Efficient algorithms for differential equation satisfied by Feynman integrals

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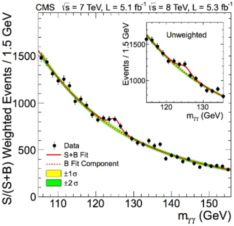
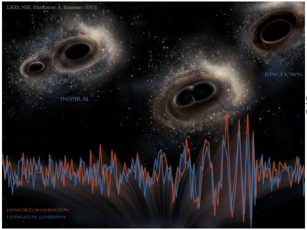
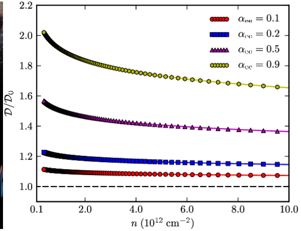
based on 2209.10962 and 2306.05263

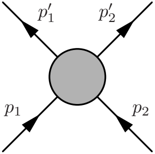
with Charles Doran, Andrew Harder, Pierre Lairez, Eric Pichon-Pharabod

and work to appear with Leonardo de la Cruz









Scattering amplitudes are the fundamental tools for making contact between quantum field theory description of nature and experiments

- Comparing particule physics model against datas from accelators
- Post-Minkowskian expansion for Gravitational wave physics
- Various condensed matter and statistical physics systems

Feynman Integrals: parametric representation

Feynman integral are given by projective space integrals

$$I_\Gamma(\underline{\nu}, D; \underline{s}, \underline{m}) = \int_{\Delta_n} \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega - \frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\omega}} \prod_{i=1}^{n} x_i^{\nu_i - 1} \Omega_0 \qquad \omega = \sum_{i=1}^{n} \nu_i - \frac{LD}{2}$$

with the volume form on $\mathbb{P}^{n-1}$

$$\Omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geq 0, \ldots, x_n \geq 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

Feynman Integrals: parametric representation

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times \mathcal{L}(\underline{m}^2;\underline{x}) - \mathcal{V}_\Gamma(\underline{s},\underline{x})$$

- Homogeneous polynomial of degree L with $u_{a_1,\dots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) := \sum_{\substack{a_1+\cdots+a_n=L \\ 0\leq a_i\leq 1}} u_{a_1,\dots,a_n} \prod_{i=1}^n x_i^{a_i}$$

- the mass hyperplane

$$\mathcal{L}(\underline{m}^2;\underline{x}) := \sum_{n=1}^n m_i^2 x_i$$

- Homogeneous polynomial of degree $L+1$

$$\mathcal{V}_\Gamma(\underline{x}) := \sum_{\substack{a_1+\cdots+a_n=L+1 \\ 0\leq a_i\leq 1}} S_{a_i,\cdots,a_n} \prod_{i=1}^n x_i^{a_i}$$

Feynman Integrals: parametric representation

The integrand is an algebraic differential form in $H^{n-1}(\mathbb{P}^{n-1}\backslash\mathbb{X}_\Gamma)$ on the complement of the graph hypersurface

$$\mathbb{X}_\Gamma := \{\mathcal{U}_\Gamma(\underline{x}) \times \mathcal{F}_\Gamma(\underline{x}) = 0, \underline{x} \in \mathbb{P}^{n-1}\}$$

- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities

Feynman integral and periods

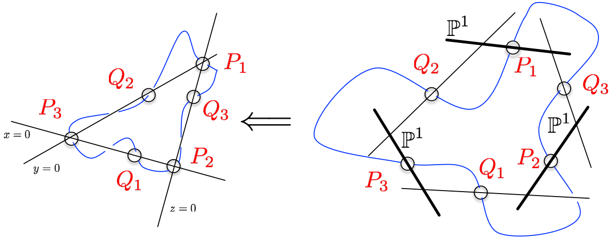
The domain of integration Δ_n is not an homology cycle because

$$\partial\Delta_n \cap \mathbb{X}_\Gamma = \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$$

we have to look at the relative cohomology

$$H^\bullet(\mathbb{P}^{n-1}\backslash X_\Gamma; \mathit{\Pi}_n\backslash\mathit{\Pi}_n\cap\mathbb{X}_\Gamma)$$

The normal crossings divisor $\mathit{\Pi}_n := \{x_1\cdots x_n = 0\}$ and $\mathbb{X}_\Gamma$ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



Differential equation

The Feynman integral *are* period integrals of the relative cohomology after performing the appropriate blow-ups

$$\mathfrak{M}(\underline{s}, \underline{m}^2) := H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\varPi_n} \backslash \widetilde{\varPi_n} \cap \widetilde{X_\Gamma})$$

Since the integrand varies with the physical variables $\{S_{\underline{a}^i}, m_1^2, \ldots, m_n^2\}$ one needs to study a variation of (mixed) Hodge structure

One can show that the Feynman integral are **holonomic D-finite functions** [Bitoun et al.;Smirnov et al.]

