December 4, 2023

Workshop: Computer Algebra for Functional Equations in Combinatorics and Physics, IHP, Paris, France

Summation Tools for Combinatorics and Elementary Particle Physics

Carsten Schneider

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University Linz

JOHANNES KEPLER UNIVERSITÄT LINZ

FWF Der Wissenschaftsfonds.

Outline

1. A warm-up example
2. The difference ring machinery for symbolic summation
3. Challenging applications

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Bigg)}_{f(j)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \quad (= H_n)$$

Arose in the context of

I. Bierenbaum, J. Blümlein, and S. Klein, **Evaluating two-loop massive operator matrix elements with Mellin-Barnes integrals**. 2006

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Bigg(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}\Bigg)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

↑ summation package Sigma

$$g(j) = \frac{(j+k+1)(j+n+1)j!k!(j+k+n)!\Big(S_1(j)-S_1(j+k)-S_1(j+n)+S_1(j+k+n)\Big)}{kn(j+k+1)!(j+n+1)!(k+n+1)!}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

FIND $g(j)$:

$$\boxed{f(j) = g(j+1) - g(j)}$$

Summing the telescoping equation over j from 0 to a gives

$$\sum_{j=0}^{a} f(j) = g(a+1) - g(0)$$

$$= \underbrace{\frac{(a+1)!(k-1)!(a+k+n+1)!(S_1(a)-S_1(a+k)-S_1(a+n)+S_1(a+k+n))}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}}_{a\to\infty}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum}=\sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!}+\frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= **res = SigmaReduce[mySum]**

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!}+\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!}+\frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $$\mathbf{mySum} = \sum_{j=0}^{a}\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S[1,j]+S[1,j+k]+S[1,j+n]-S[1,j+k+n])}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big);$$

In[3]:= $\mathbf{res} = \mathbf{SigmaReduce}[\mathbf{mySum}]$

Out[3]= $$\frac{(a+1)!(k-1)!(a+k+n+1)!\,(S[1,a]-S[1,a+k]-S[1,a+n]+S[1,a+k+n])}{n(a+k+1)!(a+n+1)!(k+n+1)!} + \frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)n!} + \frac{(2a+k+n+2)a!k!(a+k+n)!}{(a+k+1)(a+n+1)(a+k+1)!(a+n+1)!(k+n+1)!}$$

In[4]:= $\mathbf{SigmaLimit}[\mathbf{res}, \{n\}, a]$

Out[4]= $$\frac{1}{n!}\frac{S[1,k]+S[1,n]-S[1,k+n]}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{j=0}^{\infty} f(j) = \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\underbrace{\Big(\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}\Big)}_{f(j)}$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $k \geq 1$.

Telescoping

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(k)}.$$

FIND $g(k)$:

$$\boxed{g(k+1) - g(k)} = \boxed{f(k)}$$

for all $k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{f(n,k)}$$

for all $k \geq 1$.

no solution ☹

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n)f(n,k)+c_1(n)\,f(n+1,k)}$$

for all $k \geq 1$.

Sigma *computes*: $c_0(n) = -n,\ c_1(n) = (n+2)$ and

$$g(n,k) = \frac{kS_1(k)+(-n-1)S_1(n)-kS_1(k+n)-2}{(k+n+1)(n+1)^2}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} \Big[c_0(n)\, f(n,k) + c_1(n)\, f(n+1,k) \Big]}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{\sum_{k=1}^{a} c_0(n)\, f(n,k) + \sum_{k=1}^{a} c_1(n)\, f(n+1,k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=: f(n,k)}.$$

FIND $g(n, k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n, k+1) - g(n, k)} = \boxed{c_0(n) f(n, k) + c_1(n) f(n+1, k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n, a+1) - g(n, 1)} = \boxed{c_0(n) \sum_{k=1}^{a} f(n, k) + c_1(n) \sum_{k=1}^{a} f(n+1, k)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k) + S_1(n) - S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1) - g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1) - g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

Zeilberger's creative telescoping paradigm

GIVEN

$$\mathsf{A}(n) := \sum_{k=1}^{a} \underbrace{\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}}_{=:\, f(n,k)}.$$

FIND $g(n,k)$ and $c_0(n)$, $c_1(n)$:

$$\boxed{g(n,k+1)-g(n,k)} = \boxed{c_0(n) f(n,k) + c_1(n)\, f(n+1,k)}$$

for all $k \geq 1$.

Summing this equation over k from 1 to a gives:

$$\boxed{g(n,a+1)-g(n,1)} = \boxed{c_0(n)\,\mathsf{A}(n) + c_1(n)\,\mathsf{A}(n+1)}$$

$$\frac{(a+1)(S_1(a)+S_1(n)-S_1(a+n))}{(n+1)^2(a+n+2)} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)} \quad = \quad -n\mathsf{A}(n) + (2+n)\mathsf{A}(n+1)$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence finder

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

recurrence solver

$$A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)} \in \left\{ c \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)} \middle| c \in \mathbb{R} \right\}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

$$(n+2)\mathbf{A}(n+1) - n\mathbf{A}(n) = \frac{(n+1)S_1(n)+1}{(n+1)^3}$$

Summation package Sigma

(based on difference field/ring algorithms/theory
see, e.g., Abramov, Karr 1981, Bronstein 2000, Schneider 2001/2004/2005a–c/2007/2008/2010a–c)

$$\boxed{A(n) = \sum_{k=1}^{\infty} \frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}} = \boxed{0 \times \frac{1}{n(n+1)} + \frac{S_1(n)^2+S_2(n)}{2n(n+1)}}$$

where

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}$$

In[5]:= $$\mathbf{mySum} = \sum_{\mathbf{k}=1}^{\mathbf{a}} \frac{\mathbf{S[1, k] + S[1, n] - S[1, k + n]}}{\mathbf{kn(k + n + 1)}};$$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\text{SUM}[n]+(1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\text{SUM}[n] + (1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec = GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\text{SUM}[n]+(1+n)(2+n)\text{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!}+\frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec = LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\text{SUM}[n]+(1+n)(2+n)\text{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol = SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\left\{\left\{0, \frac{1}{n(n+1)}\right\}, \left\{1, \frac{S[1,n]^2 + \sum_{i=1}^{n}\frac{1}{i^2}}{2n(n+1)}\right\}\right\}$

In[5]:= $\mathbf{mySum} = \sum_{k=1}^{a} \frac{S[1,k] + S[1,n] - S[1,k+n]}{kn(k+n+1)};$

Compute a recurrence

In[6]:= $\mathbf{rec} = \mathbf{GenerateRecurrence[mySum, n][[1]]}$

Out[6]= $-n\mathrm{SUM}[n]+(1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(a+1)\left(S[1,a]+S[1,n]-S[1,a+n]\right)}{(n+1)^2(a+n+2)n!} + \frac{a(a+1)}{(n+1)^3(a+n+1)(a+n+2)n!}$

In[7]:= $\mathbf{rec} = \mathbf{LimitRec[rec, SUM[n], \{n\}, a]}$

Out[7]= $-n\mathrm{SUM}[n] + (1+n)(2+n)\mathrm{SUM}[n+1] == \frac{(n+1)S[1,n]+1}{(n+1)^3}$

Solve a recurrence

In[8]:= $\mathbf{recSol} = \mathbf{SolveRecurrence[rec, SUM[n]]}$

Out[8]= $\{\{0, \frac{1}{n(n+1)}\}, \{1, \frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}\}\}$

Combine the solutions

In[9]:= $\mathbf{FindLinearCombination[recSol, \{1, \{1/2\}\}, n, 2]}$

Out[9]= $\frac{S[1,n]^2 + \sum_{i=1}^{n} \frac{1}{i^2}}{2n(n+1)}$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Big(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\,(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n))}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(j)}\Big)$$

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f(j) = \frac{1}{n!}\sum_{k=1}^{\infty}\frac{S_1(k)+S_1(n)-S_1(k+n)}{kn(k+n+1)}$$
$$= \frac{1}{n!}\frac{S_1(n)^2+S_2(n)}{2n(n+1)}$$

where

$$S_1(n) = \sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n) = \sum_{i=1}^{n}\frac{1}{i^2}$$

A warm-up example: simplify

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Bigg(\underbrace{\frac{(2j+k+n+2)j!k!(j+k+n)!}{(j+k+1)(j+n+1)(j+k+1)!(j+n+1)!(k+n+1)!} + \frac{j!k!(j+k+n)!\left(-S_1(j)+S_1(j+k)+S_1(j+n)-S_1(j+k+n)\right)}{(j+k+1)!(j+n+1)!(k+n+1)!}}_{f(n,k,j)}\Bigg)$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f(n,k,j) = \frac{S_1(n)^2+3S_2(n)}{2n(n+1)!}$$

where

$$S_1(n)=\sum_{i=1}^{n}\frac{1}{i} \qquad S_2(n)=\sum_{i=1}^{n}\frac{1}{i^2}$$

Part 2: The difference ring machinery for symbolic summation

Part 2: The difference ring machinery for symbolic summation

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \dots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \dots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$S_{2,1}(n) = \sum_{i=1}^{n} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j} \qquad \text{(harmonic sums)}$$

