Computation of Sums and Integrals by Reduction-Based Creative Telescoping

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I. Creative Telescoping

Integrals and Sums of Special Functions

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x)\,dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

$$\frac{1}{2\pi i}\oint \frac{(1+2xy+4y^2)\exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1+4y^2)^{\frac{3}{2}}}\,dy = \frac{H_n(x)}{\lfloor n/2 \rfloor!}$$

$$\int_{-1}^{1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

$$\lambda^{\nu} \sum_{n=0}^{\infty} \frac{(1-\lambda^2)^n (z/2)^n}{n!} J_{\nu+n}(z) = J_{\nu}(\lambda z)$$

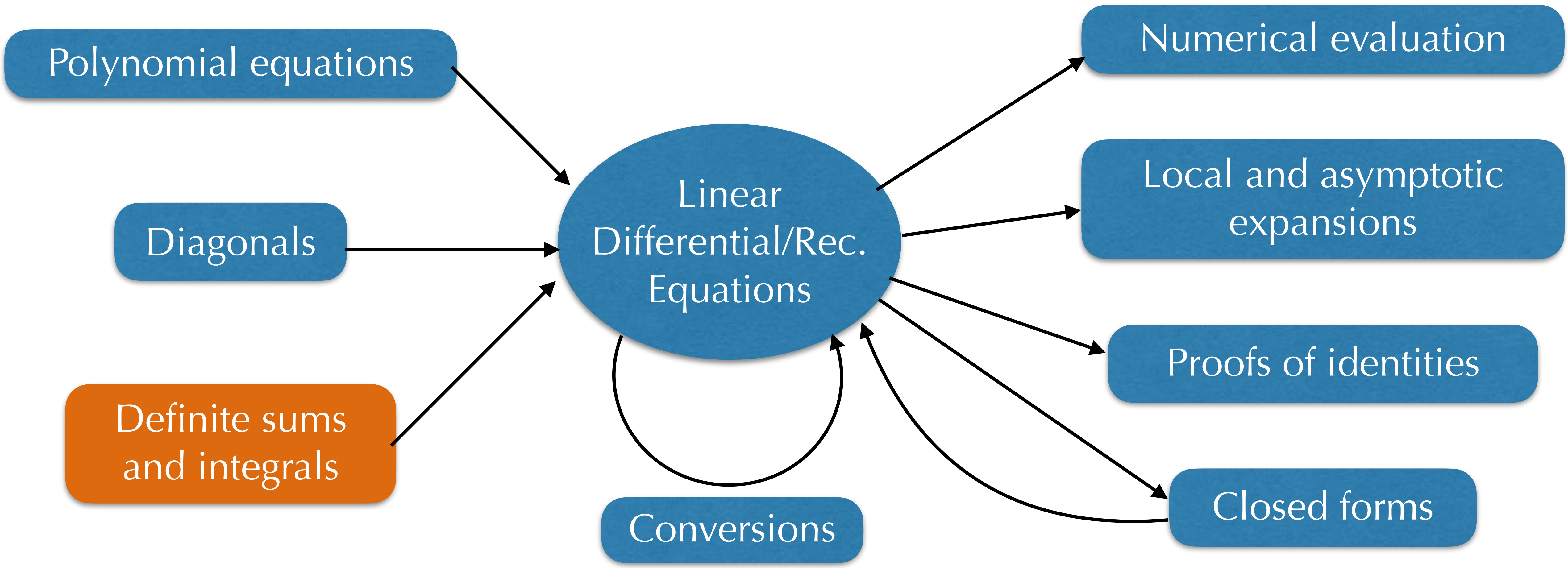
$$\sum_{k\geq 0} P_k^{(a,b)}(x) P_k^{(a,b)}(y) \frac{(a+b+1)_k\, k!}{(a+1)_k (b+1)_k} t^k = ?$$

Aims:
1. Prove them automatically
2. Find the rhs given the lhs

Note: at least one free variable

First: **find LDEs** (or LREs)

Context: LDEs as a Data-Structure



Algebraic Setup

Ore algebras:

$\mathbb{O}_r := \mathbb{K}(x_1, \ldots, x_r)\langle D_1, \ldots, D_r \rangle$ with commuting $D_i \in \{S_{x_i}, \partial_{x_i}\}$, $i = 0, \ldots, r$.

