Examples of use of functional equations obtained from the elimination theory in nonlinear models

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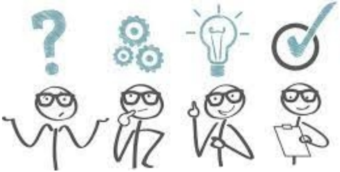
Topical day on Elimination for Functional Equations

Outline

Diagnosis: the process of determining the nature of a disease or disorder and distinguishing it from other possible conditions.



The models

$$\begin{cases} \dot{x}(t,p,f) = g(x(t,p),u(t),f,p), \\ y(t,p,f) = h(x(t,p),u(t),f,p), \\ x(t_0,p,f) = x_0, \\ t_0 \leq t \leq T. \end{cases} \tag{1}$$

Definitions

- ✓ **A fault** is an unpermitted deviation of at least one parameter of the system from the acceptable standard condition.
- ✓ **Fault diagnosability** establishes which faults can be discriminated using the available sensors in a system.
- ✓ **Fault diagnosis** consists in fault detection of the malfunction of a system and the fault isolation of the faulty component.

$f = 0$ means no fault. In the case of uncontrolled models $u = 0$.

N. Verdière, S. Orange, Diagnosability in the case of multi-faults in nonlinear models, Journal of Process Control, Vol 69, pp. 1-7, 2018.

Example: Mass ($m = 1$) attached to an elastic spring (force k), u external force ($\not\equiv 0$), $d \geq 1$

$$\ddot{y} + k(f_1 - 1)^2 y - (d + f_2)u = 0 \qquad \phi(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$$

Remarks

- ✗ Output y known $\Rightarrow \phi = (\phi_1, \phi_2)$ can be estimated.
- ✗ Components of ϕ = set of polynomial equations whose indeterminates are k, d, f_1 f_2.

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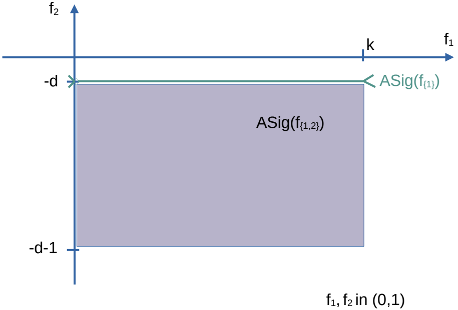
Algebraic signature

$ASig(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$, in particular:

- ✗ $ASig(f_{\{1\}}) = (k(f_1 - 1)^2, -d)$ and $ASig(f_{\{1,2\}}) = (k(f_1 - 1)^2, -d - f_2)$.
- ✗ For all f_1, $f_2 \in (0, 1)$, $ASig(f_{\{1\}}) \cap ASig(f_{\{1,2\}}) = ?$



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- ✗ For all f_1, $f_2 \in (0, 1)$, $ASig(f_{\{1\}}) \cap ASig(f_{\{1,2\}}) = \emptyset$

Definitions

- ✗ Two sets of faults are said **algebraic discriminable** if there exists an algebraic signature, such that, for all input u, the two signatures have an empty intersection.
- ✗ If all the distinct sets of faults are algebraic discriminable, the model is said **algebraically diagnosable**.

Example: Mass ($m = 1$) attached to an elastic spring (force k), u external force ($\neq 0$), $d \geq 1$

$$\ddot{y} + k(f_1 - 1)^2 y - (d + f_2)u = 0 \qquad \phi(f) = \left(k(f_1 - 1)^2, -d - f_2\right)$$

Algebraic signature

$$ASig(f) = \left(k(f_1 - 1)^2, -d - f_2\right).$$

Remark

The current algebraic signature depends on the unknown faults!

By manipulation: $\boxed{ASig(f) = (\phi_1 - k, \phi_2 + d)}$

Remarks

- ✓ New algebraic signature: each of its component depends only on ϕ_k and the parameters of the system;
- ✓ By construction, one of the component of the algebraic signature vanishes when at least one specific (multiple) fault occurs.

Towards algorithms to obtain a calculable algebraic signature and a discriminable table ...

