Some problems I'd like solved
(and a few that I've solved)
from a user of computer algebra

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Computer Algebra for Functional Equations in Combinatorics and Physics
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A big project in collaboration with

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Xi Chen, Bishal Deb, Veronica Bitonti, . . .

Overview

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- Total positivity (over a partially ordered commutative ring)

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- (Tentative) conclusion

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- (Contrast with positive semidefiniteness.)
- But . . . In many areas of mathematics, there is a preferred basis.

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Applications:

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Planar discrete potential theory and the planar Ising model
- Stieltjes moment problem
- Enumerative combinatorics

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Generalize the theory of total positivity from matrices of real numbers to matrices with entries in a partially ordered commutative ring.

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(i) $0, 1 \in \mathcal{P}$.

(ii) *If* $a, b \in \mathcal{P}$, *then* $a + b \in \mathcal{P}$ *and* $ab \in \mathcal{P}$.

(iii) $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

We write $a \geq b$ as a synonym for $a - b \in \mathcal{P}$.

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Total positivity is then defined in the usual way.

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A polynomial is nonnegative if all its coefficients are nonnegative.

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2) Nonconstant polynomials are not invertible in $\mathbb{R}[\mathbf{x}]$.

And even in the formal-power-series ring $\mathbb{R}[[\mathbf{x}]]$, $1 + x \geq 0$ but $(1+x)^{-1} = 1 - x + x^2 - \ldots \not\geq 0$

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Let $P_1(\mathbf{x}), \ldots, P_k(\mathbf{x})$ and $Q(\mathbf{x})$ be polynomials in indeterminates $\mathbf{x}$.

Can Q be written as a polynomial with nonnegative coefficients in $P_1, \ldots, P_k$?

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- If we knew which multi-indices $\mathbf{m}$ could contribute, this would be a problem in linear programming $\longrightarrow$ feasible up to quite high dimension.
- How to know? (Look at nonzero monomials . . . ??)

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Given a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$, we define its *Hankel matrix*

$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j \ge 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

Hankel-total positivity

Main Characterization (Stieltjes 1894, Gantmakher–Krein 1937)

For a sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure μ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \ge 0$.

[That is, $\boldsymbol{a}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \ge 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

From numbers to polynomials

[or, From counting to counting-with-weights]

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An industry in combinatorics: cf. Sokal–Zeng 2020 and Deb–Sokal 2022

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Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n \geq 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \geq 0$, but it is *stronger*.

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- The continued-fraction approach gives a *sufficient but not necessary* condition for coefficientwise Hankel-total positivity.
- More general approach: *production matrices* — still *sufficient but far from necessary*.

Classical continued fractions

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- Stieltjes-type continued fractions (S-fractions):

$$\sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes–Rogers polynomial}} \; t^n \quad = \quad \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

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- Jacobi-type continued fractions (J-fractions):

$$\sum_{n=0}^{\infty} \underbrace{J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})}_{\substack{\text{Jacobi–Rogers} \\ \text{polynomial}}} \; t^n \;=\; \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}}$$

Classical continued fractions

- Stieltjes-type continued fractions (S-fractions):

$$\sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes–Rogers polynomial}} t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

- Jacobi-type continued fractions (J-fractions):

$$\sum_{n=0}^{\infty} \underbrace{J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})}_{\text{Jacobi–Rogers polynomial}} t^n = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \cdots}}}$$

- This is combinatorialists' notation. Analysts take $t^n \to \dfrac{1}{z^{n+1}}$

Classical continued fractions (lattice-path interpretation)

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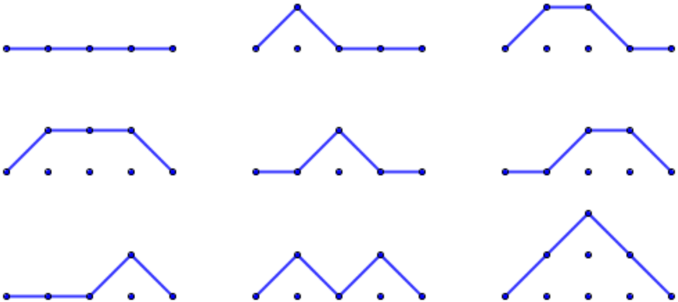

All the Motzkin paths of length $n = 4$.