A Feynman integrals satisfies inhomogenous differential equations with respect to any set of variables $\underline{z} \in \{S_{\underline{a}}, m_1^2, \ldots, m_n^2\}$

$$\mathscr{L}_\Gamma(\underline{z})\, I_\Gamma = \mathscr{S}_\Gamma$$

Generically there is an inhomogeneous term $\mathscr{S}_\Gamma \neq 0$ due to the boundary components $\partial\Delta_n$

Feynman integral D-module

We want to address the questions

1. To what class of functions belong Feynman integrals?
2. What is the geometrical algebraic origin of the motive $\mathfrak{M}(\underline{s}, \underline{m}^2)$?
3. Derivation of the (D-module of) differential equations ?
   $\mathscr{L}_\Gamma(\underline{z})\, I_\Gamma = \mathscr{S}_\Gamma$

In this talk we focus in the question 3 and present some new methods for deriving such system of differential equation and its underlying (algebraic) geometry

Feynman Integrals differential equations

For a given subset of the physical parameters $\underline{z} := (z_1, \ldots, z_r) \subset \{\underline{s}, \underline{m}^2\}$ we want to derive **minimal order** differential equations

$$\mathscr{L}_\Gamma(\underline{s}, \underline{m}^2, \partial_{\underline{z}}) \int_\sigma \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega - \frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^\omega} \prod_{i=1}^{n} x_i^{\nu_i - 1} \, \Omega_0 = \mathscr{S}_{\sigma,\Gamma}(\underline{z})$$

One way to achieve this is to construct a Gröbner basis of operators $T_{\underline{z}}$ that annihilate the integrand of the Feynman integral

$$T_{\underline{z}} \left(\frac{\mathcal{U}_\Gamma(\underline{x})^{\omega - \frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^\omega} \prod_{i=1}^{n} x_i^{\nu_i - 1} \, \Omega_0 \right) = 0$$

such that

$$T_{\underline{z}} = \mathscr{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial}_{\underline{z}}) + \sum_{i=1}^{n} \partial_{x_i} Q_i(\underline{s}, \underline{m}^2, \underline{\partial}_{\underline{z}}; \underline{x}, \underline{\partial}_{\underline{x}})$$

Feynman Integrals differential equations

where the finite order differential operator

$$\mathscr{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial_z}) = \sum_{\substack{0 \le a_i \le o_i \\ 1 \le i \le r}} p_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i}$$

$$Q_i(\underline{s}, \underline{m}^2, \underline{\partial_z}) = \sum_{\substack{0 \le a_i \le o'_i \\ 1 \le i \le r}} \sum_{\substack{0 \le b_i \le \tilde{o}_i \\ 1 \le i \le n}} q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2, \underline{x}) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i} \prod_{i=1}^{n} \left(\frac{d}{dx_i}\right)^{b_i}$$

- The orders o_i, o'_i, $\tilde{o}_i$ are positive integers
- $p_{a_1,\dots,a_r}(\underline{S}, \underline{m}^2)$ polynomials in the kinematic variables
- $q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}^2, \underline{x})$ rational functions in the kinematic variable and the projective variables $\underline{x}$.

Feynman Integrals differential equations

Integrating over a cycle γ gives

$$0 = \oint_\gamma T_{\underline{z}} \Omega_\Gamma = \mathscr{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma + \oint_\gamma d\beta_\Gamma$$

For a cycle $\partial\gamma = \emptyset$ then $\oint_\gamma d\beta_\Gamma = 0$ and we get

$$\mathscr{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma = 0$$

For the Feynman integral I_Γ we have

$$0 = \int_{\Delta_n} T_{\underline{z}} \Omega_\Gamma = \mathscr{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma + \int_{\Delta_n} d\beta_\Gamma$$

since $\partial\Delta_n \neq \emptyset$

$$\mathscr{L}_\Gamma(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma = \mathscr{S}_\Gamma$$

So we need the telescoper and the certificate

The Rational case

We start with the case of a rational differential form with $D \in 2\mathbb{N}^*$

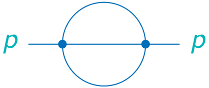
$$\Omega_\Gamma = \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega-\frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^\omega} \prod_{i=1}^n x_i^{\nu_i-1}\,\Omega_0 \qquad \omega = \sum_{i=1}^n \nu_i - \frac{LD}{2}$$

- We as well assume that all the mass parameters are all vanishing $m_1, \cdots, m_n \neq 0$
- And that $\omega > 0$, i.e. $\sum_{i=1}^n \nu_i > LD/2$

So that the integral of Ω_Γ on the positive orthan is a convergent integral

The sunset graph

The two-loop sunset graph in $D = 2$



$$I_{\ominus}(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \frac{dx_1\, dx_2\, dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

The polar hypersurface of the integral is an elliptic curve $\mathcal{F}_{\ominus}(\underline{x}) = 0$

$$\mathcal{F}_{\ominus}(\underline{x}) = (x_1x_2 + x_1x_3 + x_2x_3)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3) - p^2x_1x_2x_3$$

The sunset graph : Griffiths-Dwork method

One can obtain a differential equation annihilating acting on the integral using the Griffiths-Dwork method

Let define the integrand in differential form

$$\Omega_{\ominus} = \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{\mathcal{F}_{\ominus}(\underline{x})} = \frac{\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})}$$

consider

$$\frac{\partial \Omega_{\ominus}}{\partial p^2} = x_1 x_2 x_3 \frac{\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})^2}; \qquad \frac{\partial^2 \Omega_{\ominus}}{(\partial p^2)^2} = 2(x_1 x_2 x_3)^2 \frac{\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})^3}$$