J. Blümlein and S. Kurth, Phys. Rev. D **60** (1999) 014018 [arXiv:hep-ph/9810241];
J.A.M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \dots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

Special cases:

$$\sum_{j=1}^{n} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} \qquad \text{(binomial sums)}$$

J. Ablinger, J. Blümlein, C. G. Raab and CS, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822].

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \dots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

A more general example:

$$\sum_{k=1}^{n} \left(\prod_{i=1}^{k} \frac{1+i+i^2}{i+1} \right) \left(\sum_{j=1}^{k} \frac{1}{j \binom{4j}{3j}^2} \right) \left(\sum_{j=1}^{k} \begin{bmatrix} 2j \\ j \end{bmatrix}_q \right)$$

$$-2(1+n)^3(3+n)n!^2F(n) + (1+n)\left(8+9n+2n^2\right)n!F(n+1) - F(n+2) = 0$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\left\{c_1\prod_{i=1}^{n} i! + c_2\left(-2^n n!\prod_{i=1}^{n} i! + \tfrac{3}{2}\prod_{i=1}^{n} i!\sum_{i=1}^{n} 2^i i!\right) \mid c_1, c_2 \in \mathbb{K}\right\}$$

$$\begin{aligned}&\big(1+S_1(n)+nS_1(n)\big)^2\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2F(n)\\&-(1+n)(3+2n)S_1(n)\big(3+2n+2S_1(n)+3nS_1(n)+n^2S_1(n)\big)^2F(n+1)\\&\qquad+(1+n)^2(2+n)^3S_1(n)\big(1+S_1(n)+nS_1(n)\big)F(n+2)=0\end{aligned}$$

$$\Big\downarrow \texttt{Sigma.m}$$

$$\Big\{c_1\,S_1(n)\prod_{l=1}^{n}S_1(l)+c_2S_1(n)^2\prod_{l=1}^{n}S_1(l)\mid c_1,c_2\in\mathbb{K}\Big\}$$

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n) = \sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n), \dots, a_d(n), h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n) + \cdots + a_d(n)F(n+d) = h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

Simple sum

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\boxed{\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}}$$

$$||$$

$$\boxed{\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\boxed{\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\boxed{\sum_{r=0}^{j+1}\binom{j+1}{r}\Big(\frac{(-1)^r(-j+n-2)!r!}{(n+1)(-j+n+r-1)(-j+n+r)!}+\frac{(-1)^{n+r}(j+1)!(-j+n-2)!(-j+n-1)_r r!}{(n-1)n(n+1)(-j+n+r)!(-j-1)_r(2-n)_j}\Big)}$$

$$||$$

$$\Big(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Big)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\displaystyle\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$\sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!}$$

$$||$$

$$\sum_{j=0}^{n-2}\Bigg(\Bigg(\frac{n^2-n+1}{(n-1)^2n^2(n+1)(2-n)_j}+\frac{\sum_{i=1}^{j}\frac{(2-n)_i}{(-i+n-1)^2(i+1)!}}{(n+1)(2-n)_j}+$$
$$\frac{(-1)^{j+n}(-j-2)(-j+n-2)!}{(j-n+1)(n+1)^2n!}\Bigg)(j+1)!-\frac{1}{(n+1)^2(-j+n-1)}\Bigg)$$

$$||$$

$$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S_{-2}(n)}{n+1}+\frac{S_1(n)}{(n+1)^2}+\frac{S_2(n)}{-n-1}$$

Note: $S_a(n)=\sum_{i=1}^{N}\frac{\mathsf{sign}(a)^i}{i^{|a|}},\ a\in\mathbb{Z}\setminus\{0\}$.

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

In[1]:= $<<$ **Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

In[2]:= $<<$ **HarmonicSums.m**

HarmonicSums by Jakob Ablinger © RISC-Linz

In[3]:= $<<$ **EvaluateMultiSums.m**

EvaluateMultiSums by Carsten Schneider © RISC-Linz

In[4]:= $$\mathbf{mySum} = \sum_{j=0}^{n-2}\sum_{r=0}^{j+1}\sum_{s=0}^{n-j+r-2}\frac{(-1)^{r+s}\binom{j+1}{r}\binom{-j+n+r-2}{s}(-j+n-2)!r!}{(n-s)(s+1)(-j+n+r)!};$$

In[5]:= **EvaluateMultiSum[mySum, {}, {n}, {1}]**

Out[5]= $$\frac{-n^2-n-1}{n^2(n+1)^3}+\frac{(-1)^n\left(n^2+n+1\right)}{n^2(n+1)^3}-\frac{2S[-2,n]}{n+1}+\frac{S[1,n]}{(n+1)^2}+\frac{S[2,n]}{-n-1}$$

This summation machinery is based on

1. S.A. Abramov. The rational component of the solution of a first-order linear recurrence relation with a rational right-hand side. U.S.S.R. Comput. Maths. Math. Phys., 15:216–221, 1975.
2. M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
3. S.A. Abramov. Rational solutions of linear differential and difference equations with polynomial coefficients. U.S.S.R. Comput. Math. Math. Phys., 29(6):7–12, 1989.
4. S.A. Abramov and M. Petkovšek. D'Alembertian solutions of linear differential and difference equations. In ISSAC'94, pages 169–174. 1994.
5. P. Paule. Greatest factorial factorization and symbolic summation. J. Symbolic Comput. **20**(3), 235–268 (1995)
6. M. Petkovšek, H. S. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, MA, 1996.
7. M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. J. Pure Appl. Algebra, 139(1-3):109–131, 1999.
8. P. A. Hendriks and M. F. Singer. Solving difference equations in finite terms. *J. Symbolic Comput.*, 27(3):239–259, 1999.
9. M. Bronstein. On solutions of linear ordinary difference equations in their coefficient field. *J. Symbolic Comput.*, 29(6):841–877, 2000.
10. CS. Symbolic summation in difference fields. J. Kepler University, May 2001. PhD Thesis.
11. CS. A collection of denominator bounds to solve parameterized linear difference equations in $\Pi\Sigma$-extensions. *An. Univ. Timişoara Ser. Mat.-Inform.*, 42(2):163–179, 2004.
12. CS. Symbolic summation with single-nested sum extensions. In J. Gutierrez, editor, *Proc. ISSAC'04*, pages 282–289. ACM Press, 2004.
13. CS. Degree bounds to find polynomial solutions of parameterized linear difference equations in $\Pi\Sigma$-fields. *Appl. Algebra Engrg. Comm. Comput.*, 16(1):1–32, 2005.
14. CS. Product representations in $\Pi\Sigma$-fields. *Ann. Comb.*, 9(1):75–99, 2005.
15. CS. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
16. CS. Finding telescopers with minimal depth for indefinite nested sums and product expressions. In *Proc. ISSAC'05*, pages 285–292. ACM Press, 2005.
17. M. Kauers and C. Schneider. Indefinite summation with unspecified summands. Discrete Math., 306(17):2021–2140, 2006.
18. CS. Simplifying Sums in $\Pi\Sigma$-Extensions. *J. Algebra Appl.*, 6(3):415–441, 2007.
19. CS. A refined difference field theory for symbolic summation. *J. Symbolic Comput.*, 43(9):611–644, 2008.
20. CS. A Symbolic Summation Approach to Find Optimal Nested Sum Representations. In A. Carey, D. Ellwood, S. Paycha, and S. Rosenberg, editors, *Motives, Quantum Field Theory, and Pseudodifferential Operators*, pages 285–308. 2010.
21. CS. Parameterized Telescoping Proves Algebraic Independence of Sums. *Ann. Comb.*, 14(4):533–552, 2010.
22. CS. Structural Theorems for Symbolic Summation. *Appl. Algebra Engrg. Comm. Comput.*, 21(1):1–32, 2010.
23. CS. Simplifying Multiple Sums in Difference Fields. In: Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, J. Blümlein, C. Schneider (ed.), Texts and Monographs in Symbolic Computation, pp. 325-360. Springer, 2013.
24. CS. Fast Algorithms for Refined Parameterized Telescoping in Difference Fields. In *Computer Algebra and Polynomials*, Lecture Notes in Computer Science (LNCS), Springer, 2014.
25. CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
26. CS. Summation Theory II: Characterizations of $R\Pi\Sigma$-extensions and algorithmic aspects. J. Symb. Comput. 80(3), pp. 616-664. 2017.
27. E.D. Ocansey, CS. Representing (q-)hypergeometric products and mixed versions in difference rings. In: Advances in Computer Algebra, C. Schneider, E. Zima (ed.), Springer Proceedings in Mathematics & Statistics 226. 2018.
28. S.A. Abramov, M. Bronstein, M. Petkovšek, CS. On Rational and Hypergeometric Solutions of Linear Ordinary Difference Equations in $\Pi\Sigma^*$-field extensions. J. Symb. Comput. 107, pp. 23-66. 2021.
29. CS. Term Algebras, Canonical Representations and Difference Ring Theory for Symbolic Summation. In: Anti-Differentiation and the Calculation of Feynman Amplitudes, J. Blümlein and C. Schneider (ed.), Texts and Monographs in Symbolic Computuation. 2021. Springer.
30. J. Ablinger, CS, Solving linear difference equations with coefficients in rings with idempotent representations. In: Proc. ISSAC'21, pp. 27-34. 2021.
31. CS. Refined telescoping algorithms in $R\Pi\Sigma$-extensions to reduce the degrees of the denominators. In: Proc. ISSAC '23, pp. 498-507. 2023.
32. E.D. Ocansey, CS, Representation of hypergeometric products of higher nesting depths in difference rings. J. Symb. Comput. 120, pp. 1-50. 2024.