Notation:
$\partial_x : f(x) \mapsto f'(x)$
$S_n : u_n \mapsto u_{n+1}$
$\Delta_k : v_k \mapsto v_{k+1} - v_k$

Annihilating ideal of f: $\operatorname{Ann} f := \{P \in \mathbb{O}_r \mid P \cdot f = 0\}$.

Notation:
$\tilde{D}_r : \begin{cases} \tilde{\partial}_{x_r} = \partial_{x_r}, \\ \tilde{S}_{x_r} = \Delta_{x_r}. \end{cases}$

WANTED:

$$T(x_1, \ldots, x_{r-1}, D_1, \ldots, D_{r-1}) - \tilde{D}_r C(x_1, \ldots, x_r, D_1, \ldots, D_r) \in \operatorname{Ann} f$$

telescoper
(diff,shift under int,sum sign)

certificate
(int,sum by parts)

Example: Legendre Polynomials

```
> F:=Sum(2^(-n)*binomial(n,k)*binomial(n,n-k)*(x+1)^k*(x-1)^(n-k),k=0..n);
```

$$F := \sum_{k=0}^{n} 2^{-n} \binom{n}{k} \binom{n}{n-k} (x+1)^k (x-1)^{n-k}$$

$f_{n,k}(x)$

```
> CreativeTelescoping(F,[n::shift,x::diff],certificate='cert');
```

$$[(n+1)\,D_n + \left(1-x^2\right) D_x - xn - x, \left(x^2-1\right) D_x^2 + 2xD_x - n^2 - n]$$

D_n denotes the shift S_n

```
> normal(cert);
```

$$\left[\frac{(x-1)\,k^2\,(2k-3n-3)}{2(k^2-2kn+n^2-2k+2n+1)}, \frac{2k^2}{1+x}\right]$$

(r_1, r_2)

Meaning:

$$\begin{cases} (n+1)f_{n+1,k} + (1-x^2)f'_{n,k} - x(n+1)f_{n,k} & = \Delta_k(r_1 f_{n,k}), \\ (x^2-1)f''_{n,k} + 2xf'_{n,k} - n(n+1)f_{n,k} & = \Delta_k(r_2 f_{n,k})\,. \end{cases}$$

rhs telescope by summation

Example of an Integral

$$\int_{-1}^{1} \frac{\mathrm{e}^{-\mathrm{px}} \mathrm{T}_{\mathrm{n}}(\mathrm{x})}{\sqrt{1-\mathrm{x}^2}} \mathrm{dx} = (-1)^{\mathrm{n}} \pi \mathrm{I}_{\mathrm{n}}(\mathrm{p})$$

```
> f:=exp(-p*x)*ChebyshevT(n,x)/sqrt(1-x^2);
```

$$f := \frac{e^{-px}\,\mathrm{ChebyshevT}(n,x)}{\sqrt{1-x^2}}$$

```
> CreativeTelescoping(Int(f,x=-1..1),[n::shift,p::diff]);
```