Since we know we have the geometry of an elliptic curve we are looking for a second order differential operator acting on $\eta_{\ominus}$

$$\mathscr{L}_{\ominus}(p^2) = \frac{\partial^2}{(\partial p^2)^2} + q_1(p^2, \underline{m}^2) \frac{\partial}{\partial p^2} + q_0(p^2, \underline{m}^2)$$

The sunset graph : Griffiths-Dwork method

Remark that $(x_1x_2x_3)^2$ lies in the Jacobian ideal for $\mathcal{F}_\ominus(\underline{x})$

$$(x_1x_2x_3)^2 = \sum_{i=1}^{3} C_i^{(1)}(\underline{x})\partial_{x_i}\mathcal{F}_\ominus(\underline{x})$$

with $C_i^{(1)}(\underline{x})$ homogeneous of degree 4 in the (x_1, x_2, x_3) variables
Following Griffiths one introduces the differential form

$$\beta_1 = \frac{(x_2\, C_3^{(1)}(\underline{x}) - x_3\, C_2^{(1)}(\underline{x}))dx_1}{\mathcal{F}_\ominus(\underline{x})^2} + \frac{(x_1\, C_3^{(1)}(\underline{x}) - x_3\, C_1^{(1)}(\underline{x}))dx_2}{\mathcal{F}_\ominus(\underline{x})^2} + \frac{(x_1\, C_2^{(1)}(\underline{x}) - x_2\, C_1^{(1)}(\underline{x}))dx_3}{\mathcal{F}_\ominus(\underline{x})^2}$$

such that

$$d\beta_1 = 2\frac{\sum_{i=1}^{3} C_i^{(1)}(\underline{x})\partial_{x_i}\mathcal{F}_\ominus(\underline{x})\Omega_0}{\mathcal{F}_\ominus(\underline{x})^3} - \frac{\sum_{i=1}^{3} \partial_{x_i} C_i^{(1)}(\underline{x})\Omega_0}{\mathcal{F}_\ominus(\underline{x})^2}$$

The sunset graph : Griffiths-Dwork method

$$\mathscr{L}_{\ominus}(p^2)\Omega_{\ominus} = \frac{q_1(p^2,\underline{m}^2)x_1x_2x_3 + \sum_{i=1}^{3}\partial_{x_i}C_i^{(1)}(\underline{x})}{\mathcal{F}_{\ominus}(\underline{x})^2}\Omega_0 + d\beta_1$$

We can again reduce this second order pole using that there exist a polynomial $q_1(p^2,\underline{m}^2)$ such that

$$q_1(p^2,\underline{m}^2)x_1x_2x_3 + \sum_{i=1}^{3}\partial_{x_i}C_i^{(1)}(\underline{x}) = \sum_{i=1}^{3}C_i^{(2)}\partial_{x_i}\mathcal{F}_{\ominus}(\underline{x})$$

with $C_i^{(2)}$ of degree 1. One introduces the 1-form β_2

$$\beta_2 = \sum_{i=1}^{3}\epsilon^{ijk}\frac{x_jC_k^{(2)}(\underline{x})dx_i}{\mathcal{F}_{\ominus}(\underline{x})}$$

The sunset graph : Griffiths-Dwork method

such that

$$d\beta_2 = \frac{\sum_{i=1}^{3} C_i^{(2)}(\underline{x})\partial_{x_i}\mathcal{F}_{\ominus}(\underline{x})}{\mathcal{F}_{\ominus}(\underline{x})^2} - \frac{\sum_{i=1}^{3} \partial_{x_i} C_i^{(2)}(\underline{x})\Omega_0}{\mathcal{F}_{\ominus}(\underline{x})}$$

We have achieved that

$$\left(\frac{\partial^2}{(\partial p^2)^2} + q_1(p^2, \underline{m}^2)\frac{\partial}{\partial p^2} + \sum_{i=1}^{3} \partial_{x_i} C_i^{(2)}(\underline{x})\right)\eta_{\ominus} = d(\beta_1 + \beta_2)$$

because the $C_i^{(2)}(\underline{x})$ are of degree 1 in (x_1, x_2, x_3) then $q_0(p^2, \underline{m}^2) = \partial_{x_i} C_i^{(2)}(\underline{x})$ only depends on $p^2, \underline{m}^2$

The sunset graph : Griffiths-Dwork method

We then conclude that the minimal operator acting on the sunset integral is the Picard-Fuchs operator

$$\mathscr{L}_{p^2} = \frac{\partial^2}{(\partial p^2)^2} + q_1(p^2, \underline{m}^2)\frac{\partial}{\partial p^2} + q_0(p^2, \underline{m}^2)$$

which acts on the integrals as

$$\mathscr{L}_{p^2} I_\ominus(p^2) = \int_{x_i \geq 0} \mathscr{L}_{p^2} \frac{\Omega_0}{\mathcal{F}_\ominus(\underline{x})} = \int_{x_i \geq 0} d(\beta_1 + \beta_2) \neq 0$$

We have constructed by the telescoper $T_{p^2} = \mathscr{L}_\ominus(p^2)$ and the certificate $C_\ominus = d(\beta_1 + \beta_2)$