Towards an algorithm to obtain a calculable algebraic signature ...

$$P(y,u,p,f) = m_0(y,u) + \sum_{k=1}^{q} \gamma_k(p,f) m_k(y,u) = 0 \quad \text{and} \quad \begin{cases} \gamma_1(p,f) = \phi_1, \\ \vdots \\ \gamma_q(p,f) = \phi_q, \end{cases}$$

Towards an algorithm to obtain a calculable algebraic signature ...

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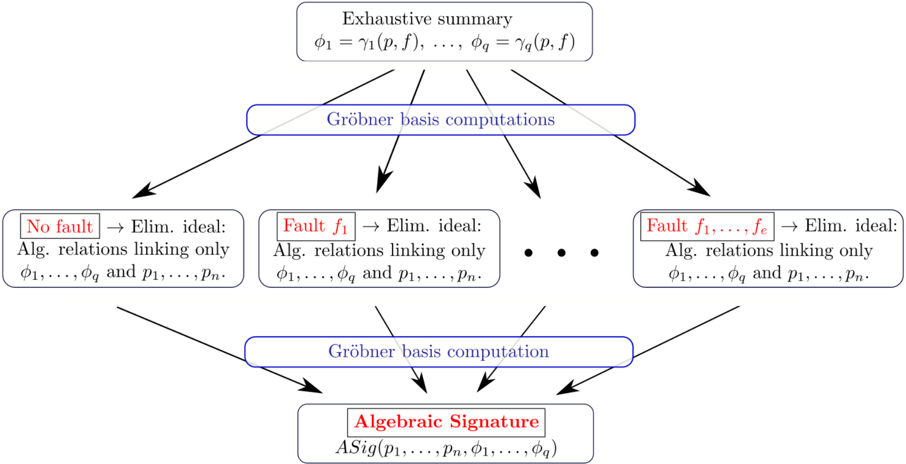
Algorithm *Algebraic-Signature*: Groebner basis computation

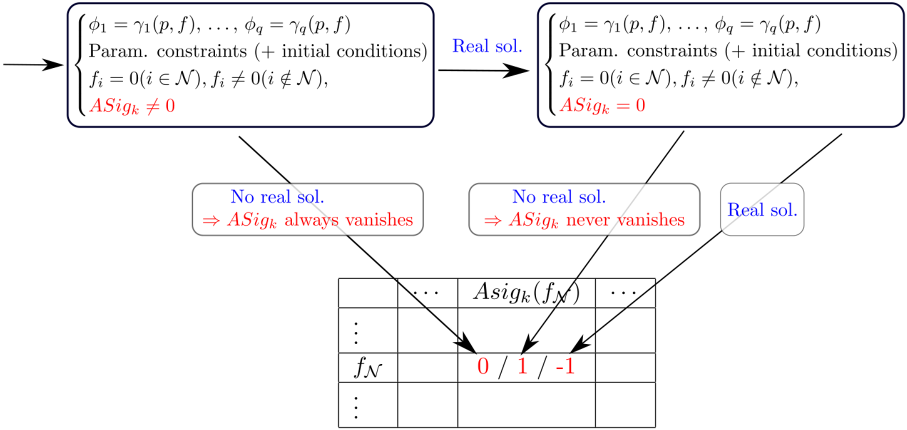


Towards an algorithm to obtain a discriminable table ...

$$\begin{array}{rccc} ASig: & \mathbb{R}^e & \longrightarrow & (R[\phi_1,\ldots,\phi_q])^l \\ & f & \mapsto & (ASig_1(\phi),\ \ldots,\ ASig_l(\phi))\,. \end{array}$$

Procedure *ExpectedValuesOfASign*: determination of a discriminable alg. signature

Inputs: Alg. signature, exhaustive summary, single faults list, param. constraints.
Outputs: lists composed of expected values of algebraic signatures.



