpha_k \approx$	$\gamma_k \approx$	β_k	$\beta_k \approx$
1	2	2.000	0.708	$-\frac{3}{4}$	−0.750	0.564	$-\frac{1}{2}$	−0.5
2	$4\cos(\frac{\pi}{5})^2$	2.618	0.561	$-\frac{6}{5}-\frac{1}{20\cos(\frac{\pi}{5})^2}$	−1.276	0.447	-1	−1.0
3	3	3.000	0.605	$-\frac{16}{9}$	−1.778	0.493	$-\frac{3}{2}$	−1.5
4	$4\cos(\frac{\pi}{7})^2$	3.246	0.873	$-\frac{15}{7}-\frac{3}{28\cos(\frac{\pi}{7})^2}$	−2.275	0.726	-2	−2.0
5	$4\cos(\frac{\pi}{8})^2$	3.414	1.625	$-\frac{21}{8}-\frac{1}{8\cos(\frac{\pi}{8})^2}$	−2.772	1.379	$-\frac{5}{2}$	−2.5
6	$4\cos(\frac{\pi}{9})^2$	3.532	3.782	$-\frac{28}{9}-\frac{5}{36\cos(\frac{\pi}{9})^2}$	−3.268	3.260	-3	−3.0
7	$4\cos(\frac{\pi}{10})^2$	3.618	10.708	$-\frac{18}{5}-\frac{3}{20\cos(\frac{\pi}{10})^2}$	−3.766	9.350	$-\frac{7}{2}$	−3.5

Construction of $R_1(z)$



Let $R_1(z) = \sum_{\ell \geq 0} r_{1,n} \frac{z^n}{n!}$ be the EGF of relaxed binary trees with bounded right height ≤ 1.

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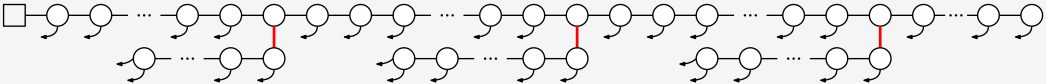

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$$R_1(z) = \sum_{n \geq 0} R_{1,\ell}(z)$$

where $R_{1,\ell}(z)$ is the EGF for relaxed binary trees with exactly ℓ left-subtrees, i.e. ℓ left-edges from level 0 to level 1.

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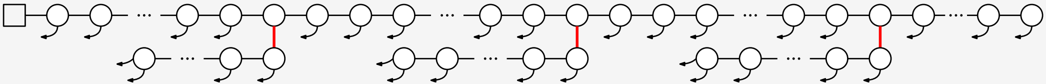

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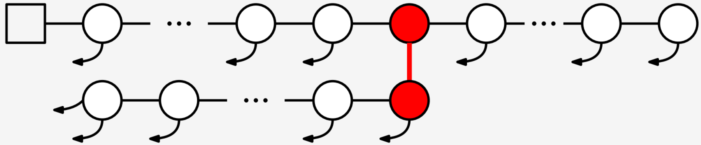

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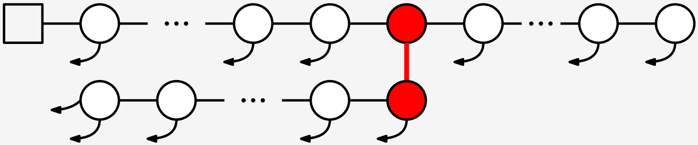
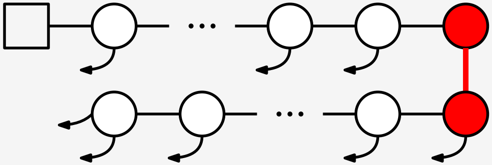


Symbolic specification

1 delete initial sequence



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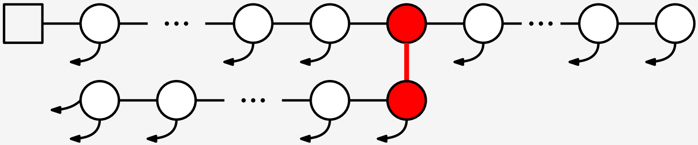
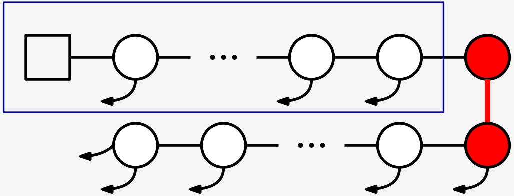