The differential operator $\mathscr{L}_{p^2}$ is the Picard–Fuchs operator of the elliptic curve defined by the graph polynomial $\mathcal{F}_\ominus(x_1, x_2, x_3) = 0$

The sunset graph : Griffiths-Dwork method

If one considers the family of elliptic curve E

$$y^2 = 4x^3 - g_2(t)x - g_3(t); \qquad \Delta(t) = g_2(t)^3 - 27g_3(t)^2$$

the periods satisfy the differential system of equations

$$\frac{d}{dt}\begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix} = \begin{pmatrix} -\frac{1}{12}\frac{d}{dt}\log\Delta(t) & \frac{3\delta(t)}{2\Delta(t)} \\ -\frac{g_2(t)\delta(t)}{8\Delta(t)} & \frac{1}{12}\frac{d}{dt}\log\Delta(t) \end{pmatrix}\begin{pmatrix} \int_\gamma \frac{dx}{y} \\ \int_\gamma \frac{xdx}{y} \end{pmatrix}$$

with $\delta(t) = 3g_3(t)\frac{d}{dt}g_2(t) - 2g_2(t)\frac{d}{dt}g_3(t)$

The Picard–Fuchs operator acting on the period integral $\int_\gamma dx/y$ is

$$\mathscr{L}_{\rm ell} = 144\Delta(t)^2\delta(t)\frac{d^2}{dt^2} + 144\Delta(t)\left(\delta(t)\frac{d\Delta(t)}{dt} - \Delta(t)\frac{d\delta(t)}{dt}\right)\frac{d}{dt}$$
$$+27g_2(t)\delta(t)^3 + 12\frac{d^2\Delta(t)}{dt^2}\delta(t)\,\Delta(t) - \left(\frac{d\Delta(t)}{dt}\right)^2\delta(t) - 12\frac{d\delta(t)}{dt}\Delta(t)\,\frac{d\Delta(t)}{dt}.$$

This matches the differential operator derived using the Griffiths–Dwork method

Extended Griffiths-Dwork algorithms

In general the graph hypersurface does not have isolated singularities (which is the generic case) therefore the “naïve” implementation of the Griffiths-Dwork algorithm does not work

One could use the implementation of Doron Zeilberger (1990) creative telescoping algorithm by F. Chyzak or C. Koutschan but the algorithm takes a very long time for graph with many edges

$$\Omega_\Gamma = \frac{\mathcal{U}_\Gamma(\underline{x})^{\omega-\frac{D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\omega}} \prod_{i=1}^{n} x_i^{\nu_i-1}\,\Omega_0$$

We used `period` by Pierre Lairez of an extended Griffiths-Dwork algorithm that handles singular hypersurfaces

Extended Griffiths-Dwork: syzygies

In this example we saw the pole reduction

$$\frac{\partial^2\eta_{\ominus}}{(\partial p^2)^2}=\frac{2(x_1x_2x_3)^2}{\mathcal{F}_{\ominus}(\underline{x})^3}\Omega_0=\frac{\sum_{i=1}^3\partial_{x_i}C_i}{\mathcal{F}_{\ominus}(\underline{x})^2}\Omega_0+d\beta_1$$

For singular hypersurface $X_\Gamma\subset\mathbb{P}^{n-1}$ the Jacobian reduction may not be enough to reduce the pole order when $k\geq n$

Other reduction rules come from the *syzygies* of the derivatives $\frac{\partial\mathcal{F}_\Gamma}{\partial x_i}$, i.e. tuples $(B_1,\dots,B_n)$ be homogeneous of degree $k\deg\mathcal{F}_\Gamma-n+1$ such that $\sum_i B_i\frac{\partial\mathcal{F}_\Gamma}{\partial x_i}=0$ such that $(\xi_i=(-1)^{i-1}dx_1\cdots\widehat{dx_i}\cdots dx_n)$

$$\frac{\sum_i\frac{\partial B_i}{\partial x_i}}{\mathcal{F}_\Gamma^k}\Omega_0=d\left(\sum_i\frac{B_i}{\mathcal{F}_\Gamma^k}\xi_i\right)\Longrightarrow\int_\gamma\frac{\sum_i\frac{\partial B_i}{\partial x_i}}{\mathcal{F}_\Gamma^k}\Omega_0=0.$$

In singular cases, these relations are missed by the Griffiths–Dwork reduction, we need the extended Griffiths–Dwork reduction implemented by [Lairez]

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\qquad \deg A = k \deg \mathcal{F}_\Gamma - n$

1. Compute a basis of the space S_k of all syzygies of degree $k \deg \mathcal{F}_\Gamma - n + 1$ quotiented by the space of trivial syzygies

$$D_{ij} = -D_{ji}, \qquad B_i = \sum_{j=1}^{n} D_{ij} \frac{\partial \mathcal{F}_\Gamma}{\partial x_j} \Longrightarrow \sum_i B_i \frac{\partial \mathcal{F}_\Gamma}{\partial x_i} = 0$$

are irrelevant because already used by the Griffiths–Dwork reduction

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\qquad \deg A = k \deg \mathcal{F}_\Gamma - n$