Part 2: The difference ring machinery for symbolic summation

Part 2: The difference ring machinery for symbolic summation

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \underbrace{\underbrace{\mathbb{Q}(x)}_{\text{rat. fu. field}} [s]}_{\text{polynomial ring}}$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{array}{rlcl} \text{ev}': & \mathbb{Q}(x) \times \mathbb{N} & \to & \mathbb{Q} \\ & (\frac{p(x)}{q(x)}, n) & \mapsto & \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function

$$\begin{aligned} \mathrm{ev}' : \quad & \mathbb{Q}(x) \times \mathbb{N} && \to \quad \mathbb{Q} \\ & \left(\tfrac{p(x)}{q(x)}, n\right) && \mapsto \quad \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\mathrm{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} \quad \to \quad \mathbb{Q}$$

$$\mathbf{ev(s, n) = S_1(n)}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

$$\begin{aligned} \mathrm{ev}' : \quad \mathbb{Q}(x) \times \mathbb{N} &\to \mathbb{Q} \\ \left(\tfrac{p(x)}{q(x)}, n\right) &\mapsto \begin{cases} \frac{p(n)}{q(n)} & \text{if } q(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$$\begin{aligned} \mathrm{ev} : \quad \mathbb{Q}(x)[s] \times \mathbb{N} &\to \mathbb{Q} \\ \left(\sum_{i=0}^{d} f_i\, s^i, n\right) &\mapsto \sum_{i=0}^{d} \mathrm{ev}'(f_i, n) S_1(n)^i \end{aligned} \qquad \mathbf{ev(s, n) = S_1(n)}$$

Definition: $(\mathbb{A}, \mathrm{ev})$ is called an eval-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \qquad = \qquad \langle 0, 1, 2, 3, \ldots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \rangle$$

Simplify
$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\tau(x)\tau(\tfrac{1}{x}) \quad = \quad \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \rangle$$
$$\|$$
$$\langle 0, 1, 1, 1, \dots \rangle$$

Simplify
$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}} \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

It is almost a ring homomorphism :

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \\ & & \| \\ & & \langle 0, 1, 1, 1, \dots \rangle \\ & & \nparallel \\ \tau(x\frac{1}{x}) = \tau(1) & = & \langle 1, 1, 1, 1, \dots \rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccc} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}}/\sim \\ & f & \mapsto & \langle \mathrm{ev}(f, n) \rangle_{n \geq 0} \end{array}$$

$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is a ring homomorphism :

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0, 1, 2, 3, \dots \rangle \langle 0, 1, \frac{1}{2}, \frac{1}{3}, \dots \rangle \\ & & \| \\ & & \langle 0, 1, 1, 1, \dots \rangle \\ & & \| \\ \tau(x\frac{1}{x}) = \tau(1) & = & \langle 1, 1, 1, 1, \dots \rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$

Consider the map

$$\begin{array}{rccl} \tau : & \mathbb{A} & \to & \mathbb{Q}^{\mathbb{N}}/\sim \\ & f & \mapsto & \langle \text{ev}(f,n) \rangle_{n \geq 0} \end{array}$$

$(a_n) \sim (b_n)$ iff $a_n = b_n$ from a certain point on

It is an injective ring homomorphism (ring embedding):

$$\begin{array}{rcc} \tau(x)\tau(\frac{1}{x}) & = & \langle 0,1,2,3,\dots\rangle\langle 0,1,\frac{1}{2},\frac{1}{3},\dots\rangle \\ & & || \\ & & \langle 0,1,1,1,\dots\rangle \\ & & || \\ \tau(x\,\frac{1}{x}) = \tau(1) & = & \langle 1,1,1,1,\dots\rangle \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) =?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{rlcl} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

$$\sigma : \quad \mathbb{Q}(x)[s] \quad \to \quad \mathbb{Q}(x)[s] \qquad s \mapsto s + \tfrac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\text{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\left(s + \frac{1}{x+1}\right)^i \end{array}$$

$$s \mapsto s + \frac{1}{x+1}$$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

Definition: $(\mathbb{A}, \sigma)$ with a ring $\mathbb{A}$ and automorphism σ is called a difference ring; the set of constants is

$$\text{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism

$$\begin{array}{llll} \sigma' : & \mathbb{Q}(x) & \to & \mathbb{Q}(x) \\ & r(x) & \mapsto & r(x+1) \end{array}$$

$$\begin{array}{llll} \sigma : & \mathbb{Q}(x)[s] & \to & \mathbb{Q}(x)[s] \\ & \sum_{i=0}^{d} f_i\, s^i & \mapsto & \sum_{i=0}^{d} \sigma'(f_i)\left(s + \frac{1}{x+1}\right)^i \end{array}$$

$s \mapsto s + \frac{1}{x+1}$

$$\mathbf{S_1(n+1) = S_1(n) + \frac{1}{n+1}}$$

In this example:

$$\mathrm{const}_\sigma \mathbb{A} = \{c \in \mathbb{A} \mid \sigma(c) = c\} = \mathbb{Q}$$

This is a special case of an $R\Pi\Sigma$-ring

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) = S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

$$\Updownarrow$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

shift operator

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) \Updownarrow= S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\begin{array}{ccc} \mathbb{K}(x)[s] & \xrightarrow{\quad\sigma\quad} & \mathbb{K}(x)[s] \\ \downarrow \tau & = & \downarrow \tau \\ \mathbb{K}^{\mathbb{N}}/\sim & \xrightarrow{\quad S\quad} & \mathbb{K}^{\mathbb{N}}/\sim \end{array}$$

Simplify

$$\sum_{k=0}^{a} S_1(k) = ?$$

built on Karr's DF theory of $\Pi\Sigma$-fields

1. a formal ring $\mathbb{A} = \mathbb{Q}(x)[s]$
2. an evaluation function $\mathrm{ev} : \mathbb{A} \times \mathbb{N} \to \mathbb{Q}$
3. a ring automorphism $\sigma : \mathbb{A} \to \mathbb{A}$

ev and σ interact:

$$\mathrm{ev}(\sigma(s), n) = \mathrm{ev}(s + \tfrac{1}{x+1}, n) \Updownarrow= S_1(n) + \tfrac{1}{n+1} = \mathrm{ev}(s, n+1)$$

$$\tau(\sigma(s)) = \langle 1, 1 + \tfrac{1}{2}, 1 + \tfrac{1}{2} + \tfrac{1}{3}, \dots \rangle = S(\langle 0, 1, 1 + \tfrac{1}{2}, \dots \rangle) = S(\tau(s))$$

τ is an injective difference ring homomorphism:

$$\boxed{(\mathbb{K}(x)[s], \sigma)} \overset{\tau}{\simeq} \boxed{(\underbrace{\tau(\mathbb{Q}(x))}_{\text{rat. seq.}}[\langle S_1(n)\rangle_{n\geq 0}], S)} \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\sum_{k=0}^{a} S_1(k) =?$$

$$\begin{array}{ccccc} (\mathbb{A},\sigma) & \overset{\tau}{\simeq} & (\tau(\mathbb{A}),S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim,S) \\ & & \| & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}] & & \end{array}$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\begin{array}{ccccc} (\mathbb{A}, \sigma) & \overset{\tau}{\simeq} & (\tau(\mathbb{A}), S) & \leq & (\mathbb{K}^{\mathbb{N}}/\sim, S) \\ & & || & & \\ & & \tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}] & & \end{array}$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \;\leq\; (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$||$$

$$\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}]$$

$$\sum_{k=0}^{a} S_1(k) = ?$$

Given: $f(k) = S_1(k)$
Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \quad \leq \quad (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$||$$

$$\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}]$$

$$\sum_{k=0}^{a} S_1(k) =?$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k)\rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$\Updownarrow \quad \tau$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \;\leq\; (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$\tau(\mathbb{A}) = \tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}]$$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k) \rangle_{k \geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \leq (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$||$$

$$\tau(\mathbb{Q}(x))[\langle S_1(k) \rangle_{k \geq 0}]$$

$$\sum_{k=0}^{a} S_1(k) = g(a+1) - g(0) = (a+1)S_1(a+1) - (a+1)$$

Given: $f(k) = S_1(k)$

Find: $g = \langle g(k)\rangle_{k\geq 0} \in \tau(\mathbb{A})$ s.t.