Example: $\ddot{x} + k(f_1 - 1)^2 x - (d + f_2)u = 0$, $ASig(f) = (\phi_1 - k, \phi_2 + d)$.

$$C_{p,f} = \{0 < k < 4,\ 1 \le d,\ 0 \le f_1 < 2,\ 0 \le f_2 < 2\}$$

f	$ASig_1(f)$	$ASig_2(f)$
$f_{\{\}}$	0	0
$f_{\{1\}}$	1	0
$f_{\{2\}}$	0	1
$f_{\{1,2\}}$	1	1

$$C_{p,f} = \emptyset$$

f	$ASig_1(f)$	$ASig_2(f)$
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$f_{\{1,2\}}$	-1	1

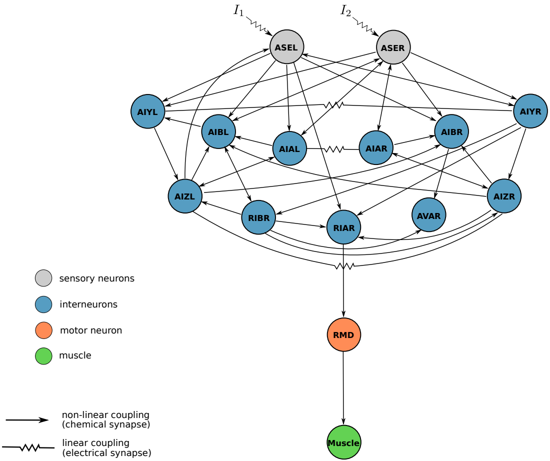
Remarks

- ✓ Importance of the constraints
- ✓ From *ASig*: numerical procedures developed to detect and isolate (multiple) faults acting on the system.

Conclusion and perspective

- ✓ Diagnosability study: from the data collected on the physical system, can the chosen mathematical model permit to discriminate predefined faults that may occur on the system?
- ✓ New example of the interest of functional relations obtained from differential algebra and the semialgebraic approach.
- ✓ Precomputations lead to efficient numerical procedures to detect and isolate (multiple) faults.
- ✓ Reflection to consider the model uncertainties.

An example in neuroscience : A neural network underlying a chemotaxis behavior in *C. elegans*

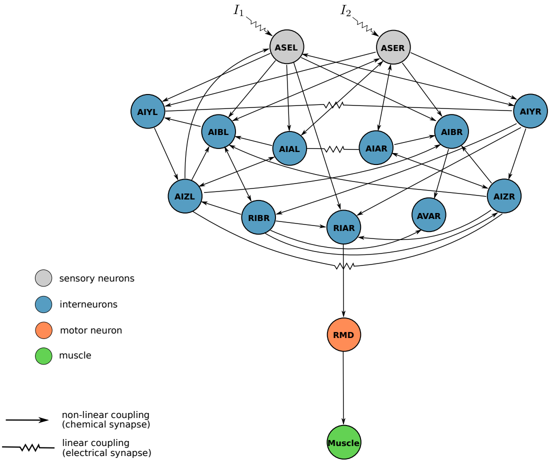


C. elegans:

- ✓ relatively simple nervous system
- ✓ shares numerous fundamental biological features with humans (similar neurotransmitters, channels, and developmental genes).

S. Orange, N. Verdière, L. Naudin, An a priori study for the reconstruction of some variables of interest in nonlinear complex networks with an application in neuroscience, Chaos, Solitons and Fractals, 2023.

An example in neuroscience : A neural network underlying a chemotaxis behavior in *C. elegans*



🤔 Challenge of Loïs: reconstruct the behavior of the muscle from sensory neurons.

Remarks

- ✓ Specific variables associated with each neuron
- ✓ Complicated in practice to measure them all directly
- ✓ Knowledge of sensory neurons sufficient?

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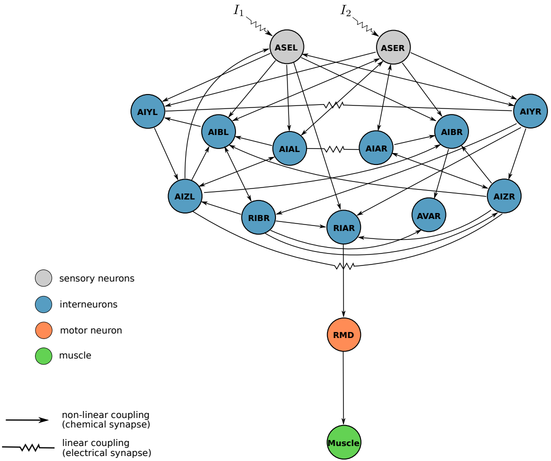

Currently:

- ✓ Global observability (= ability to retrieve the state of the whole system from known inputs and some measured outputs) ⇒ state reconstructors (I. Sendiña-Nadal, C. Letellier, Observability analysis and state reconstruction for networks of nonlinear systems, 2022)
- ✓ Functional observability ⇒ target state reconstruction in *linear* dynamical networks (N. Montanari and al., Functional observability and target state estimation in large-scale networks, 2022)

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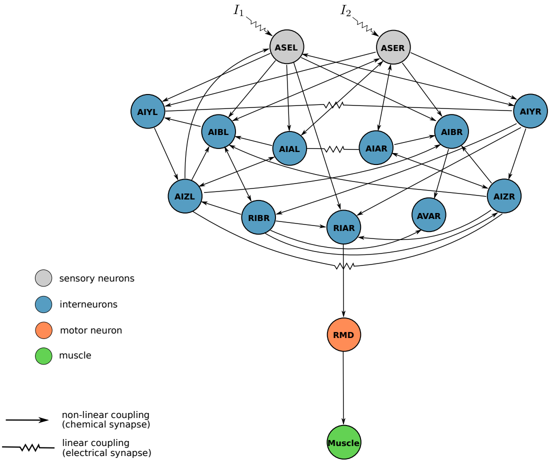
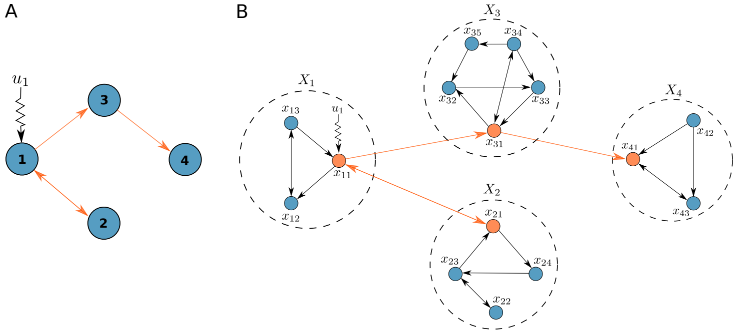


Develop an a priori study and an algorithm to determine minimal sets of nodes needed to be observed in a *nonlinear* network for the reconstruction of certain variables of interest.

Example of a complex network composed of 4 nodes ($N = 4$)



Assumptions

- Internal dynamics of the ith nodes: FitzHugh–Nagumo, Hindmarsh–Rose, Hodgkin–Huxley models....

$$\begin{cases} \dot{x}_{i,1} &= f_{i,1}(X_i, \Theta_i) + \sum_{j \in \mathcal{N}_i^-} c_j(x_{i,1}, x_{j,1}) + u_i \\ \dot{x}_{i,2} &= f_{i,2}(X_i, \Theta_i), \\ \vdots & \\ \dot{x}_{i,n} &= f_{i,n}(X_i, \Theta_i), \end{cases}$$

- $f_{i,j}(X_i, \Theta_i)$ are linear combinations of the state variables $x_{i,2}, \ldots, x_{i,n}$

$x_{i,1}$: variable of interest

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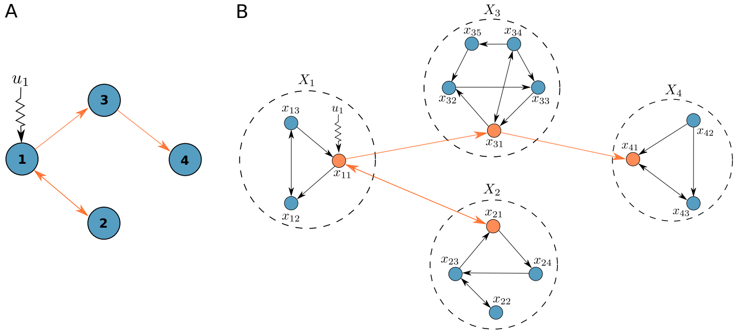