2. Compute a normal form R of A modulo the Jacobian ideal plus the space $dV = \left\{ \sum_i \frac{\partial B_i}{\partial x_i} \,\middle|\, \underline{B} \in V \right\}$, that is for some polynomials B_i and C_i

$$A = R + \underbrace{\sum_i \frac{\partial B_i}{\partial x_i}}_{\in dV} + \underbrace{C_1 \frac{\partial \mathcal{F}_\Gamma}{\partial x_1} + \cdots + C_n \frac{\partial \mathcal{F}_\Gamma}{\partial x_n}}_{\in \text{Jacobian ideal}},$$

Extended Griffiths-Dwork : syzygies

Given a form $\Omega = \frac{A}{\mathcal{F}_\Gamma^k} d\underline{x}$ $\deg A = k \deg \mathcal{F}_\Gamma - n$

3. This leads to the following relation

$$(k-1)\frac{A}{\mathcal{F}_\Gamma^k} d\underline{x} = \frac{\sum_i \frac{\partial C_i}{\partial x_i}}{\mathcal{F}_\Gamma^{k-1}} d\underline{x} - d\left(\sum_i \frac{B_i}{\mathcal{F}_\Gamma^k}\xi_i + \sum_i \frac{C_i}{\mathcal{F}_\Gamma^{k-1}}\xi_i\right).$$

Then

$$\int_\gamma \frac{A(\underline{x})}{\mathcal{F}_\Gamma(\underline{x})^k} d\underline{x} = -\frac{1}{k-1}\int_\gamma \frac{\sum_i \frac{\partial C_i}{\partial x_i}}{\mathcal{F}_\Gamma^{k-1}} d\underline{x},$$

The extended Griffiths–Dwork reduction presented above is not always enough and may need further extensions, i.e. syzygies of syzygies.
There is a hierarchy of extensions which eventually collapse to the strongest possible reduction.
However, for all the computations presented here, we only needed the first extension.

Pole conditions [Picard (1899)]

In the construction we will only consider the case where $\beta(\underline{x}, t)$ is holomorphic on $\mathbb{P}^{n-1} \setminus X_\Gamma$, that is is β_Γ does not have poles that are not present in Ω_Γ.

Consider the rational function $F(x_1, x_2)$

$$\frac{ax_1 + bx_2 + c}{\left(\alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2 + \delta x_1 + \eta x_2 + \zeta\right)^2} = \partial_{x_1} \frac{N_1(x_1, x_2)}{D_1(x_1, x_2)} + \partial_{x_2} \frac{N_2(x_1, x_2)}{D_2(x_1, x_2)}$$

where $a, b, c, \alpha, \beta, \gamma, \delta, \eta, \eta$ are constants and polynomials $N_i(x_1, x_2)$ and $D_i(x_1, x_2)$ with $i = 1, 2$.

The denominators have poles at $x_2^0 = (a\delta - 2\alpha c)/(2\alpha b - a\gamma)$ which is not a pole of the left-hand-side.

This means one can find a cycle γ passing by x_2^0 such that the integral of $\int_\gamma F(x_1, x_2)$ is finite and non-vanishing.

Minimality of the Picard–Fuchs operator

This dimension $\dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n \backslash \mathbb{V}(\mathcal{U}_\Gamma) \cup \mathbb{V}(\mathcal{F}_\Gamma)\right)$ gives an upper bound on the order of the minimal order differential operator acting on the Feynman integral. [Bitoun et al.]

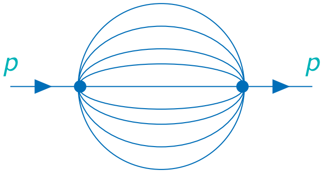
The extended Griffiths–Dwork algorithm leads to a minimal order differential operator

$$\mathscr{L}_\Gamma \Omega_\Gamma = d\beta_\Gamma$$

annihilating (in cohomology) the Feynman integral differential form Ω_Γ with the condition that the certificate β_Γ is an holomorphic form on $\mathbb{P}^{n-1}\backslash Z_\Gamma$. $\mathscr{L}_\Gamma$ is the minimal differential order differential operator satisfying this condition.

Using the algorithm by [Chyzak, Goyer, Mezzarobba] we test the irreducibility of the the Picard–Fuchs operator and factorize when it is reducible.

Sunset graph Picard–Fuchs operator



$$\Omega_n^{\ominus}(t,\underline{m}^2) := \frac{\Omega_0}{\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x})} \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_{\ominus})$$

$$\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x}) := x_1\cdots x_n\left(\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{j=1}^{n} m_j^2 x_j\right)-t\right)$$

The graph hypersurface $\mathcal{F}_n^{\ominus}(t,\underline{m}^2;x)=0$ defines a Calabi-Yau manifold of dimension $n-1$

For generic physical parameters configurations we find a minimal order Picard–Fuchs operator [Lairez, Vanhove]

$$\mathscr{L}_t = \sum_{r=0}^{o_n} q_r(t,\underline{m}^2)\left(\frac{d}{dt}\right)^r \qquad o_n = 2^n - \binom{n+1}{\left\lfloor\frac{n+1}{2}\right\rfloor};\, n\geq 2.$$