$$g(k+1) - g(k) = S_1(k)$$

Output: $g(k) = k\,S_1(k) - k$

$$\Updownarrow \quad \tau$$

Find: $\bar{g} \in \mathbb{A}$:

$$\sigma(\bar{g}) - \bar{g} = s$$

Output: $\bar{g} = x\,s - x$

$$(\mathbb{A}, \sigma) \quad \overset{\tau}{\simeq} \quad (\tau(\mathbb{A}), S) \quad \leq \quad (\mathbb{K}^{\mathbb{N}}/\sim, S)$$

$$||$$

$$\tau(\mathbb{Q}(x))[\langle S_1(k)\rangle_{k\geq 0}]$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(k+1)!=(k+1)k! $\quad\leftrightarrow\quad \sigma(p_1) = (x+1)p_1$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

hypergeometric products $\leftrightarrow$ $\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg.	$\leftrightarrow$	$\sigma(p_1) = a_1\, p_1$	$a_1 \in \mathbb{K}(x)^*$
products		$\sigma(p_2) = a_2 p_2$	$a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^*$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^* \end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$$
$$\sigma(p_2) = a_2 p_2 \qquad a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^*$$
$$\vdots$$
$$\sigma(p_e) = a_e p_e \qquad a_e \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^*$$

$(-1)^k \quad \leftrightarrow \quad \sigma(z) = -z \qquad z^2 = 1$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z]}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x) = x+1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1) &= a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2) &= a_2p_2 && a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*\\ &\vdots\\ \sigma(p_e) &= a_ep_e && a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*\end{aligned}$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow \quad \sigma(\mathbf{z}) = \alpha\,\mathbf{z} \qquad \mathbf{z}^{\lambda}=\mathbf{1}$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products	$\leftrightarrow$	$\sigma(p_1) = a_1\, p_1$	$a_1 \in \mathbb{K}(x)^*$
		$\sigma(p_2) = a_2 p_2$	$a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^*$
		$\vdots$	
		$\sigma(p_e) = a_e p_e$	$a_e \in \mathbb{K}(x)[p_1, p_1^{-1}]\dots[p_{e-1}, p_{e-1}^{-1}]^*$
α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}}$	$\leftrightarrow$	$\sigma(\mathbf{z}) = \alpha\, \mathbf{z}$	$\mathbf{z}^\lambda = \mathbf{1}$
$S_1(k+1) = S_1(k) + \frac{1}{k+1}$	$\leftrightarrow$	$\sigma(s_1) = s_1 + \frac{1}{x+1}$	

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1]$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\sigma(p_1) = a_1\, p_1 \qquad a_1 \in \mathbb{K}(x)^*$$

$$\sigma(p_2) = a_2 p_2 \qquad a_2 \in \mathbb{K}(x)[p_1, p_1^{-1}]^*$$

$$\vdots$$

$$\sigma(p_e) = a_e p_e \qquad a_e \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^*$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow$

$$\sigma(\mathbf{z}) = \alpha\, \mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$$

(nested) sum $\leftrightarrow$

$$\sigma(s_1) = s_1 + f_1 \qquad f_1 \in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z]$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2]}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^* \end{aligned}$$

α is a primitive λth root of unity $\quad \alpha^{\mathbf{k}} \leftrightarrow$

$$\sigma(\mathbf{z}) = \alpha\, \mathbf{z} \qquad \mathbf{z}^{\lambda} = \mathbf{1}$$

(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A}, \sigma)$

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2][s_3] \dots}$$

- with an automorphism where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and where

$$\sigma(x) = x + 1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned} \sigma(p_1) &= a_1\, p_1 & a_1 &\in \mathbb{K}(x)^* \\ \sigma(p_2) &= a_2 p_2 & a_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}]^* \\ &\vdots \\ \sigma(p_e) &= a_e p_e & a_e &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_{e-1}, p_{e-1}^{-1}]^* \end{aligned}$$

α is a primitive λth root of unity $\alpha^{\mathbf{k}}$ $\leftrightarrow$ $\sigma(\mathbf{z}) = \alpha\, \mathbf{z}$ $\quad \mathbf{z}^{\lambda} = \mathbf{1}$

(nested) sum $\leftrightarrow$

$$\begin{aligned} \sigma(s_1) &= s_1 + f_1 & f_1 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z] \\ \sigma(s_2) &= s_2 + f_2 & f_2 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1] \\ \sigma(s_3) &= s_3 + f_3 & f_3 &\in \mathbb{K}(x)[p_1, p_1^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \\ &\vdots \end{aligned}$$

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x)=x+1$$

(nested) hyperg. products $\leftrightarrow$

$$\begin{aligned}\sigma(p_1)&=a_1\,p_1 && a_1\in\mathbb{K}(x)^*\\ \sigma(p_2)&=a_2p_2 && a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*\\ &\vdots\\ \sigma(p_e)&=a_ep_e && a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*\end{aligned}$$

α is a primitive λth root of unity $\alpha^{\mathbf{k}}$ $\leftrightarrow$ $\sigma(\mathbf{z})=\alpha\,\mathbf{z}$ $\quad\mathbf{z}^{\lambda}=\mathbf{1}$

(nested) sum $\leftrightarrow$

$$\begin{aligned}\sigma(s_1)&=s_1+f_1 && f_1\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z]\\ \sigma(s_2)&=s_2+f_2 && f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]\\ \sigma(s_3)&=s_3+f_3 && f_3\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]\\ &\vdots\end{aligned}$$

such that $\text{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

Further details: Symbolic summation in an $R\Pi\Sigma$-ring $(\mathbb{A},\sigma)$

- a ring (containing $\mathbb{Q}$) (Karr81,CS16,CS17,CS18)

$$\boxed{\mathbb{A}:=\mathbb{K}(x)[p_1,p_1^{-1}][p_2,p_2^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2][s_3]\dots}$$

- with an automorphism where $\sigma(c)=c$ for all $c\in\mathbb{K}$ and where

$$\sigma(x)=x+1$$

(nested) hyperg. products $\leftrightarrow$

$$\sigma(p_1)=a_1\,p_1 \qquad a_1\in\mathbb{K}(x)^*$$

$$\sigma(p_2)=a_2p_2 \qquad a_2\in\mathbb{K}(x)[p_1,p_1^{-1}]^*$$

$$\vdots$$

$$\sigma(p_e)=a_ep_e \qquad a_e\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_{e-1},p_{e-1}^{-1}]^*$$

α is a primitive λth root of unity

GIVEN $f\in\mathbb{A}$;
FIND, in case of existence, a $g\in\mathbb{A}$ such that

$$\sigma(g)-g=f.$$

(nested) su

$$\dots e,p_e^{-1}][z]$$

$$\sigma(s_2)=s_2+f_2 \qquad f_2\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1]$$

$$\sigma(s_3)=s_3+f_3 \qquad f_3\in\mathbb{K}(x)[p_1,p_1^{-1}]\dots[p_e,p_e^{-1}][z][s_1][s_2]$$

$$\vdots$$

such that $\mathrm{const}_\sigma\mathbb{A}=\{c\in\mathbb{A}|\sigma(c)=c\}=\mathbb{K}$.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\mathrm{const}_{\sigma}\mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}] \dots [p_e, p_e^{-1}][z][s_1][s_2] \dots [s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\text{const}_\sigma \mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]\dots[s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\mathrm{const}_\sigma\mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]\dots[s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\mathrm{const}_\sigma \mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

3. There is an $\boxed{\text{embedding } \tau}$ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Remark 1: Related results have been worked out in the Galois theory of difference equations (van der Put/Singer, 1997) and (Hardouin/Singer, 2008)

Further details (2): Galois theory for $R\Pi\Sigma$-extensions

- a ring (containing $\mathbb{Q}$)

$$\boxed{\mathbb{A} := \mathbb{K}(x)[p_1, p_1^{-1}][p_2, p_2^{-1}]\dots[p_e, p_e^{-1}][z][s_1][s_2]\dots[s_r]}$$

- with an automorphism as given in the previous slide.

Theorem. The following statements are equivalent:

1. $\mathrm{const}_\sigma\mathbb{A} = \mathbb{K}$.