$$[pD_n + pD_p - n, pD_n^2 - 2nD_n - 2D_n - p]$$

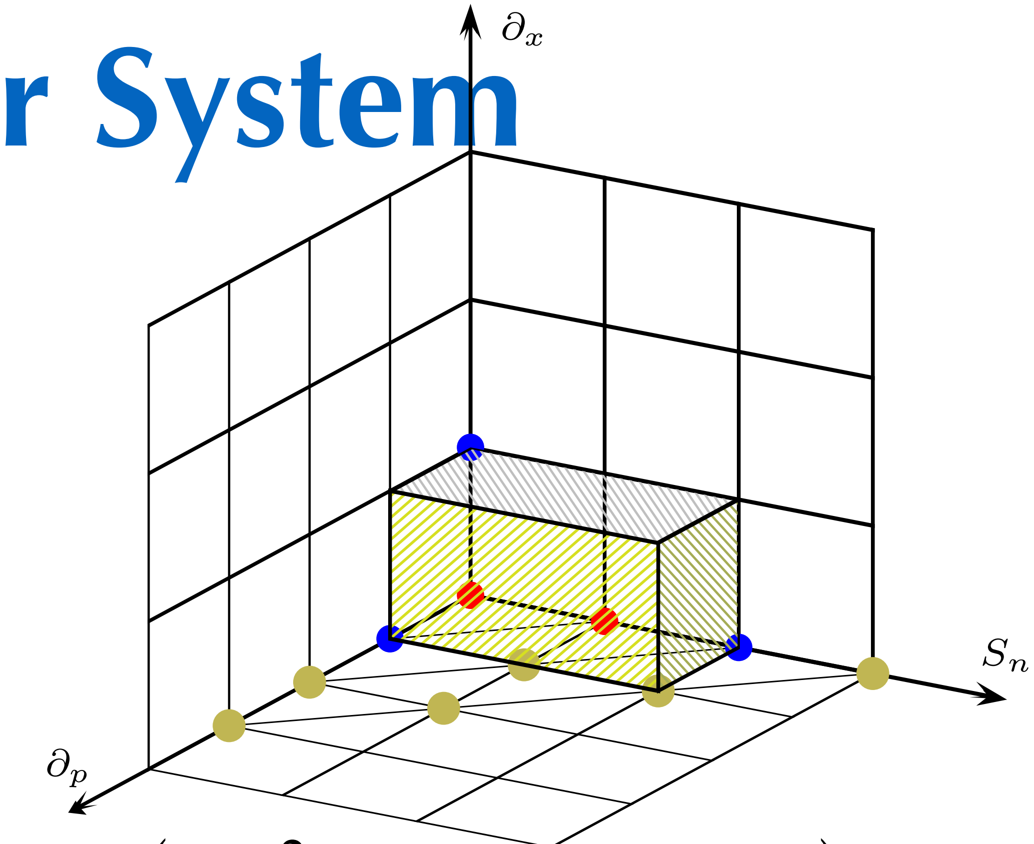
Implying: the integral $F_n(p)$ satisfies

Deformation of the contour gets rid of the certificate

$$pF_{n+1} + pF_n' - nF_n = 0, \quad pF_{n+2} - 2(n+1)F_{n+1} - pF_n = 0$$

II. Chyzak's Generalization of Zeilberger's Algorithm

From CT to Linear System



$$\int_{-1}^{1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

f

Ann f generated by the operators

$$\partial_p + x1, \quad S_n^2 - 2xS_n + 1, \quad (x^2-1)\partial_x - nS_n + \left(p(x^2-1) + (n+1)x\right)1$$

Undetermined coefficients

certificate

$$\text{telescoper} \quad \sum_{(k,m)} t_{k,m}(n,p)\partial_p^k S_n^m - \partial_x\left(c_0(n,p,x) + c_1(n,p,x)S_n\right) = 0 \bmod \operatorname{Ann} f$$

Reduces to

$$\begin{cases} \dfrac{\partial c_0}{\partial x} - \dfrac{p(x^2-1)+(n+1)x}{x^2-1}c_0 - \dfrac{n+1}{x^2-1}c_1 = \sum_{(k,m)} t_{k,m}u_{k,m}^{(0)}, \\[2ex] \dfrac{\partial c_1}{\partial x} + \dfrac{n}{x^2-1}c_0 + \dfrac{nx - p(x^2-1)}{x^2-1}c_1 = \sum_{(k,m)} t_{k,m}u_{k,m}^{(1)}. \end{cases}$$