Symbolic specification

1 delete initial sequence
2 decompose



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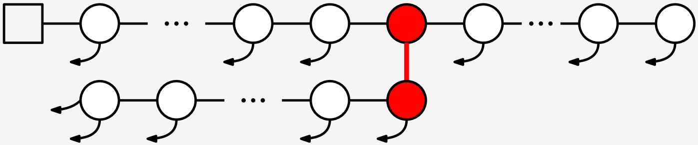
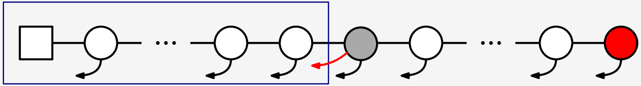


Symbolic specification

1. delete initial sequence
2. decompose
3. append and add pointer



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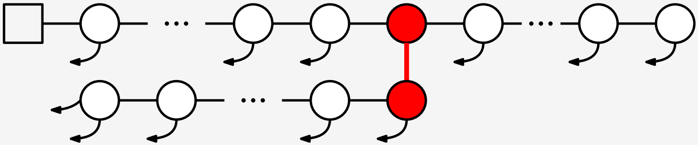

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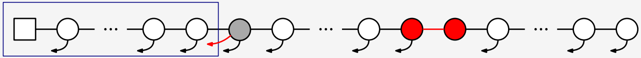
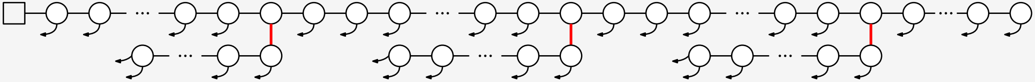


$R_{1,1}(z)$

$$R_{1,1}(z) = \underbrace{S}_{\substack{\text{init.}\\ \text{seq.}}} \circ \underbrace{I}_{\substack{\text{lvl 0}\\ \text{node}}} \circ \underbrace{S \circ P}_{\substack{\text{red pointer}\\ \text{and seq.}}} \left(\underbrace{zR_{1,0}(z)}_{\substack{\text{grey node +}\\ \text{last seq.}}} \right)$$

$$R_{1,1}(z) = \frac{1}{1-z} \int \frac{1}{1-z} z \left(zR_{1,0}(z)\right)' \, dz$$

Construction of $R_{1,\ell}(z)$



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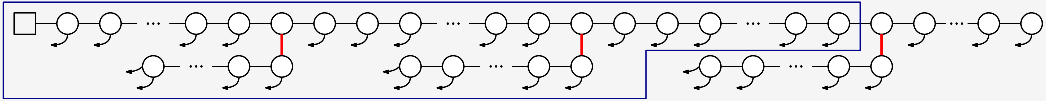
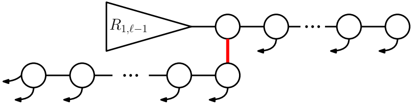


Observation

Same structure as for $R_{1,1}(z)$



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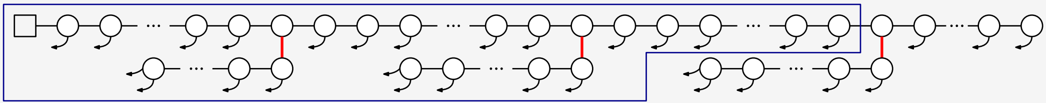

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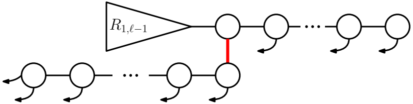

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Recall that $R_1(z) = \sum_{\ell \geq 0} R_{1,\ell}(z)$. Summing the previous equation (formally) for $\ell \geq 1$ gives

$$\frac{1-2z}{1-z} R_1'(z) - \frac{1}{1-z} R_1(z) - \left((1-z) R_{1,0}(z)\right)' = 0.$$

A special class of ODEs

Consider an ordinary generating function of the kind

$$\partial^r Y(z) + a_1(z)\partial^{r-1} Y(z) + \cdots + a_r(z)Y(z) = 0, \tag{2}$$

where the $a_i \equiv a_i(z)$ are meromorphic in a simply connected domain Ω. Let $\omega_\zeta(f)$ be the order of the pole of f at ζ.

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where the $a_i \equiv a_i(z)$ are meromorphic in a simply connected domain Ω. Let $\omega_\zeta(f)$ be the order of the pole of f at ζ.

Definition (Regular singularity)

The differential equation (2) is said to have a singularity at ζ if at least one of the $\omega_\zeta(f)$ is positive. The point ζ is said to be a *regular singularity* if

$$\omega_\zeta(a_1) \leq 1, \qquad \omega_\zeta(a_2) \leq 2, \qquad \ldots, \qquad \omega_\zeta(a_r) \leq r,$$

and an *irregular singularity* otherwise.