   (i.e., $(\mathbb{A}, \sigma)$ is an $R\Pi\Sigma$-ring)

2. $(\mathbb{A}, \sigma)$ is simple.

   (i.e., there is no ideal in $\mathbb{A}$ which is closed under σ except $\{0\}$ and $\mathbb{A}$)

3. There is an embedding τ from $(\mathbb{A}, \sigma)$ into the ring of sequences.

CS. A Difference Ring Theory for Symbolic Summation. J. Symb. Comput. 72, pp. 82-127. 2016.
CS. Characterizations of $R\Pi\Sigma$-extensions. J. Symb. Comput. 80, pp. 616-664. 2017.

Remark 2: Theory covers also the q-hypergeometric, mutli-basic and mixed cases

Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

$$\sigma(s_i) = s_i + \frac{1}{(x+1)^i} \qquad i = 1, 2, 3, \dots$$

Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

$$\sigma(s_i) = s_i + \frac{1}{(x+1)^i} \qquad i = 1, 2, 3, \dots$$

2. There is an embedding of the polynomial ring $\mathbb{Q}(x)[s_1, s_2, \dots]$ into $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$s_1 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n\geq 0}, \qquad s_2 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n\geq 0} \qquad \dots$$

Example: algebraic independence of sequences

1. $(\mathbb{Q}(x)[s_1, s_2, \dots], \sigma)$ is an $R\Pi\Sigma$-ring with

$$\sigma(s_i) = s_i + \frac{1}{(x+1)^i} \qquad i = 1, 2, 3, \dots$$

2. There is an embedding of the polynomial ring $\mathbb{Q}(x)[s_1, s_2, \dots]$ into $\mathbb{Q}^{\mathbb{N}}/\sim$ with

$$s_1 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i} \right\rangle_{n\geq 0}, \qquad s_2 \mapsto \left\langle \sum_{i=1}^{n} \frac{1}{i^2} \right\rangle_{n\geq 0} \qquad \dots$$

$\Rightarrow$ The generalized harmonic numbers

$$S_1(n) = \sum_{i=1}^{n} \frac{1}{i}, \quad S_2(n) = \sum_{i=1}^{n} \frac{1}{i^2}, \quad S_3(n) = \sum_{i=1}^{n} \frac{1}{i^3}, \quad \dots$$

are algebraically independent among each other over the rational sequences.

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$$\Big\downarrow \quad \texttt{SigmaReduce[A,n]}$$

$B(n)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,n]`

$B(n)$: nested product-sum expression (sums/products not in the denominator)

- such that

$$A(\lambda) = B(\lambda) \qquad \text{for all } \lambda \in \mathbb{N} \text{ with } \lambda \geq \delta$$

(δ can be computed explicitly)

- and such that

  the arising sums and products in $B(n)$ (except the alternating sign) are **algebraically independent**
  (i.e., they do not satisfy any polynomial relation)

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,n]`

$B(n)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(n)$ is usually much smaller

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,n]`

$B(n)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(n)$ is usually much smaller

Application 2: we solve the zero-recognition problem:

$$A(n) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B = 0$$

Simplification of nested product-sum expressions

$A(n)$: nested product-sum expression (sums/products not in the denominator)

$\downarrow$ `SigmaReduce[A,n]`

$B(n)$: nested product-sum expression (sums/products not in the denominator)

Application 1: the expression $B(n)$ is usually much smaller

Application 2: we solve the zero-recognition problem:

$$A(n) \text{ evaluates to 0 from a certain point on} \quad \Leftrightarrow \quad B = 0$$

Application 3: we get canonical form representations

1. Creative telescoping (for the special case of hypergeometric terms see Zeilberger's algorithm (1991))

GIVEN a definite sum

$$F(n)=\sum_{k=0}^{n} f(n,k);$$

$f(n,k)$: indefinite nested product-sum in k;
n: extra parameter

FIND a recurrence for $F(n)$

2. Recurrence solving

GIVEN a recurrence

$a_0(n),\dots,a_d(n),h(n)$:
indefinite nested product-sum expressions.

$$a_0(n)F(n)+\dots+a_d(n)F(n+d)=h(n);$$

FIND all solutions expressible by indefinite nested products/sums
(Abramov/Bronstein/Petkovšek/CS, 2021)

3. Find a "closed form"

F(n)=combined solutions in terms of indefinite nested sums.

Part 3: Challenging applications

- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

Part 3: Challenging applications

- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

On January 22, 2020 I received the following email by Doron Zeilberger:

```
Dear Carsten,

I (and Shalosh) just posted a paper

https://arxiv.org/abs/2001.06839

with a challenge to you (see the middle of page 4)

Can you (and Sigma) extend theorem 5 of that paper
to the general case with k absent-minded passengers?
....
If you and Sigma can do the fourth moment, and derive
the asymptotic in n (with a fixed but arbitrary k), I will
donate $100$ to the OEIS in your honor.
...
Best wishes,
Doron
```

On January 22, 2020 I received the following email by Doron Zeilberger:

```
Dear Carsten,

I (and Shalosh) just posted a paper

https://ar

with

Can y
to the
....
If you and Sig                                  nd derive
the asymptotic in n (wi          but arbitrary k), I will
donate $100$ to the OEIS in your honor.
...
Best wishes,
Doron
```

This email provoked various heavy calculations by means of computer algebra that solved fully the above challenge (based on beautiful results of Doron).
In the following only the symbolic summation aspect is illustrated.

$n \geq 2$ passengers take step-wise their seats in a plane with n seats.

$n \geq 2$ passengers take step-wise their seats in a plane with n seats.

1. The first $k \geq 1$ passengers are absent-minded, i.e., they loose their seat tickets and take a seat uniformly at random.

$n \geq 2$ passengers take step-wise their seats in a plane with n seats.

1. The first $k \geq 1$ passengers are absent-minded, i.e., they loose their seat tickets and take a seat uniformly at random.
2. Each of the remaining $n - k$ passengers takes the dedicated seat if it is still free; otherwise, they choose uniformly at random one of the still available free seats.

$n \geq 2$ passengers take step-wise their seats in a plane with n seats.

1. The first $k \geq 1$ passengers are absent-minded, i.e., they loose their seat tickets and take a seat uniformly at random.
2. Each of the remaining $n-k$ passengers takes the dedicated seat if it is still free; otherwise, they choose uniformly at random one of the still available free seats.

$\downarrow$ [Henze/Last:arXiv:1809.10192]

The expected value for the passengers sitting in the wrong seat is

$$E(X_n) = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n}$$

and the variance is

$$V(X_n) = \frac{k(n-1)}{n^2} + \sum_{i=1}^{-k+n} \frac{(1-i-k+n)\left(1-\frac{1-i-k+n}{1-i+n}\right)}{1-i+n} + 2\left(\frac{(k-1)k}{2(n-1)n^2} + \sum_{i=1}^{k}\sum_{j=1}^{-k+n} \frac{\frac{1-j-k+n}{-j+n} - \frac{1-j-k+n}{1-j+n}}{n}\right)$$

In[6]:= $\mathbf{E = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};}$

In[7]:= **EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]**

In[6]:= $\mathbf{E = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};}$

In[7]:= $\mathbf{EvaluateMultiSum[V, \{\}, \{k, n\}, \{1, 2\}, \{n, Infinity\}]}$

Out[7]= $\frac{-kS[1,k] + kS[1,n] + k(n-1)}{n}$

In[6]:= $\mathbf{E = \frac{k(n-1)}{n} + \sum_{i=1}^{-k+n} \frac{k}{1-i+n};}$

In[7]:= $\mathbf{EvaluateMultiSum[V, \{\}, \{k, n\}, \{1, 2\}, \{n, Infinity\}]}$

Out[7]= $\frac{-kS[1,k] + kS[1,n] + k(n-1)}{n}$

In[8]:= $$\mathbf{V = \frac{k(n-1)}{n^2} + \sum_{i=1}^{-k+n} \frac{(1-i-k+n)\left(1-\frac{1-i-k+n}{1-i+n}\right)}{1-i+n} + 2\left(\frac{(k-1)k}{2(n-1)n^2} + \sum_{i=1}^{k}\sum_{j=1}^{-k+n} \frac{\frac{1-j-k+n}{-j+n} - \frac{1-j-k+n}{1-j+n}}{n}\right);}$$

In[9]:= $\mathbf{EvaluateMultiSum[V, \{\}, \{k, n\}, \{1, 2\}, \{n, Infinity\}]}$

In[6]:= $$\mathbf{E}=\frac{\mathbf{k}(\mathbf{n}-1)}{\mathbf{n}}+\sum_{\mathbf{i}=1}^{-\mathbf{k}+\mathbf{n}}\frac{\mathbf{k}}{1-\mathbf{i}+\mathbf{n}};$$