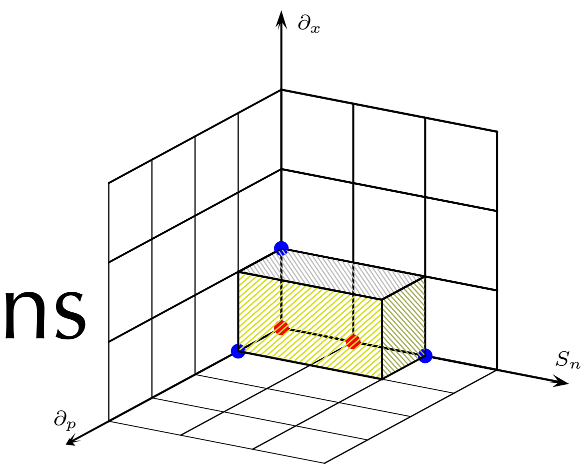
Reduce and extract coeffs of 1 and S_n

$\partial_p^k S_n^m$ reduces to $u_{k,m}^{(0)}1 + u_{k,m}^{(1)}S_n$

Look for $t_{k,m}$ s.t. a rational solution exists

Increase support until a soln is found

Solve Linear System



Ex: Telescoper in $1, S_n, \partial_p$? Starting point: reductions

$$1 \mapsto 1, \quad S_n \mapsto S_n, \quad \partial_p \mapsto -x$$

System: $$\begin{cases} \dfrac{\partial c_0}{\partial x} - \dfrac{p(x^2-1)+(n+1)x}{x^2-1}c_0 - \dfrac{n+1}{x^2-1}c_1 = t_{0,0} - xt_{1,0}, \\ \dfrac{\partial c_1}{\partial x} + \dfrac{n}{x^2-1}c_0 + \dfrac{nx - p(x^2-1)}{x^2-1}c_1 = t_{0,1}\,. \end{cases}$$

Unknown:
c_0, c_1 in $\mathbb{Q}(n,p,x)$
$t_{i,j} \in \mathbb{Q}(n,p)$

1. Uncoupling leads to:

A 1st order system of dim n can always be uncoupled to an equation of order $\leq n$

$$(x^2-1)c_0'' + (x+2p-2px^2)c_0' + (p^2(x^2-1) - px - (n+1)^2)c_0 =$$
$$(px^3 - (n+3)x^2 - px + 1)t_{1,0} - (px^2 - (n+2)x - p)t_{0,0} + (n+1)t_{0,1}$$

2. Indicial equation at ± 1 : $\alpha(\alpha - 1/2) = 0 \Rightarrow$ denominator wrt $x = 1$

3. Bound on the degree: 1

4. Linear system in the coeffs of c_0 and $t_{0,0}, t_{1,0}, t_{0,1}$ gives

$$p\partial_p + pS_n - n - \partial_x(x - S_n) \in \operatorname{Ann} f$$

Chyzak's Algorithm (2000)

Algorithm CreativeTelescoping
Input: a Gröbner basis G of a D-finite ideal $I \subset \mathbb{O}_r = \mathbb{K}(x_1, \ldots, x_r)\langle D_1, \ldots, D_r \rangle$
a set M of monomials in $D_1, \ldots, D_{r-1}$
Output: a telescoper $\sum_{m \in M} t_m m \in I + \tilde{D}_r \mathbb{O}_r$ with $t_m \in \mathbb{K}(x_1, \ldots, x_{r-1})$ if one exists

// $Q := (Q_1, \ldots, Q_n)$ *is a basis of* $\mathbb{O}_r / I$ *(obtained from* G*)*
1. Compute a matrix A s.t. $\tilde{D}_r Q = AQ \bmod I$ (by reduction)
2. Compute a matrix B s.t. $M = BQ \bmod I$
3. Setup the system $\partial_{x_r}(C) + CA = BT$ (differential case)
   or $S_{x_r}(C)A - C = BT$ (shift case)
4. Find its rational solutions // *e.g., by uncoupling and parameterized Liouville/Abramov*
5. Return it if it is nonzero, FAIL otherwise

Increase support M until a soln is found

Almkvist-Zeilberger (1990): $r = 2, n = 1$, differential
Zeilberger (1990): $r = 2, n = 1$, shift

Koutschan's Heuristic (2010)

// $Q := (Q_1, \ldots, Q_n)$ is a basis of $\mathbb{O}_r/I$ (obtained from G)

1. Compute a matrix A s.t. $D_r Q = AQ \bmod I$ (by reduction)
2. Compute a matrix B s.t. $M = BQ \bmod I$
3. Setup the system $D_r(C) + AC = BT$ (differential case)

   or $AD_r(C) - C = BT$ (shift case)
4. Find its rational solutions *// e.g., by uncoupling and parameterized Liouville/Abramov*
5. Return it if it is nonzero, FAIL otherwise

Heuristic: a multiple of the denominator is predicted from the leading terms of the Gröbner basis G.