The three-loop sunset graph

$$
\begin{tikzcd}
 & \begin{matrix}\mathscr{L}_3\\ Pic=19\end{matrix} & \\
\begin{matrix}\mathscr{L}_4\\ Pic=18\end{matrix} \arrow[ur, "m_4=m_1"] & & \begin{matrix}\mathscr{L}_4\\ Pic=18\end{matrix} \arrow[ul, "m_4=m_1"'] \\
 & \begin{matrix}\mathscr{L}_5\\ Pic=17\end{matrix} \arrow[ul, "m_3=m_1"] \arrow[ur, "m_3=m_4"'] & \\
 & \begin{matrix}\mathscr{L}_6\\ Pic=16\end{matrix} \arrow[u, "m_2=m_1"] &
\end{tikzcd}
$$

$$\mathscr{L}_r = (\alpha \frac{d}{dp^2} + \beta) \circ \mathscr{L}_{r-1}$$

At three-loop we have a $K3$ surface of $19 \leq Pic \leq 16$ depending on the mass configuration

For generic mass parameters the Picard-Fuchs operators

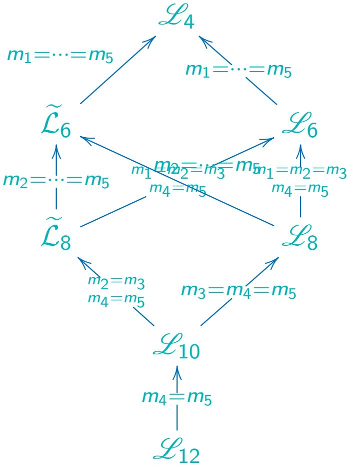
$$\mathscr{L}_6 = \sum_{r=0}^{6} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 6 and degree 25

$$q_6(p^2) = \tilde{q}_6(p^2) \times \prod_{\epsilon_i = \pm 1} \left(p^2 - (\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3 + \epsilon_4 m_4)^2\right)$$

with $\tilde{q}_6(p^2)$ degree 17 with apparent singularities

The four-loop sunset graph



The geometry is the one of the Calabi-Yau threefold.

For generic kinematics we have an order 12 degree 121 operator

$$\mathcal{L}_t^{[1^5]} = \sum_{r=0}^{12} q_r^{[1^5]}(t, \underline{m}^2) \left(\frac{d}{dt}\right)^r .$$

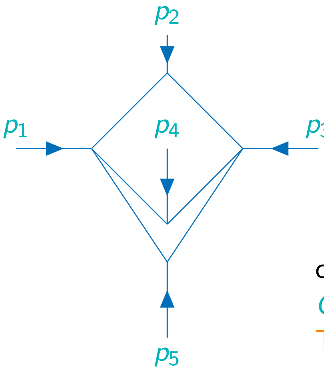
The degree of the apparent singularities is a polynomial of degree 98

The five-loop and six-loop sunset graph

- **The six mass configuration** $m_1 \neq m_2 \neq m_3 \neq m_4 \neq m_5 \neq m_6$ **denote** $[1^6]$**:** the Picard–Fuchs operator of order 29 and degree of the polynomial $q_{29}(t)$ is 521.
- **The seven mass configuration** $m_1 \neq m_2 \neq m_3 \neq m_4 \neq m_5 \neq m_6 \neq m_7$**:** the Picard–Fuchs operator of order 58 with a degree 2273
- Results compatible with a CY $n-1$-fold

Results obtained using Pierre Lairez `period`

Tardigrade



The rational differential form in $\mathbb{P}^5$

$$\Omega(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_6(\underline{x})\mathscr{L}_6(\underline{m}^2,\underline{x}) - t\mathcal{V}(\underline{s},\underline{x})\right)^2}$$

$$\mathcal{U}_6(\underline{x}) = (x_1+x_2)(x_3+x_4) + (x_1+x_2)(x_5+x_6) + (x_3+x_4)(x_5+x_6)$$

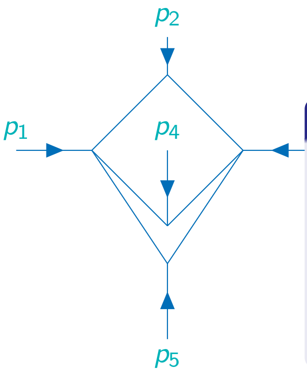
$\mathcal{V}(\underline{s},\underline{x}) = \sum_{1\leq i,j,k\leq 6} C_{ijk} y_i y_j y_k$ with linear changes $(x_{2i-1}, x_{2i}) \to (y_{2i-1}, y_{2i})$ and $i = 1,2,3$
C_{ijk} symmetric traceless i.e. $C_{iij} = 0$
The algorithm gives an irreducible Picard–Fuchs operator of order 11 with an head polynomial of degree up to 215. [Lairez, Vanhove]

Tardigrade

The rational differential form in $\mathbb{P}^5$

$$\Omega(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_6(\underline{x})\mathscr{L}_6(\underline{m}^2, \underline{x}) - t\mathcal{V}(\underline{s}, \underline{x})\right)^2}$$

The motive associated to this graph

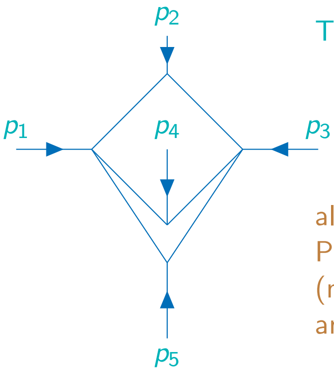


Theorem (Tardigrade motive)

Let $X_{(2,2,2);D}$ be the tardigrade hypersurface for generic mass and momentum parameters and $D \geq 2$. Then there is a quartic K3 surface with six A_1 singularities so that $\mathrm{Gr}_4^W \mathrm{H}^4(X_{(2,2,2);D}; \mathbb{Q})$ is isomorphic to $\mathrm{H}^2(S; \mathbb{Q})(-1)$ for a $K3$ surface S up to mixed Tate factors.