In[7]:= **EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]**

Out[7]= $$\frac{-\mathtt{kS}[1,\mathtt{k}]+\mathtt{kS}[1,\mathtt{n}]+\mathtt{k}(\mathtt{n}-1)}{\mathtt{n}}$$

In[8]:= $$\mathbf{V}=\frac{\mathbf{k}(\mathbf{n}-1)}{\mathbf{n}^2}+\sum_{\mathbf{i}=1}^{-\mathbf{k}+\mathbf{n}}\frac{(1-\mathbf{i}-\mathbf{k}+\mathbf{n})\left(1-\frac{1-\mathbf{i}-\mathbf{k}+\mathbf{n}}{1-\mathbf{i}+\mathbf{n}}\right)}{1-\mathbf{i}+\mathbf{n}}+2\left(\frac{(\mathbf{k}-1)\mathbf{k}}{2(\mathbf{n}-1)\mathbf{n}^2}+\sum_{\mathbf{i}=1}^{\mathbf{k}}\sum_{\mathbf{j}=1}^{-\mathbf{k}+\mathbf{n}}\frac{\frac{1-\mathbf{j}-\mathbf{k}+\mathbf{n}}{-\mathbf{j}+\mathbf{n}}-\frac{1-\mathbf{j}-\mathbf{k}+\mathbf{n}}{1-\mathbf{j}+\mathbf{n}}}{\mathbf{n}}\right);$$

In[9]:= **EvaluateMultiSum[V, {}, {k, n}, {1, 2}, {n, Infinity}]**

Out[9]= $$-\frac{\mathtt{k}(2+\mathtt{n})\mathtt{S}[1,\mathtt{k}]}{\mathtt{n}}+\frac{\mathtt{k}(2+\mathtt{n})\mathtt{S}[1,\mathtt{n}]}{\mathtt{n}}+\mathtt{k}^2\mathtt{S}[2,\mathtt{k}]-\mathtt{k}^2\mathtt{S}[2,\mathtt{n}]+\frac{2\mathtt{k}-\mathtt{k}^2-2\mathtt{n}-2\mathtt{kn}+2\mathtt{k}^2\mathtt{n}+2\mathtt{n}^2-\mathtt{kn}^2}{(\mathtt{n}-1)\mathtt{n}^2}$$

Other highlights related to combinatorial problems

- Plane Partitions VI: Stembridge's TSPP Theorem
  (joint with G.E. Andrews, P. Paule; 2005)
- Unfair permutations
  (joint with H. Prodinger, S. Wagner, 2011)
- Asymptotic and exact results on the complexity of the Novelli-Pak-Stoyanovskii algorithm
  (joint with R. Sulzgruber; 2017)
- Evaluation of binomial double sums involving absolute values
  (joint with C. Krattenthaler; 2020)

Part 3: Challenging applications

- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

[Arose in the context to explore rational approximations of $\zeta(4)$]

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}\,.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Conjecture (Wadim Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m \Big(t + \frac{n}{2}\Big) \frac{(t-n)_m}{m!} \, \frac{(t-2n+m)_{2n-m}}{(2n-m)!} \times \frac{(t+n+1)_n}{(t)_{n+1}} \, \frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}} \left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2n-m+j}{n} \frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3} \sum_{\nu=n-m+1}^{\infty} \frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6} \sum_{\nu=1}^{\infty} \frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

[Arose in the context to explore rational approximations of $\zeta(4)$]

Theorem (CS, Sigma, Zudilin) For integers $n \geq m \geq 0$, define two rational functions

$$R(t) = R_{n,m}(t) = (-1)^m\Big(t + \frac{n}{2}\Big)\frac{(t-n)_m}{m!}\,\frac{(t-2n+m)_{2n-m}}{(2n-m)!} \\ \times \frac{(t+n+1)_n}{(t)_{n+1}}\,\frac{(t+n+1)_{2n-m}}{(t)_{2n-m+1}}\left(\frac{n!}{(t)_{n+1}}\right)^2$$

and

$$\tilde{R}(t) = \tilde{R}_{n,m}(t) = \frac{n!\,(t-n)_{2n-m}}{(t)_{n+1}(t)_{2n-m+1}}\sum_{j=0}^{n}\binom{n}{j}^2\binom{2n-m+j}{n}\frac{(t-j)_n}{n!}.$$

Then

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu} = \frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}.$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

with

$$\alpha_0(n,m)=(2n-m)^5,$$

$$\alpha_1(n,m)=-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3 \\ +30n^2m-14nm^2+2m^3+8n^2-10nm+2m^2-4n+m),$$

$$\alpha_2(n,m)=-(2n-m-1)^3(4n-m)(m+2).$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$
$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

$$
\begin{aligned}
S(n,m) = \sum_{j=0}^{n} \sum_{\nu=1}^{\infty} \Bigg(& \frac{\binom{n}{j}^2 \binom{j-m+2n}{n} (1+\nu)_{-m+2n} (1-j+\nu+n)_{-1+n}}{(1+\nu+n)_n (1+\nu+n)_{-m+2n} (\nu+n)^4 (\nu-m+2n)^3} \\
& \times \Bigg((\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) \\
& \quad + 2S_1(\nu+n) - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - \nu(j-\nu-n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) - S_1(\nu+2n) \\
& \quad - S_1(\nu-m+3n) - S_1(-j+\nu+n) + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& - (j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n} - S_1(\nu) + 2S_1(\nu+n) \\
& \quad - S_1(\nu+2n) - S_1(\nu-m+3n) - S_1(-j+\nu+n) \\
& \quad + S_1(\nu-m+2n) + S_1(-j+\nu+2n)\Big) \\
& + \nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{(j-\nu-2n)^2} - S_2(\nu) + 2S_2(\nu+n) \\
& \quad - S_2(\nu+2n) - S_2(\nu-m+3n) - S_2(-j+\nu+n) \\
& \quad + S_2(\nu-m+2n) + S_2(-j+\nu+2n)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+4(j+n)(\nu+n)-3(\nu+n)^2+n(-m+n)-j(m+2n)\Big)\\
&-2(\nu+n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&-3(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}-S_1(\nu)\\
&\quad+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&-(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)\\
&\quad+S_1(\nu-m+2n)+S_1(-j+\nu+2n)\Big)\\
&+2jn(m-n)+2(j+n)(\nu+n)^2-(\nu+n)^3-(\nu+n)\big(n(m-n)+j(m+2n)\big)\Big)\\
&\quad\times\big(-S_1(\nu+n)+S_1(\nu+2n)\big)\\
&+(\nu+n)(\nu-m+2n)\Big(-\nu(j-\nu-n)(\nu+n)(\nu-m+2n)\Big(-\frac{1}{-j+\nu+2n}\\
&\quad-S_1(\nu)+2S_1(\nu+n)-S_1(\nu+2n)-S_1(\nu-m+3n)-S_1(-j+\nu+n)
\end{aligned}
$$

$$
\begin{aligned}
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m - n) + 2(j + n)(\nu + n)^2 - (\nu + n)^3 - (\nu + n)\big(n(m - n) + j(m + 2n)\big)\Big) \\
&\times \big(- S_1(\nu) + S_1(\nu - m + 2n)\big) \\
&- (\nu + n)(\nu - m + 2n)\Big(- \nu(j - \nu - n)(\nu + n)(\nu - m + 2n)\Big(- \frac{1}{-j + \nu + 2n} \\
&- S_1(\nu) + 2S_1(\nu + n) - S_1(\nu + 2n) - S_1(\nu - m + 3n) - S_1(-j + \nu + n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m - n) + 2(j + n)(\nu + n)^2 - (\nu + n)^3 - (\nu + n)\big(n(m - n) + j(m + 2n)\big)\Big) \\
&\times \big(- S_1(\nu + n) + S_1(\nu - m + 3n)\big) \\
&+ (\nu + n)(\nu - m + 2n)\Big(- \nu(j - \nu - n)(\nu + n)(\nu - m + 2n)\Big(- \frac{1}{-j + \nu + 2n} \\
&- S_1(\nu) + 2S_1(\nu + n) - S_1(\nu + 2n) - S_1(\nu - m + 3n) - S_1(-j + \nu + n) \\
&+ S_1(\nu - m + 2n) + S_1(-j + \nu + 2n)\Big) \\
&+ 2jn(m - n) + 2(j + n)(\nu + n)^2 - (\nu + n)^3 \\
&- (\nu + n)\big(n(m - n) + j(m + 2n)\big)\Big) \\
&\times \Big(- \frac{1}{-j + \nu + 2n} - S_1(-j + \nu + n) + S_1(-j + \nu + 2n)\Big)\Big)\Big)
\end{aligned}
$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \quad \text{Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$