Very efficient in practice.
Does not guarantee minimality of the telescoper

Room for Improvement (1): Repeated Computations

// *$Q := (Q_1, \ldots, Q_n)$ is a basis of $\mathbb{O}_r/I$ (obtained from G)*

1. Compute a matrix A s.t. $D_r Q = AQ \bmod I$ (by reduction)
2. Compute a matrix B s.t. $M = BQ \bmod I$
3. Setup the system $D_r(C) + AC = BT$ (differential case)
   or $AD_r(C) - C = BT$ (shift case)
4. Find its rational solutions // *e.g., by uncoupling and parameterized Liouville/Abramov*
5. Return it if it is nonzero, FAIL otherwise

Only B and T depend on M

Increase support M until a soln is found

The homogeneous part of the system does not depend on M

Reduction-based creative telescoping avoids some of this repetition

Room for Improvement (2): Certificates are Big

$$C_n := \sum_{r,s} \underbrace{(-1)^{n+r+s}\binom{n}{r}\binom{n}{s}\binom{n+s}{s}\binom{n+r}{r}\binom{2n-r-s}{n}}_{f_{n,r,s}}$$

$$(n+2)^3 C_{n+2} - 2(2n+3)(3n^2+9n+7)C_{n+1} - (4n+3)(4n+4)(4n+5)C_n$$
$$= \Delta_r(\cdots) + \Delta_s(\cdots) = 180\text{ kB} \simeq 2\text{ pages}$$

$$I(z) = \oint \frac{(1+t_3)^2 dt_1 dt_2 dt_3}{t_1t_2t_3(1+t_3(1+t_1))(1+t_3(1+t_2)) + z(1+t_1)(1+t_2)(1+t_3)^4}$$

$$z^2(4z+1)(16z-1)I'''(z) + 3z(128z^2+18z-1)I''(z) + (444z^2+40z-1)I'(z) + 2(30z+1)I(z)$$
$$= \frac{d}{dt_1}(\cdots) + \frac{d}{dt_2}(\cdots) + \frac{d}{dt_3}(\cdots) = 1\,080\text{ kB} \simeq 12\text{ pages}$$

and sometimes also unnecessary

Reduction-based creative telescoping can avoid some of this computation

Test-Set

$$\int J_{m+n}(2tx)T_{m-n}(x)\frac{dx}{\sqrt{1-x^2}},$$

$$\int \frac{n^2+x+1}{n^2+1}\left(\frac{(x+1)^2}{(x-4)(x-3)^2(x^2-5)^3}\right)^n e^{\frac{x^3+1}{x(x-3)(x-4)^2}}\sqrt{x^2-5}\,dx,$$

$$\int C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,dx,$$

$$\int x^\ell C_m^{(\mu)}(x)C_n^{(\nu)}(x)(1-x^2)^{\nu-1/2}\,dx,$$

$$\int (a+x)^{\gamma+\lambda-1}(a-x)^{\beta-1}C_m^{(\gamma)}(x/a)C_n^{(\lambda)}(x/a)\,dx,$$

$$\int C_n^{(\lambda)}(x)C_m^{(\lambda)}(x)C_\ell^{(\lambda)}(x)(1-x^2)^{\lambda-1/2}\,dx,$$

$$\int xJ_1(ax)I_1(ax)Y_0(x)K_0(x)\,dx$$

$$\sum_k \binom{n}{k}^2\binom{n+k}{k}^2,$$

$$\sum_n J_n^2(x),$$

$$\sum_n \frac{J_{2n+1/2}(x)}{\sqrt{x}}P_{2n}(u)\frac{(4n+1)(2n)!}{2^{2n}n!^2},$$