Determined in [Doran, Harder, Pichon-Pharabod, Vanhove]

Tardigrade



The rational differential form in $\mathbb{P}^5$

$$\Omega(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_6(\underline{x})\mathscr{L}_6(\underline{m}^2, \underline{x}) - t\mathcal{V}(\underline{s}, \underline{x})\right)^2}$$

The singularities of the graph polynomials are all of type A_1 and one can apply Eric Pichon-Pharabod program lefschetz-family to (numerically) determine the transcendental lattice and confirm that we have a $K3$ of Picard Rank 11

The non-rational case

We now consider the non-rational case $D \in \mathbb{R}$

$$\Omega_\Gamma = \left(\frac{\mathcal{U}_\Gamma(\underline{x})}{\mathcal{F}_\Gamma(\underline{x})}\right)^{\sum_i \nu_i} \left(\frac{\mathcal{U}_\Gamma(\underline{x})^{L+1}}{\mathcal{F}_\Gamma(\underline{x})^L}\right)^D \prod_{i=1}^n x_i^{\nu_i - 1}\,\Omega_0$$

- We relax all assumption on the mass parameters who can all vanish $m_1, \cdots, m_n \in \mathbb{R}$
- We have degree 0 rational form

$$R(\underline{x}) := \frac{\mathcal{U}_\Gamma(\underline{x})^{L+1}}{\mathcal{F}_\Gamma(\underline{x})^L}$$

Feynman Integrals: divergences

As a function of the powers of the propagators $\underline{\nu}$ and the dimension D the integral has singularities located on hyperplane defined by $\sum_{i=1}^{n} a_i \nu_i + a_0 D = 0$ with $(a_0, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$

One can perform a Laurent expansion near say $D_c = 4$ dimensions

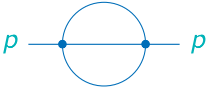
$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) = \sum_{r \geq -2L} (D - D_c)^r I_\Gamma^{(r)}(\underline{s}, \underline{m}^2; \underline{\nu})$$

where $I_\Gamma^{(r)}(\underline{s}, \underline{m}^2; \underline{\nu})$ are convergent integrals.

Eugene R. Speer
Generalized Feynman Amplitudes
Princeton University Press, (1969)

The sunset graph

The two-loop sunset graph in $D = 2 - 2\epsilon$ with $\epsilon \in \mathbb{R}$



$$I_{\ominus}(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \left(\frac{\mathcal{U}_{\ominus}^3(\underline{x})}{\mathcal{F}_{\ominus}^2(\underline{x})}\right)^{\epsilon} \frac{dx_1 dx_2 dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

with

$$\mathcal{U}_{\ominus}(\underline{x}) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

The polar hypersurface of the integral is still the elliptic curve $\mathcal{F}_{\ominus}(\underline{x}) = 0$

$$\mathcal{F}_{\ominus}(\underline{x}) = (x_1 x_2 + x_1 x_3 + x_2 x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - p^2 x_1 x_2 x_3$$

The sunset graph : Griffiths-Dwork method I

We consider differentiation with respect to a single physical parameter $z \in \{\vec{m}, \vec{s}\}$

We consider the derivative

$$\left(\frac{d}{dz}\right)^a \Omega_\Gamma^\epsilon = \frac{\Gamma(1+a+\epsilon)}{\Gamma(1+2\epsilon)} \frac{(x_1x_2x_3)^a}{\mathcal{F}_\ominus^{a+1}} \left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \Omega_n^{(0)}$$

we reduce the numerator in the Jacobian ideal of $\mathcal{F}_\ominus$

$$\frac{\Gamma(1+a+\epsilon)}{\Gamma(1+2\epsilon)} (x_1x_2x_3)^a = \vec{C}_{(a)} \cdot \vec{\nabla} \mathcal{F}_\ominus.$$

The sunset graph : Griffiths-Dwork method II

Integration by part gives

$$\left(\frac{d}{dz}\right)^a \Omega_\Gamma^\epsilon = \frac{\vec{\nabla}\cdot\vec{C}_{(a)}}{a\mathcal{F}_\ominus^a}\left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \Omega_0 + \epsilon\frac{\vec{C}_{(a)}\cdot\vec{\nabla}\log(\mathcal{U}_\ominus^3/\mathcal{F}_\ominus^2)}{a\mathcal{F}_\ominus^a}\left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \Omega_0 + d\beta_{(a)}$$

or equivalently

$$\left(\frac{d}{dz}\right)^a \Omega_\Gamma^\epsilon = \frac{\vec{\nabla}\cdot\vec{C}_{(a)}}{(a+2\epsilon)\mathcal{F}_\ominus^a}\left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \Omega_0 + 3\epsilon\frac{\vec{C}_{(a)}\cdot\vec{\nabla}\log(\mathcal{U}_\ominus)}{(a+2\epsilon)\mathcal{F}_\ominus^a}\left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \Omega_0 + d\beta_{(a)}$$