$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$S(n,m)=\sum_{j=0}^{n}\underbrace{\sum_{\nu=1}^{\infty}F(n,m,j,\nu)}_{T(n,m,j)}$$

$$\Big\downarrow \text{ Sigma.m with DR-creative telesoping}$$

$$a_0(n,m,j)\,T(n,m,j)+a_1(n,m,j)\,T(n,m,j+1)$$
$$+a_2(n,m,j)\,T(n,m,j+2)=a_3(n,m,j)$$
$$T(n,,m+1)=b_0(n,m,j)\,T(n,m,j)+b_1(n,m,j)\,T(n,m,j+1)=b_2(n,m,j)$$

$$\Big\downarrow \text{ Sigma.m with Holonomic-DR approach}$$

$$(2n-m)^5S(n,m)$$
$$-(4n-2m-1)(6n^4-24n^3m+22n^2m^2-8nm^3+m^4-24n^3+30n^2m-14nm^2$$
$$+2m^3+8n^2-10nm+2m^2-4n+m)S(n,m+1)$$
$$-(2n-m-1)^3(4n-m)(m+2)S(n,m+2)=R(n,m)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\Big|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\Big|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

SigmaReduce

$$\text{RHS}=\frac{1}{6}\Bigg(\overbrace{\sum_{j=0}^{n}\sum_{\nu=1}^{\infty}G_1(n,m,j,\nu)}^{=S(n,m)}+\sum_{j=0}^{n-1}\sum_{\nu=j+1}^{n}G_2(n,m,j,\nu)$$

$$+\sum_{j=1}^{n}\sum_{\nu=1}^{j}G_3(n,m,j,\nu)\Bigg)$$

Proof tactic: Both sides of

$$-\frac{1}{3}\sum_{\nu=n-m+1}^{\infty}\frac{dR(t)}{dt}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{d^2\tilde{R}(t)}{dt^2}\bigg|_{t=\nu}$$

satisfy the same recurrence:

$$\alpha_0(n,m)Z(n,m)+\alpha_1(n,m)Z(n,m+1)+\alpha_2(n,m)Z(n,m+2)=0$$

Finally, check 2 initial values: another round of non-trivial summation...

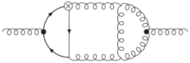
Highlights related to number theory

- Apéry's double sum is plain sailing indeed (2007)
- When is 0.999... equal to 1?
  (joint with R. Pemantle; 2007)
- Gaussian hypergeometric series and extensions of supercongruences
  (joint with R. Osburn; 2009)
- A case study for $\zeta(4)$
  (joint with W. Zudilin; 2021)
- Error bounds for the asymptotic expansion of the partition function
  [compare Hardy–Ramanujan, Wright, Rademacher, Lehmer, O'Sullivan]
  (joint with K. Banerjee, P. Paule, C.-S. Radu; 2023)

Part 3: Challenging applications

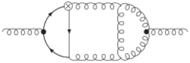
- combinatorics
- special functions
- number theory
- statistics
- numerics
- computer science
- elementary particle physics (QCD)

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$\longrightarrow$

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

Feynman integrals

$$\int_0^1 x^N \, dx$$

Feynman integrals

$$\int_0^1 x^N (1+x)^N \, dx$$

Feynman integrals

$$\int_0^1 \frac{x^N(1+x)^N}{(1-x)^{1+\varepsilon}}\,dx$$

Feynman integrals

$$\int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \dots \, dx_1 \, dx_2$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \; dx_1 \, dx_2 \, dx_3$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1+x_1)^N}{(1-x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5$$

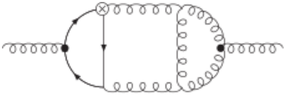
Feynman integrals

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{x_1^N (1 + x_1)^N}{(1 - x_1)^{1+\varepsilon}} \ldots \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 \, dx_6$$

Feynman integrals

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \frac{x_1^N(1+x_1)^{N-j+k}}{(1-x_1)^{1+\varepsilon}}\ldots\ dx_1\ dx_2\ dx_3\ dx_4\ dx_5\ dx_6$$

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$\Vert$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

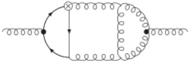
$$\Big[\left[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k$$

$$+\left[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)\right]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times\left[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3\right]^{N-3-j}dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)

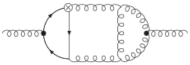


behavior of particles

$$\longrightarrow \quad \int \Phi(N, \epsilon, x)\, dx$$

Feynman integrals

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$$\int \Phi(N, \epsilon, x)\, dx$$

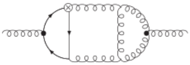
Feynman integrals

DESY

$$\sum f(N, \epsilon, k)$$

complicated multi-sums

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

DESY

$$\sum f(N,\epsilon,k)$$

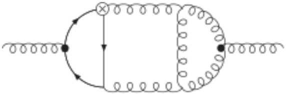
complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions

Feynman integrals



a 3-loop massive ladder diagram [arXiv:1509.08324]

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\binom{N-1}{j+2}\binom{j+1}{k+1} \qquad = $$

$$\times \int_0^1\int_0^1\int_0^1\int_0^1\int_0^1\int_0^1 \theta(1-x_5-x_6)(1-x_2)(1-x_4)x_2^{-\varepsilon}$$

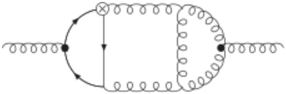
$$(1-x_2)^{-\varepsilon}x_4^{\varepsilon/2-1}(1-x_4)^{\varepsilon/2-1}x_5^{\varepsilon-1}x_6^{-\varepsilon/2}$$

$$\Big[[-x_3(1-x_4)-x_4(1-x_5-x_6+x_5x_1+x_6x_3)]^k$$

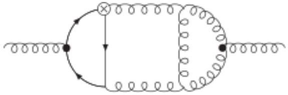
$$+[x_3(1-x_4)-(1-x_4)(1-x_5-x_6+x_5x_1+x_6x_3)]^k\Big]$$

$$\times(1-x_5-x_6+x_5x_1+x_6x_3)^{j-k}(1-x_2)^{N-3-j}$$

$$\times[x_1-(1-x_5-x_6)-x_5x_1-x_6x_3]^{N-3-j}\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5\,dx_6$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$



$$= F_{-3}(N)\varepsilon^{-3} + F_{-2}(N)\varepsilon^{-2} + F_{-1}(N)\varepsilon^{-1} + \boxed{F_0(N)}$$

Simplify

$$||$$

$$\sum_{j=0}^{N-3}\sum_{k=0}^{j}\sum_{l=0}^{k}\sum_{q=0}^{-j+N-3}\sum_{s=1}^{-l+N-q-3}\sum_{r=0}^{-l+N-q-s-3}(-1)^{-j+k-l+N-q-3}\times$$
$$\times\frac{\binom{j+1}{k+1}\binom{k}{l}\binom{N-1}{j+2}\binom{-j+N-3}{q}\binom{-l+N-q-3}{s}\binom{-l+N-q-s-3}{r}r!(-l+N-q-r-s-3)!(s-1)!}{(-l+N-q-2)!(-j+N-1)(N-q-r-s-2)(q+s+1)}$$
$$\Big[4S_1(-j+N-1)-4S_1(-j+N-2)-2S_1(k)$$
$$-(S_1(-l+N-q-2)+S_1(-l+N-q-r-s-3)-2S_1(r+s))$$
$$+2S_1(s-1)-2S_1(r+s)\Big]$$ + **3 further 6–fold sums**

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 + \frac{(17N+5)S_1(N)^3}{3N(N+1)} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$
$$+\Big(-\frac{4(13N+5)}{N^2(N+1)^2} + \Big(\frac{4(-1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$
$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$
$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$
$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$
$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$
$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$
$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$
$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$
$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$$

$$\frac{7}{12}S_1(N)^4 \ldots [illegible] \frac{(17N+5)S_1(N)^3}{[illegible]} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+ \Big(- [illegible] \frac{[illegible]1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{3} - (-1)^N\Big)S_3(N)$$

$$+ \big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)}\Big)S_1(N) + \Big(\frac{3}{4} + (-1)^N\Big)S_2(N)^2$$

$$- 2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-1)^N\big)S_1(N) + \frac{4(-1)^N}{N+1}\Big)$$

$$+ \Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+ \frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+ \Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+ \Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$- \frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+ 32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 \cdots \frac{(17N+5)S_1(N)^3}{\cdots} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(-\cdots\frac{\cdots 1)^N(2N+1)}{\cdots N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{\cdots} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} \cdots \Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(-\cdots \frac{\cdots^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N(5-3N)}{2N^2(N+1)} - \frac{5}{2N^2}\Big)S_2(N) + S_{-2}(N)\Big(10S_1(N)^2 + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N-1)}{N(N+1)}\Big)S_1(N) + \frac{8(-1)^N(3N+1)}{N(N+1)^2} + \big(-22+6(-1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N(9N+5)}{N(N+1)} - \frac{29}{3N}\Big)S_3(N) + \Big(\frac{19}{2} - 2(-1)^N\Big)S_4(N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(-1)^N(9N+5)}{N(N+1)} - \frac{2}{N}\Big)S_{2,1}(N) + \big(20+2(-1)^N\big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$\boxed{F_0(N)} =$$