$$\sum_n C_n^{(k)}(x)C_n^{(k)}(y)\frac{u^n}{n!},$$

$$\sum_n J_n(x)C_n^{(k)}(y)\frac{u^n}{n!},$$

$$\sum_k \frac{(a+b+1)_k}{(a+1)_k(b+1)_k}P_k^{(a,b)}(x)P_k^{(a,b)}(y),$$

$$\sum_k \frac{(a+b+1)_k k!}{(a+1)_k(b+1)_k}P_k^{(a,b)}(x)P_k^{(a,b)}(y)t^k.$$

Timings

Integrals	Chyzak's algo.	Reduction-Based	Koutschan's heuristic
1	10 s.	14 s.	1.9 s.
2	> 1 h	1.2 s.	> 1 h
3	355 s.	1.5 s.	2.1 s.
4	> 4h	106 s.	3.4 s.
5	> 1h	45 s.	56 s.
6	245 s.	> 1h	1.7 s.
7	21 s.	> 1h	5.1 s.

Sums	Chyzak's algo.	Reduction-Based	Koutschan's heuristic
1	0.1 s.	0.1 s.	0.3 s.
2	0.2 s.	0.1 s.	0.1 s.
3	6.8 s.	13 s.	2.3 s.
4	58 s.	2.1 s.	4.9 s.
5	75 s.	7.5 s.	2.9 s.
6	> 4 h	279 s.	83 s.
7	> 4 h	6473 s.	17 s.

Koutschan's Mathematica package `HolonomicFunctions` (first & last col.)
Our new Maple package `CreativeTelescoping` (2nd col.)

Timings on a Mac mini 2018, with 32 Gb mem.

III. Reduction-Based Creative Telescoping (2010—today)

A Brief History of Reduction-Based CT

$$\int f(x,y)\,dx,\ f \text{ rational}$$

Bostan, Chen, Chyzak, Li (2010)

multiple integrals
n vars, rational

Bostan, Lairez, S. (2013–2016)

bivariate
dim = 1

Bostan, Chen, Chyzak, Dumont, Huang, Kauers, Li, S., Xin (2013–2016)

this talk

single integral
n vars, finite dim

Bostan, Chyzak, van der Hoeven, Lairez, S. (2018–2021)

multiple
binomial sums
via gen. fcns.

Bostan, Lairez, S. (2017)

bivariate
integral bases

Chen, Du, van Hoeij, Kauers, Koutschan, Wang (2016–today)

Implementations available for most (all?) of them

single sum
n vars, finite dim

Brochet, S. (2023)

All build on earlier work on integration, summation, periods, differential equations…

Reduction-Based Creative Telescoping

$$T(x_1, \ldots, x_{r-1}, D_1, \ldots, D_{r-1}) - \tilde{D}_r\, C(x_1, \ldots, x_r, D_1, \ldots, D_r) \in \operatorname{Ann} f$$

telescoper
(diff,shift) under (int,sum) sign

certificate
(int,sum) by parts

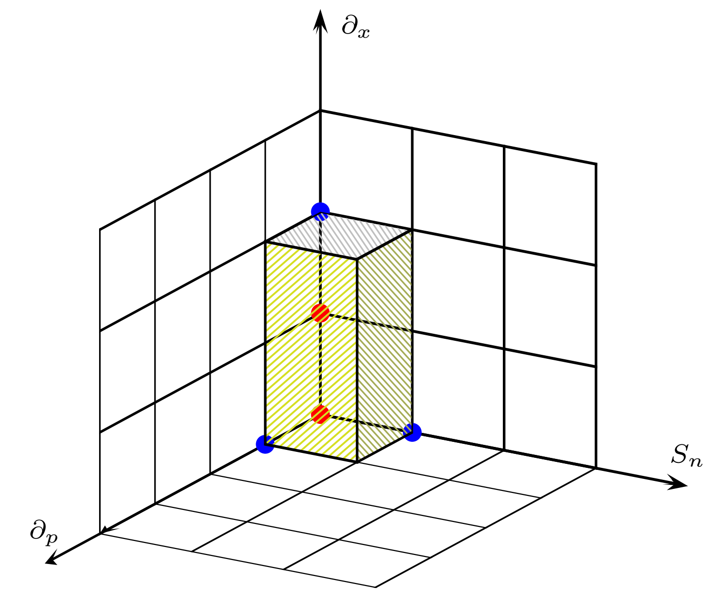
$\tilde{D}_r$ is a linear map in $\mathbb{K}(x_1, \ldots, x_r)\langle \partial_1, \ldots, \partial_r \rangle / \operatorname{Ann} f$