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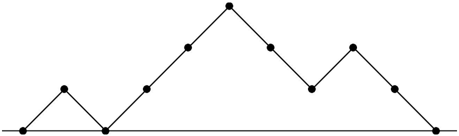

A Dyck path of length $2n = 10$

Classical continued fractions (lattice-path interpretation)

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Theorem (Flajolet 1980)

- The Jacobi–Rogers polynomial $J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length n, in which each rise gets weight 1, each level step at height i gets weight γ_i, and each fall from height i gets weight β_i.
- The Stieltjes–Rogers polynomial $S_n(\boldsymbol{\alpha})$ is the generating polynomial for Dyck paths of length $2n$, in which each rise gets weight 1 and each fall from height i gets weight α_i.

Computing classical continued fractions

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- Given a power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$ with $a_0 = 1$, how to compute

$$f(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \qquad ?$$

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- Define for $k \ge 0$ the S-fraction starting at level k:

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- Then we have the obvious recurrence

$$f_k(t) \;=\; \frac{1}{1 \;-\; \alpha_{k+1} t\, f_{k+1}(t)}$$

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Primitive algorithm.

1. Set $f_0(t) = f(t)$.
2. For $k = 1, 2, 3, \ldots$, do:
   (a) If $f_{k-1}(t) = 1$, set $\alpha_k = 0$ and terminate.
   (b) If $f_{k-1}(t) \neq 1$, set $\alpha_k = [t^1]\, f_{k-1}(t)$ and

$$f_k(t) \;=\; \alpha_k^{-1} t^{-1} \left(1 \,-\, \frac{1}{f_{k-1}(t)}\right)$$

Computing classical continued fractions

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Disadvantage of this algorithm: it requires division of power series.
But we can linearize the problem . . .

Computing classical continued fractions (bis)

- Define $g_k(t) = \prod_{i=0}^{k} f_i(t)$ for $k \geq -1$

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- Define $g_k(t) \;=\; \prod_{i=0}^{k} f_i(t) \quad$ for $k \geq -1$
- So $g_{-1}(t) = 1$ and $f_k(t) \;=\; \dfrac{g_k(t)}{g_{k-1}(t)}$
- Nonlinear 2-term recurrence for $(f_k) \longrightarrow$ linear 3-term recurrence

$$g_k(t) - g_{k-1}(t) \;=\; \alpha_{k+1}\, t\, g_{k+1}(t)$$

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Refined algorithm.

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   (b) If $g_{k-1}(t) \neq g_{k-2}(t)$, set $\alpha_k = [t^1] \bigl(g_{k-1}(t) - g_{k-2}(t)\bigr)$ and

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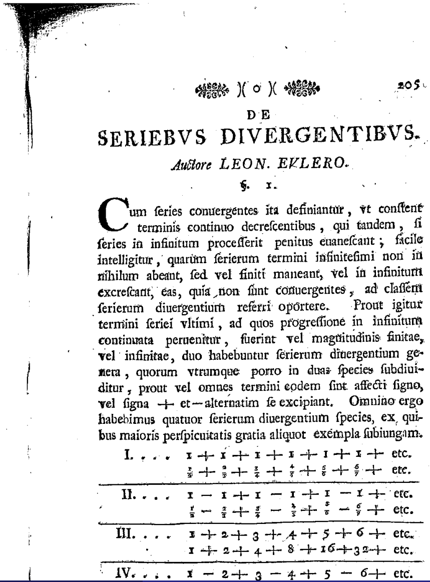
   $$g_k(t) \;=\; \alpha_k^{-1} t^{-1} \big(g_{k-1}(t) - g_{k-2}(t)\big)$$

Can also let $g_{-1}(t) = 1 + \ldots$ be arbitrary, not just $= 1$.