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$$\ell_{k,k}(z)\partial^k R_k(z) + \ell_{k,k-1}(z)\partial^{k-1}R_k(z) + \ldots + \ell_{k,0}(z)R_k(z) = 0$$

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Given an equation of the form (2) and a regular singular point ζ, the *indicial polynomial* $I(\alpha)$ at ζ is defined as

$$I(\alpha) = \alpha^{\underline{r}} + \delta_1\alpha^{\underline{r-1}} + \cdots + \delta_r, \qquad \alpha^{\underline{\ell}} := \alpha(\alpha-1)\cdots(\alpha-\ell+1),$$

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All the solutions of the differential equations behave for $z \to \zeta$ like

$$(z-\zeta)^\alpha \log(z-\zeta)^\beta$$

for some $\alpha \in \mathbb{C}, \beta \in \mathbb{N}$.

- α is a root of the indicial polynomial
- β is related to multiple roots of the indicial polynomial and roots at integer distances

A basis for our class of ODEs

Theorem

Consider a differential equation (2) *and a regular singular point* ζ *such that* $\boldsymbol{\omega_\zeta(a_i) \leq 1}$ *for all* $i = 1, \ldots, r$*, and* $\delta_1 \geq 0$*.*

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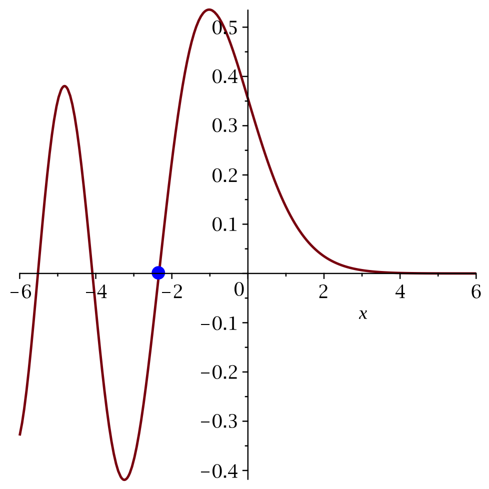
What is the Airy function?

Properties

- $\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$
- Largest root $a_1 \approx -2.338$
- $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$
- Also defined by $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$



- [Banderier, Flajolet, Schaeffer, Soria 2001]: Random Maps
- [Flajolet, Louchard 2001]: Brownian excursion area

Refined heuristic analysis

1. **Ansatz of order 1:**

$$d_{n,m} \approx h(n) f\left(\frac{m+1}{\sqrt[3]{n}}\right),$$
$$s_n = 2 + cn^{-2/3} + O(n^{-1}).$$

yields estimates $c = 2^{2/3} a_1$ such that

$$h(n) \approx 2^n e^{3a_1 (n/2)^{1/3}} \quad \text{and} \quad f(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1).$$

2. **Ansatz of order 2:**

$$d_{n,m} \approx h(n) \left(f_0\left(\frac{m+1}{\sqrt[3]{n}}\right) + n^{-1/3} f_1\left(\frac{m+1}{\sqrt[3]{n}}\right)\right),$$
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yields estimates $d = 8/3$ such that

$$h(n) \sim const \cdot 2^n e^{3a_1 (n/2)^{1/3}} n^{4/3} \quad \text{and} \quad f_0(\kappa) = \mathrm{Ai}(2^{1/3}\kappa + a_1).$$

This way we conjecture the asymptotic form for relaxed binary trees:

$$r_n = n! d_{2n,0} = \Theta\left(n! 4^n e^{3a_1 n^{1/3}} n\right).$$

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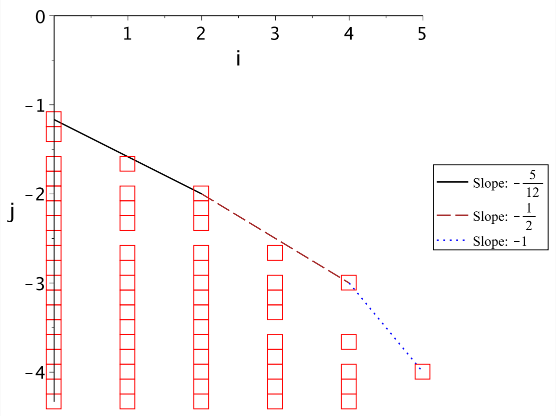
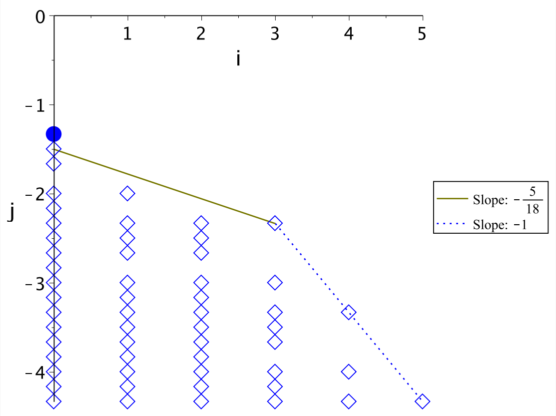
Lower bound – Case analysis