Principle:

Reduce successive monomials in $D_1, \ldots, D_{r-1}$ modulo the image of $\tilde{D}_r$ until a linear dependency is found between the reductions

Motivation: 1) save on the repeated computations
2) save on the certificate

Example



$$\int_{-1}^{1} \frac{e^{-px}T_n(x)}{\sqrt{1-x^2}}\,dx = (-1)^n \pi I_n(p)$$

Ann f generated by

$$\partial_p + x1, \qquad -nS_n + (x^2-1)\partial_x + (p(x^2-1)+(n+1)x),$$

$$(x^2-1)\partial_x^2 + (2px^2-2p+3x)\partial_x + (p^2x^2-n^2-p^2+3px+1)1$$

Modulo derivatives in x,

$$\partial_p f \equiv -xf$$

$$nS_n f \equiv (p(x^2-1)+(n+1)x)f - 2xf + \cancel{\partial_x((x^2-1)f)}$$

$$(p^2x^2-n^2-p^2+3px+1)f - (4xp+3)f + 2f \equiv 0$$

$$\partial_p^2 f \equiv x^2 f$$

Combinations of f, xf only

Conclusion: the integral $F_n(p)$ satisfies

$$F_n' + F_{n+1} = \frac{n}{p}F_n \qquad\qquad p^2F_n'' + pF_n' = (n^2+p^2)F_n$$

Working mod $\tilde{D}$ on the left in 1 variable

$$L = c_s D^s + \cdots + c_0$$

Left division by $\tilde{D}$: $uL = L^\star(u) + \tilde{D}\, P_L(u)$

Lagrange's identity
$\equiv$ repeated integration/
summation by parts

Adjoint of L: $L^\star = \begin{cases} c_0 + \cdots + (-\partial)^s c_s & \text{(differential)}, \\ c_0 + \cdots + S_n^{-s} c_s & \text{(shift)}. \end{cases}$

Applications:

(1). $\forall M, \quad uM(f) = M^\star(u) f + \tilde{D}(\cdots)$

explicit
rational fcn

(2). $L(f) = 0 \Rightarrow \forall u, \quad L^\star(u) f = \tilde{D}(\cdots)$

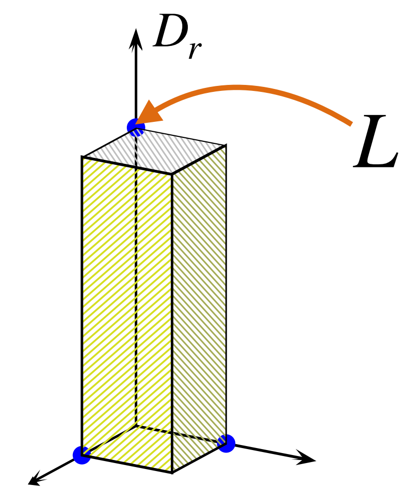
converse

(3). (2) & L minimal, $vf = \tilde{D}M(f) \Rightarrow v \in L^\star(\mathbb{K}(x))$

details
later

Computation reduced to rational fcns x f and working modulo $\operatorname{Im} L^\star$

From r variables to 1 variable by cyclic vectors



If $(1, D_r, \ldots, D_r^{s-1})$ is a basis of $\mathbb{O}_r/\operatorname{Ann} f$, then