Computing classical continued fractions (bis2)



DE

SERIEBVS DIVERGENTIBVS.

Auctore LEON. EVLERO.

§. 1.

Cum ſeries conuergentes ita definiantur, vt conſtent terminis continuo decreſcentibus, qui tandem, ſi ſeries in infinitum proceſſerit penitus euaneſcant; facile intelligitur, quarum ſerierum termini infiniteſimi non in nihilum abeant, ſed vel finiti maneant, vel in infinitum excreſcant, eas, quia non ſint conuergentes, ad claſſem ſerierum diuergentium referri oportere. Prout igitur termini ſeriei vltimi, ad quos progreſſione in infinitum continuata peruenitur, fuerint vel magnitudinis finitae, vel infinitae, duo habebuntur ſerierum diuergentium genera, quorum vtrumque porro in duas ſpecies ſubdiuiditur, prout vel omnes termini eodem ſint affecti ſigno, vel ſigna + et − alternatim ſe excipiant. Omnino ergo habebimus quatuor ſerierum diuergentium ſpecies, ex quibus maioris perſpicuitatis gratia aliquot exempla ſubiungam.

I. . . . $1 + 1 + 1 + 1 + 1 + 1 +$ etc.
$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} +$ etc.

II. . . . $1 - 1 + 1 - 1 + 1 - 1 +$ etc.
$\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \frac{5}{6} - \frac{6}{7} +$ etc.

III. . . . $1 + 2 + 3 + 4 + 5 + 6 +$ etc.
$1 + 2 + 4 + 8 + 16 + 32 +$ etc.

IV. . . . $1 - 2 + 3 - 4 + 5 - 6 +$ etc.

Computing classical continued fractions (bis2)

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- Presented to the St. Petersburg Academy in 1753
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- Euler derives the continued fraction

$$\sum_{n=0}^{\infty} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cfrac{3t}{1 - \cdots}}}}}}}$$

Computing classical continued fractions (bis2)

§. 21. Datur vero alius modus in ſummam huius ſeriei inquirendi ex natura fractionum continuarum petitus, qui multo facilius et promtius negotium conficit: ſit enim formulam generalius exprimendo:

$$A = 1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} = \frac{1}{1+B}$$

$$\text{erit } B = \frac{1x - 2x^2 + 6x^3 - 24x^4 + 120x^5 - 720x^6 + 5040x^7 - \text{etc.}}{1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.}} = \frac{x}{1+C}$$

$$\text{et } 1 + C = \frac{1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.}}{1 - 2x + 6x^2 - 24x^3 + 120x^4 - 720x^5 + 5040x^6 - \text{etc.}}$$

$$\text{Ergo } C = \frac{x - 4x^2 + 18x^3 - 96x^4 + 600x^5 - 4320x^6 + \text{etc.}}{1 - 2x + 6x^2 - 24x^3 + 120x^4 - 720x^5 + \text{etc.}} = \frac{x}{1+D}$$

$$\text{vnde } D = \frac{2x - 12x^2 + 72x^3 - 480x^4 + 3600x^5 - \text{etc.}}{1 - 4x + 18x^2 - 96x^3 + 600x^4 - \text{etc.}} = \frac{2x}{1+E}$$

$$\text{Porro } E = \frac{2x - 18x^2 + 144x^3 - 1200x^4 + \text{etc.}}{1 - 6x + 36x^2 - 240x^3 + \text{etc.}} = \frac{2x}{1+F}$$

$$\text{Atque } F = \frac{3x - 36x^2 + 360x^3 - \text{etc.}}{1 - 9x + 72x^2 - 600x^3 + \text{etc.}} = \frac{3x}{1+G}$$

$$\text{Erit } G = \frac{3x - 48x^2 + \text{etc.}}{1 - 12x + 120x^2} = \frac{3x}{1+H}$$