3. Treat $p_{n,m}$ and $p'_{n,m}$ separately and prove that all dominating terms in the respective regimes (corners of convex hull) are positive.

$$p_{n,m} = \sum \tilde{a}_{i,j} m^i n^j \qquad p'_{n,m} = \sum \tilde{a}'_{i,j} m^i n^j$$



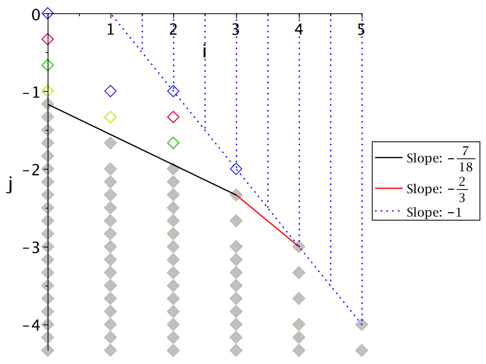


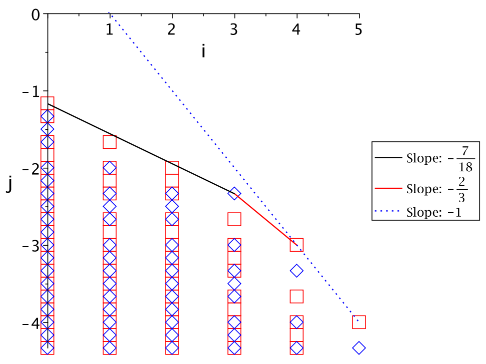
non-zero coefficients

Technicalities for compacted trees and minimal DFAs

Lots of technicalities:

- Before induction, we have to remove the negative term from the recurrence, but we have to do so precisely for asymptotics to stay the same.
- We only prove bounds for small m; we prove that large m terms don't matter
- The lower bound is negative for very large m, so we have to be careful with induction
- We only prove the bounds for sufficiently large n, but this only makes a difference to the constant term. Proof involves colorful Newton polygons:



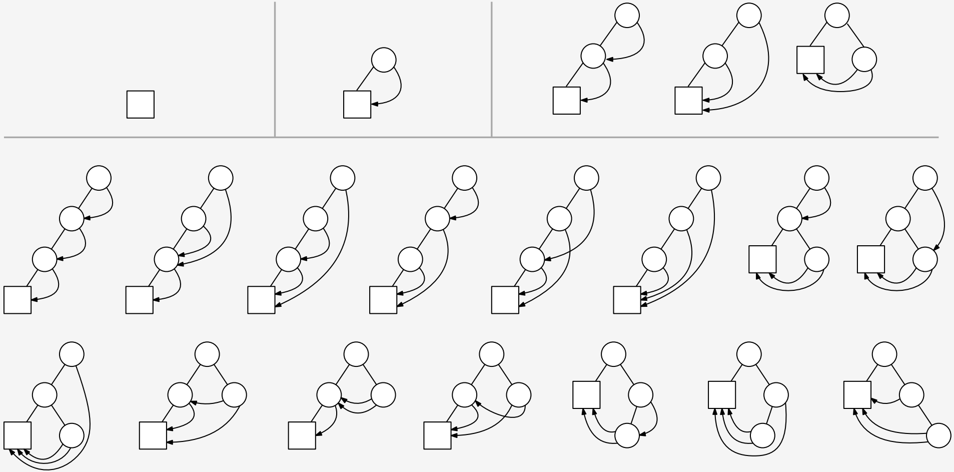


Compacted (unlabeled binary) trees

- **Size:** number of internal nodes
- $\boldsymbol{c_n}$**:** number of compacted trees of size n

$$(c_n)_{n\geq 0} = (1, 1, 3, 15, 111, 1119, 14487, \dots)$$

- Important: Subtrees are unique!



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Simple bounds

$$n! \leq c_n \leq \frac{1}{n+1}\binom{2n}{n} n!$$