$$\frac{7}{12}S_1(N)^4 \dots \frac{(17N+5)S_1(N)^3}{\dots} + \Big(\frac{35N^2-2N-5}{2N^2(N+1)^2} + \frac{13S_2(N)}{2} + \frac{5(-1)^N}{2N^2}\Big)S_1(N)^2$$

$$+\Big(- \dots \frac{\dots 1)^N(2N+1)}{N(N+1)} - \frac{13}{N}\Big)S_2(N) + \Big(\frac{29}{\dots} - (-1)^N\Big)S_3(N)$$

$$+\big(2+2(-1)^N\big)S_{2,1}(N) - 28S_{-2,1}(N) + \frac{20(-1)^N}{N^2(N+1)} \dots \Big)S_2(N)^2$$

$$-2(-1)^N S_{-2}(N)^2 + S_{-3}(N)\Big(\frac{2(3N-5)}{N(N+1)} + \big(26+4(\dots \frac{\dots^N}{N+1}\Big)$$

$$+\Big(\frac{(-1)^N \dots}{2N^2} \dots + \Big(\frac{8(-1)^N(2N+1)}{N(N+1)}$$

$$+\frac{4(3N - \dots}{N(N+ \dots} \dots -1)^N\big)S_2(N) - \frac{16}{N(N+1)}\Big)$$

$$+\Big(\frac{(-1)^N \dots}{N(\dots} \dots N) + \big(-6+5(-1)^N\big)S_{-4}(N)$$

$$+\Big(-\frac{2(- \dots}{N(N+1)} \dots \Big)S_{2,-2}(N) + \big(-17+13(-1)^N\big)S_{3,1}(N)$$

$$-\frac{8(-1)^N(2N+1)+4(9N+1)}{N(N+1)}S_{-2,1}(N) - \big(24+4(-1)^N\big)S_{-3,1}(N) + \big(3-5(-1)^N\big)S_{2,1,1}(N)$$

$$+32S_{-2,1,1}(N) + \left(\frac{3}{2}S_1(N)^2 - \frac{3S_1(N)}{N} + \frac{3}{2}(-1)^N S_{-2}(N)\right)\zeta(2)$$

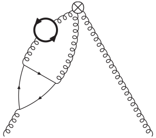
$$S_1(N) = \sum_{i=1}^{N}\frac{1}{i}$$

$$S_2(N) = \sum_{i=1}^{N}\frac{1}{i^2}$$

$$S_{-2,1,1}(N) = \sum_{i=1}^{N}\frac{(-1)^i\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{1}{k}}{j}}{i^2}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

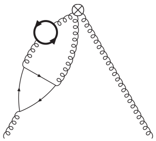
(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



All diagrams are produced with axodraw (J. Vermaseren).

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



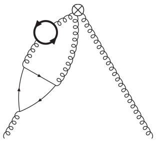
Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with

- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

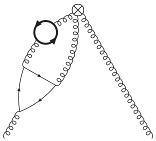
expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Typical triple sum:

$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i} \frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon} \times 2^{-2+\varepsilon} e^{-\frac{3\varepsilon\gamma}{2}} \eta^k \times$$
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

Typical triple sum:

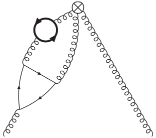
$$\sum_{j=0}^{N}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{(4+\varepsilon)(-2+N)(-1+N)N\pi(-1)^{2-k}}{2+\varepsilon}\times 2^{-2+\varepsilon}e^{-\frac{3\varepsilon\gamma}{2}}\eta^{k}\times$$
$$\frac{\Gamma(1-\frac{\varepsilon}{2}-i+j+k)\Gamma(-1-\frac{\varepsilon}{2})\Gamma(2+\frac{\varepsilon}{2})\Gamma(1+N)\Gamma(1+\varepsilon+i-k)\Gamma(-\frac{3\varepsilon}{2}+k)\Gamma(1-\varepsilon+k)\Gamma(3-\varepsilon+k)\Gamma(-\frac{1}{2}-\frac{\varepsilon}{2}+k)}{\Gamma(-\frac{3}{2}-\frac{\varepsilon}{2})\Gamma(\frac{5}{2}+\frac{\varepsilon}{2})\Gamma(2+i)\Gamma(1+k)\Gamma(2-i+j)\Gamma(2-\varepsilon+k)\Gamma(\frac{5}{2}-\varepsilon+k)\Gamma(-\frac{\varepsilon}{2}+k)\Gamma(5+\frac{\varepsilon}{2}+N)}$$

6 hours for this sum

$\sim$ 10 years of calculation time for full expression

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes-
and ${}_pF_q$-technologies
$\longrightarrow$

expression (95 MB) with

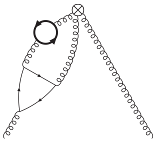
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB)
consisting of 8 multi-sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m

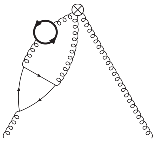
Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

sum	size of sum (with ε)	summand size of constant term	time of calculation		number of indef. sums
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{\infty}$	17.7 MB	266.3 MB	177529 s	(2.1 days)	1188
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{\infty}$	232 MB	1646.4 MB	980756 s	(11.4 days)	747
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{\infty}$	67.7 MB	458 MB	524485 s	(6.1 days)	557
$\sum_{i_1=0}^{\infty}$	38.2 MB	90.5 MB	689100 s	(8.0 days)	44
$\sum_{i_4=2}^{N-3}\sum_{i_3=0}^{i_4-2}\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}$	1.3 MB	6.5 MB	305718 s	(3.5 days)	1933
$\sum_{i_3=3}^{N-4}\sum_{i_2=0}^{i_3-1}\sum_{i_1=0}^{i_2}$	11.6 MB	32.4 MB	710576 s	(8.2 days)	621
$\sum_{i_2=3}^{N-4}\sum_{i_1=0}^{i_2}$	4.5 MB	5.5 MB	435640 s	(5.0 days)	536
$\sum_{i_1=3}^{N-4}$	0.7 MB	1.3 MB	9017s	(2.5 hours)	68

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

$\downarrow$ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

$\downarrow$ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)

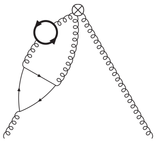
Most complicated objects: generalized binomial sums, like

$$\sum_{h=1}^{N} 2^{-2h}(1-\eta)^h \binom{2h}{h} \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i}}{i\binom{2i}{i}} \right) \left(\sum_{i=1}^{h} \frac{(1-\eta)^i \binom{2i}{i}}{2^{2i}} \right) \times$$

$$\times \left(\sum_{i=1}^{h} \frac{2^{2i}(1-\eta)^{-i} \sum_{j=1}^{i} \frac{\sum_{k=1}^{j} \frac{(1-\eta)^k}{k}}{j}}{i\binom{2i}{i}} \right).$$

Example: a 2-mass 3-loop Feynman integral [arXiv:1804.02226]

(arose in the calculation of the gluonic operator matrix element $A_{gg,Q}^{(3)}$)



Mellin-Barnes- and ${}_pF_q$-technologies $\longrightarrow$

expression (95 MB) with
- 150 single sums
- 1000 double sums
- 12160 triple sums
- 1555 quadruple sums

↓ SumProduction.m (2 hours)

expression (377 MB) consisting of 8 multi-sums

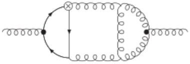
↓ EvaluateMultiSums.m (3 month)

expression (154 MB) consisting of 4110 indefinite sums

← Sigma.m (32 days)

expression (8.3 MB) consisting of 74 indefinite sums

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles → $\int \Phi(N,\epsilon,x)\,dx$

Feynman integrals

↓ **DESY**

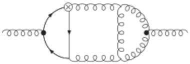
$\sum f(N,\epsilon,k)$

complicated multi-sums

← **advanced difference ring theory** (Sigma-package)

expression in special functions

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

LHC at CERN

DESY

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

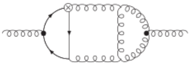
advanced difference ring theory

(Sigma-package)

expression in special functions

applicable

Evaluation of Feynman Integrals (joint with J. Blümlein, P. Marquard since 2007)



behavior of particles

$$\int \Phi(N,\epsilon,x)\,dx$$

Feynman integrals

- What did the universe look like in the first second
- Do the 4 fundamental forces unite at high energies?
- Do the properties of the new particle agree with the predicted Higgs-Boson?

DESY

applicable

$$\sum f(N,\epsilon,k)$$

complicated multi-sums

advanced difference ring theory

(Sigma-package)

expression in special functions