$$\text{Sic } H = \frac{4x - \text{etc.}}{1 - 16x} = \frac{4x}{1+I}$$

Sicque porro patebit fore $I = \frac{4x}{1+K}$, $K = \frac{1x}{5+L}$; $L = \frac{5x}{1+M}$ etc. in infinitum, ita vt harum formularum ordo facile perſpiciatur. His autem valoribus ſucceſſive ſubſtitutis, erit

$$1 - 1x + 2x^2 - 6x^3 + 24x^4 - 120x^5 + 720x^6 - 5040x^7 + \text{etc.} =$$

Computing classical continued fractions (bis2)

$$A = \cfrac{1}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{2x}{1+\cfrac{2x}{1+\cfrac{3x}{1+\cfrac{3x}{1+\cfrac{4x}{1+\cfrac{4x}{1+\cfrac{5x}{1+\cfrac{5x}{1+\cfrac{6x}{1+\cfrac{6x}{1+\cfrac{7x}{\text{etc.}}}}}}}}}}}}}}$$

§. 22. Quemadmodum autem huiusmodi fractionum continuarum valor fit inveſtigandus, alibi oſtendi: Scilicet cum ſingulorum denominatorum partes integrae ſint vnitates, ſoli numeratores in computum veniunt; ſit ergo $x = 1$, atque inveſtigatio ſummae A ſequenti modo inſtituetur:

$$A = \tfrac{0}{1},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{4}{7},\ \tfrac{8}{13},\ \tfrac{20}{34},\ \tfrac{44}{73},\ \tfrac{124}{209},\ \tfrac{300}{501},\ \text{etc.}$$

num. 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, etc:

Fractiones nimirum hic exhibitae continuo propius ad verum valorem ipſius A accedunt, et quidem alternatim eo ſunt maiores et minores; ita vt ſit:

Computing classical continued fractions (bis2)

DE LA MÉTHODE GÉNÉRALE

POUR REDUIRE TOUTES SORTES DES QUANTITÉS EN FRACTIONS CONTINUES.

PAR

B. VISCOVATOV.

Présenté le 18. Décembre 1805.

J'ai en l'honneur de présenter à l'Académie en 1802 un mémoire sous le titre : *Essai d'une méthode générale pour réduire toutes sortes de séries en fractions continues* : après ce tems ayant eu occasion de penser encore à cette matière, j'ai fait de nouvelles réflexions qui peuvent servir à perfectionner et simplifier la méthode dont il s'agit. Ce sont ces réflexions que je présente maintenant à la société savante.

1. *Réduire une fraction quelconque*

$$P = \frac{a_1 + b_1 + c_1 + d_1 + e_1 + f_1 + g_1 + etc.}{a + b + c + d + e + f + g + etc.},$$

en fraction continue

Computing classical continued fractions (bis2)

- Euler 1746
- Viscovatov 1805
- Rediscovered a few times in the 20th century
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Surely the story unfolded here emphasizes how valuable it is to study and understand the central ideas behind major pieces of mathematics produced by giants like Euler.

— *George Andrews*

Comparing efficiency of algorithms

Timing tests for

$$\sum_{n=0}^{\infty} n!\, t^n = \cfrac{1}{1 - \cfrac{1t}{1 - \cfrac{1t}{1 - \cfrac{2t}{1 - \cfrac{2t}{1 - \cfrac{3t}{1 - \cfrac{3t}{1 - \cdots}}}}}}}$$

Comparing efficiency of algorithms

N	Primitive algorithm	Refined algorithm	Ratio
100	0.20	0.15	1.33
200	0.87	0.14	6.32
300	2.20	0.29	7.47
400	4.87	0.51	9.53
500	9.41	0.79	11.86
600	17.32	1.15	15.06
700	30.26	1.58	19.17
800	51.10	2.09	24.44
900	83.48	2.69	31.07
1000	131.90	3.25	40.63
1100	200.71	4.14	48.46
1200	297.45	5.10	58.38
1300	429.43	6.21	69.18
1400	606.35	7.20	84.20
1500	840.25	8.75	95.99
1600	1128.79	9.54	118.28
1700	1490.64	11.00	135.50
1800	1947.84	12.59	154.68
1900	2505.78	14.40	174.06
2000	3176.93	15.74	201.85
3000	20896.0	43.85	476.52
4000		94.49	
5000		170.51	
6000		277.10	
7000		420.58	
8000		604.25	
9000		835.81	