Assumptions

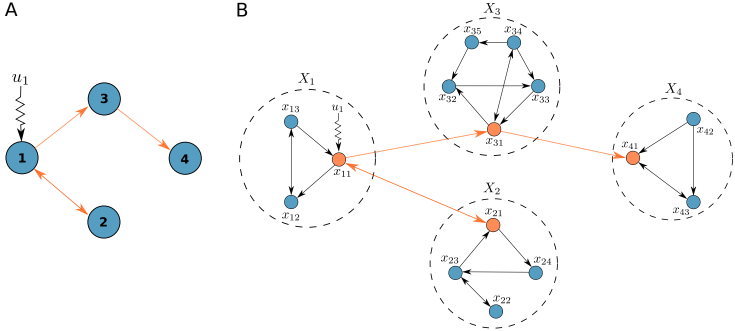
- Linear and nonlinear couplings (Electrical, chemical, mixed synapses) only involve $x_{i,1}$

$$\begin{cases} \dot{x}_{i,1} &= f_{i,1}(X_i, \Theta_i) + \sum_{j \in \mathcal{N}_i^-} c_j(x_{i,1}, x_{j,1}) + u_i \\ \dot{x}_{i,2} &= f_{i,2}(X_i, \Theta_i), \\ \vdots & \\ \dot{x}_{i,n} &= f_{i,n}(X_i, \Theta_i), \end{cases}$$

- $f_{i,j}(X_i, \Theta_i)$ are linear combinations of the state variables $x_{i,2}, \ldots, x_{i,n}$
- $\mathcal{N}_i^-$ the in-neighbors set of the node i
- $c_j : \mathbb{R}^2 \to \mathbb{R}$ is infinitely differentiable
- For all $x_{i,1} \in \mathbb{R}$, the function $x_{j,1} \to c_j(x_{i,1}, x_{j,1})$ is one to one.

$x_{i,1}$: variable of interest

Example of a complex network composed of 4 nodes ($N = 4$)



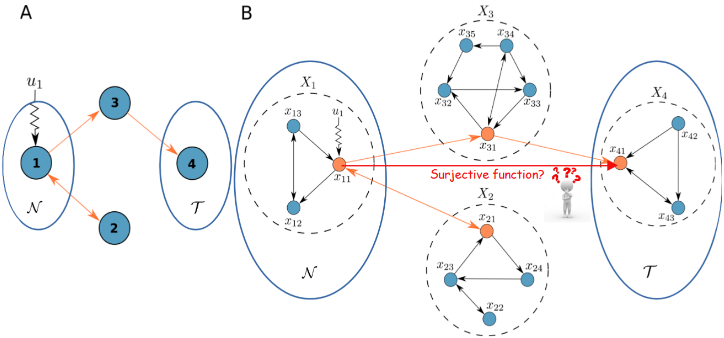
Assumptions

- Values of some variables of interest and their derivatives are known from measurements at a given time*

Specific relations

$$P_i(x_{i,1}, \Theta_i) + \sum_{k=0}^{\alpha} h_{i,k}(x_{i,1}, \Theta_i) u_i^{(k)} + \sum_{j \in \mathcal{N}_i^-} \sum_{k=0}^{\alpha} h_{i,k}(x_{i,1}, \Theta_i) c_j(x_{i,1}, x_{j,1})^{(k)} = 0$$

(*L. Naudin, N. Corson, M. Aziz-Alaoui, J. L. Jimenez Laredo, T. Démare, On the modeling of the three types of non-spiking neurons of the caenorhabditis elegans, International Journal of Neural Systems 31 (02), 2021).



Reconstructibility

Let $\mathcal{N}$ and $\mathcal{T}$ be two sets of nodes of the network ($\mathcal{T}$ = *target set*) (for example $\mathcal{N} = \{1\}$, $\mathcal{T} = \{4\}$).
The solution $(x_{i,1})_{i\in\mathcal{T}}$ is reconstructible from $(x_{i,1})_{i\in\mathcal{N}}$ if there exists a surjective function on the solution set of $(x_{i,1})_{i\in\mathcal{N}}$ in the solution set of $(x_{i,1})_{i\in\mathcal{T}}$.
The set $\mathcal{T}$ is said $\mathcal{N}$-reconstructible afterwards.