We ask that

$$\vec{C}_{(a)}\cdot\vec{\nabla}\mathcal{U}_\ominus = c_{(a)}(\underline{x})\mathcal{U}_\ominus.$$

The sunset graph : Griffiths-Dwork method III

Solving the system

$$\frac{\Gamma(1+a+\epsilon)}{\Gamma(1+2\epsilon)}(x_1x_2x_3)^a = \vec{C}_{(a)} \cdot \vec{\nabla}\mathcal{F}_{\ominus},$$
$$\vec{C}_{(a)} \cdot \vec{\nabla}\mathcal{U}_{\ominus} = c_{(a)}(\underline{x})\mathcal{U}_{\ominus}.$$

Gives the pole reduction

$$\left(\frac{d}{dz}\right)^a \Omega_\Gamma^\epsilon = \frac{\vec{\nabla}\cdot\vec{C}_{(a)} + 3\epsilon c_{(a)}(\underline{x})}{(a+2\epsilon)\mathcal{F}_{\ominus}^a}\left(\frac{\mathcal{U}_{\ominus}^3}{\mathcal{F}_{\ominus}^2}\right)^\epsilon \Omega_0 + d\beta_{(a)}.$$

- This tells us how to modify the Griffiths-Dwork pole reduction and deduce the ϵ deformed differential equation.
- This allows to treat case that are divergence for $\epsilon = 0$ which was not possible with the previous algorithm

Work in progress [de la Cruz, Vanhove]

The sunset integrals : all equal mass case

For the all equal mass case the algorithm gives (up to 20 loops)
For the all equal mass case $m_1 = \cdots = m_{l+1} = 1$ we find the sunset Feynman integral satisfies the differential equation

$$\mathscr{L}_{\ominus}^{(l),\epsilon} I_{\ominus}(\{1,\dots,1\},t,\epsilon) = -(l+1)!\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)}$$

with

$$\mathscr{L}_{\ominus}^{(l),\epsilon} = \mathscr{L}_{\ominus}^{(l),l} + \epsilon\mathscr{L}_{\ominus}^{(l),l-1} + \cdots + \epsilon^l\mathscr{L}_{\ominus}^{(l),0}$$

where the differential operator is $\mathscr{L}_{\ominus}^{(l),r}$ is of order r.

Two loop Sunset: different masses I

$$\mathscr{L}_{\ominus}^{(2),\epsilon} = \mathscr{L}_1^{(1)}\mathscr{L}_1^{(2)}\mathscr{L}_{\ominus}^{3-mass} + \epsilon\mathscr{L}_4^{(3)} + \epsilon^2\mathscr{L}_3^{(4)} + \epsilon^3\mathscr{L}_2^{(5)} + \epsilon^4\mathscr{L}_1^{(6)} + \epsilon^5\mathscr{L}_0^{(7)}$$

where $\mathscr{L}_m^{(r)}$ are irreducible differential operator of order m and $\mathscr{L}_{\ominus}^{3-mass}$ is the differential operator for the three-mass two-loop sunset integral in two dimensions.

Its actions on the Feynman integral is given by

$$\mathscr{L}_{\ominus}^{(2),\epsilon} I(\underline{m}, t; \epsilon) = \mathscr{S}(\vec{m}, t; \epsilon)$$

with the source term

$$\mathscr{S}(\vec{m}, t; \epsilon) = \frac{c_{23}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_2 m_3)^{2\epsilon}\Gamma(1+2\epsilon)} + \frac{c_{13}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_1 m_3)^{2\epsilon}\Gamma(1+2\epsilon)} + \frac{c_{12}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_1 m_2)^{2\epsilon}\Gamma(1+2\epsilon)}$$

Two loop Sunset: different masses II

The ϵ deformed operator has for highest order term

$$\mathscr{L}_{\ominus}^{(2),\epsilon}\Big|_{(d/dt)^4} = t^3 \prod_{i=1}^{4}(t-\mu_i^2)$$

$$\times \left(-(2\epsilon+5)\,t^2 - 2\left(m_1^2+m_2^2+m_3^2\right)(1+2\epsilon)\,t + (7+6\epsilon)\prod_{i=1}^{4}\mu_i \right)$$

where
$\mu_i = \{m_1+m_2+m_3, -m_1+m_2+m_3, m_1-m_2+m_3, m_1+m_2-m_3\}$ are the thresholds.

- The ϵ deformation is only affecting the apparent singularities
- The non-apparent singularities are still the roots of the discriminant of the sunset elliptic curve
- The order 4 operator is irreducible

Outlook

- We have put forward a new approach for deriving the differential equation for Feynman integrals
- We can derive the differential equations in general dimension by extending the Griffiths-Dwork reduction
- We see how the twist ϵ-factor affects only the apparent singularities
- For graphs with many edges the reduction takes a long time we have been using the `FiniteFlow` program to speed up the computation but still improvements are needed

We have a seminar on these mathematical aspects of Feynman integral run by Francis Brown, Erik Panzer, Federico Zerbini and myself at the address https://www.ihes.fr/~vanhove/motivefeynman.html