Comparing efficiency of algorithms

Euler also proved the more general continued fraction

$$\sum_{n=0}^{\infty} x^{\overline{n}}\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{(x+1)t}{1 - \cfrac{2t}{1 - \cfrac{(x+2)t}{1 - \cfrac{3t}{1 - \cdots}}}}}}}$$

where $x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$

Comparing efficiency of algorithms

N	Primitive algorithm	Refined algorithm	Ratio
10	0.02	0.02	1.21
15	0.08	0.06	1.46
20	0.27	0.12	2.25
25	0.50	0.21	2.40
30	1.04	0.36	2.85
35	3.15	0.56	5.64
40	16.13	0.77	21.07
45	57.23	1.04	55.14
50	139.52	1.41	98.66
55	283.39	1.72	164.86
60	505.61	2.15	234.67
65	1029.79	2.90	355.29
70	5390.53	3.44	1567.81
75	20714.2	4.23	4893.62
80	54919.5	4.75	11560.1
90		6.35	
100		8.60	
110		10.79	
120		13.52	
130		16.54	
140		19.97	
150		24.06	
160		28.42	
170		33.76	
180		39.46	
190		45.91	
200		52.23	
300		158.25	
400		360.65	
500		691.27	
600		1184.81	
700		1910.57	
800		2909.85	
900		4244.91	
1000		5960.16	

Hankel-TP for Stieltjes-type continued fractions

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Theorem (A.S. 2014, based on Viennot 1983)

The sequence $(S_n(\alpha))_{n \geq 0}$ of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive in the polynomial ring $\mathbb{Z}[\alpha]$.

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Proof uses the Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

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Can now specialize α to *nonnegative* elements in any partially ordered commutative ring, and get Hankel-TP.

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Many applications . . .

Hankel-TP for Jacobi-type continued fractions

What about J-type continued fractions?

Hankel-TP for Jacobi-type continued fractions

What about J-type continued fractions?

As before, we form the Hankel matrix

$$H_\infty(\boldsymbol{J}) \;=\; \big(J_{i+j}(\boldsymbol{\beta}, \boldsymbol{\gamma})\big)_{i,j \ge 0}$$

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- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \geq 0$.

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But the story is more complicated than for S-type fractions, because:

- The matrix $H_\infty(\boldsymbol{J})$ is *not* totally positive in $\mathbb{Z}[\boldsymbol{\beta},\boldsymbol{\gamma}]$.
- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \geq 0$.
- Rather, the total positivity of $H_\infty(\boldsymbol{J})$ holds only when $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy suitable *inequalities*.

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Proof uses the method of *production matrices.*

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- But it is a much more general tool.

Production matrices (proof of theorem)

- Define the augmented production matrix

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- Part (b) on the Hankel matrix needs one small further step.

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- Having such an algorithm would be *extremely* useful.

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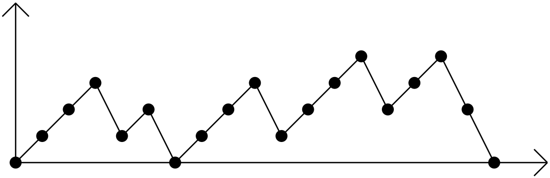

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- **Conjecture 1**: $(A_n(x))_{n \ge 0}$ is a Stieltjes moment sequence for all $x \ge 1$ (but not for $0 < x < 1$).

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- **Theorem** (conjectured by me, 2014; proven by G. Edgar, 2017): $(A_n)_{n \ge 0}$ is a Stieltjes moment sequence.
- Define Apéry polynomials $A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k$
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- **Conjecture 2**: $(A_n(1+y))_{n \ge 0}$ is coefficientwise Hankel-TP in y. (Tested up to 12×12)

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Dedicated to the memory of Philippe Flajolet (1948–2011)