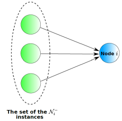
From the local specific relation

$$P_i(x_{i,1},\Theta_i) + \sum_{k=0}^{\alpha} h_{i,k}(x_{i,1},\Theta_i)u_i^{(k)} + \sum_{j\in\mathcal{N}_i^-}\sum_{k=0}^{\alpha} h_{i,k}(x_{i,1},\Theta_i)c_j(x_{i,1},x_{j,1})^{(k)} = 0$$

one gets:

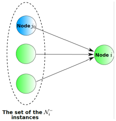
1^{st} consequence

The $\mathcal{N}$-reconstructibility of node i is deduced from the one of its in-neighbors ($\mathcal{N}_i^-$) and of the value of $x_{i,1}$ and its derivatives at a given time $\tilde{t}$.



2^{nd} consequence

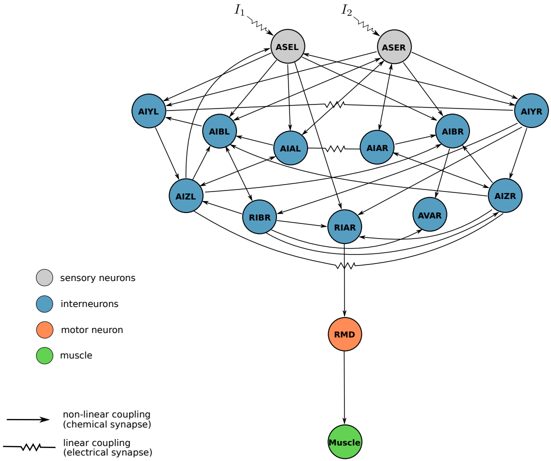
The $\mathcal{N}$-reconstructibility of node j_0 is deduced from the $\mathcal{N}$-reconstructibility of the node i, the $\mathcal{N}$-recontructibility of the nodes in $\mathcal{N}_i^- \setminus \{j_0\}$ and the value of $x_{j_0,1}$ and its derivatives at a given time $\tilde{t}$.



↪ Algorithm `TargetReconstructibilitySets`.

What are the nodes permitting the reconstruction of the node $\mathcal{T} = \{Muscle\}$?

$\Rightarrow$ Algorithm returns the minimal sets (for the inclusion) of nodes to reconstruct the variable of interest of $\mathcal{T}$.



$\mathcal{T} = \{Muscle\}$ is $\{ASEL, ASER\}$-reconstructible, if we observe one of the following set:

$\{RIAR\}$, $\{RMD\}$, $\{AIAL, AIAR\}$, $\{AIAL, AIBL\}$, $\{AIAL, AIYL\}$, $\{AIAL, AIYR, \}$, $\{AIAL, AIZL\}$, $\{AIAL, AIZR\}$, $\{AIAR, AIBL\}$, $\{AIAR, AIYL\}$, $\{AIAR, AIYR\}$, $\{AIAR, AIZL\}$, $\{AIAR, AIZR\}$, $\{AIBR, AIYL, AIZR\}$, $\{AIBR, AIYR, AIZR\}$, $\{AIBR, AIZL, AIZR\}$, $\{AIYL, AIZR, AVAR\}$, $\{AIYL, AIZR, RIBR\}$, $\{AIYR, AIZR, AVAR\}$, $\{AIYR, AIZR, RIBR\}$, $\{AIZL, AIZR, AVAR\}$, $\{AIZL, AIZR, RIBR\}$, $\{AIBL, AIBR, AIZR, AVAR\}$, $\{AIBL, AIBR, AIZR, RIBR\}$, $\{AIBL, AIZR, AVAR, RIBR\}$

Summary

- ✓ Target reconstructibility: determining which nodes are needed to infer the state of a target subset
- ✓ Theoretical results based on specific local relations and two reconstructibility properties
- ✓ Algorithm `TargetReconstructibilitySets`
- ✓ Application of our algorithm for the target reconstructability of a *C. elegans* muscle involved in a chemotaxis behavior.

Perspectives

- ✓ Develop methods quantifying the quality of these sets of nodes (⇒choose the best option to reconstruct the target nodes states)
- ✓ Development of a state reconstructor.

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- **S. Orange, N. Verdière, L. Naudin, *An a priori study for the reconstruction of some variables of interest in nonlinear complex networks with an application in neuroscience*, Chaos, Solitons and Fractals, 2023, 113644.**
- **N. Verdière, S. Orange, *Diagnosability in the case of multi-faults in nonlinear models*, Journal of Process Control, Vol 69, pp. 1-7, 2018.**

Thank you for your attention!