1. $L(f) = 0, \quad$ with $\quad L = c_s D_r^s + \cdots + c_0$
2. For $i = 1, \ldots, r-1$, $D_i =: B_i(D_r)$

one can always reduce to this situation in practice

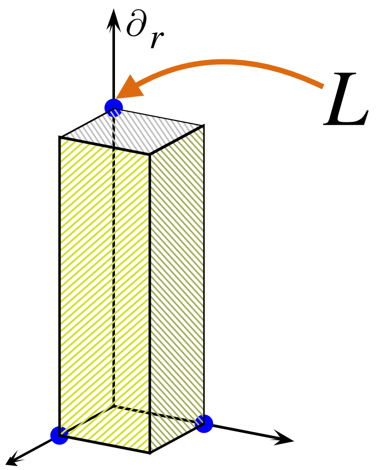
Prop. For $u \in \mathbb{K}(x_1, \ldots, x_r)$,

1. $\partial_{x_i}(uf) = \left(\dfrac{d}{dx_i}u + B_i^\star(u)\right)f, \quad S_{x_i}(uf) = B_i^\star(u(x_i+1))f$
2. $uf \in \tilde{D}_r(\mathbb{O}_r/\operatorname{Ann} f)(f) \Leftrightarrow u \in L^\star(\mathbb{K}(\underline{x}))$

Computation reduced to rational fcns x f and working modulo $\operatorname{Im} L^\star$

details later

Algorithm (simplified version)



$Q := \varnothing$ *// list of monomials already reduced*
$T := \varnothing$ *// list of telescopers*
Repeat
  $\mu :=$ NextMonomial in $D_1, \ldots, D_{r-1}$
  Compute u_μ rat fcn s.t. $\mu(f) = u_\mu f + \tilde{D}_r(\cdots)$
  Compute $F_\mu := u_\mu$ $\mathrm{mod}\ \mathrm{Im}(L^\star)$
  If there is a relation $F_\mu = \sum_{\nu \in Q} a_\nu F_\nu$ $\left(a_\nu \in \mathbb{K}(x_1, \ldots, x_{r-1})\right)$
  then $T := T \cup \{\partial_\mu - \sum a_\nu D_\nu\}$
  else $Q := Q \cup \{\mu\}$
Until user satisfied
Return T

next part

Linear algebra over polynomial matrices

IV. Univariate Reduction

Reductions of Rational Functions

Hermite reduction: $f = g + \partial_t h, \qquad f, g, h$ in $\mathbb{K}(t)$
$g = 0$ iff f is a derivative
g does not have multiple poles

Abramov reduction: $f = g + \Delta_t h, \qquad f, g, h$ in $\mathbb{K}(t)$
$g = 0$ iff f is a difference
poles of g do not differ by an integer

Variant with f, g, h hypergeometric by Abramov-Petkovšek

Goal: $f = g + L^\star(h), \qquad f, g, h$ in $\mathbb{K}(t)$
$g = 0$ iff $f \in \operatorname{Im} L^\star$
g minimal in some sense

Differential Case – Example

$$M = (1-x)^2\,\partial_x^2 + (1-x^2)\partial_x - 2(x^2+3x+1)$$

To be reduced: $F = x^2 + 5x + 9 + \dfrac{10}{x-1}$ mod Im M

Method: reduce poles by decreasing order, add special cases, reduce polynomial part

$$M\left(\frac{1}{x-1}\right) = -2x - 7 - \frac{6}{x-1} =: f_1$$

$$F + \frac{5}{3}f_1 = x^2 + \frac{5}{3}x - \frac{8}{3}$$

$$M(1) = -2(x^2+3x+1) =: f_2$$

$$+\frac{1}{2}f_2 = -\frac{4}{3}x - \frac{11}{3}$$

More reduction is possible: $M((x-1)^s) \underset{x\to 1}{\sim} (s+2)(s-5)(x-1)^s$

$$M\left(\frac{1}{(x-1)^2}\right) = -2 - \frac{8}{x-1} := f_3$$

$$-\frac{2}{3}f_3 + \frac{1}{2}f_1 = 0$$

Conclusion: $F \in M\big(\mathbb{Q}(x)\big)$

Generalized Hermite Reduction

$$M = c_m(x)\partial_x^m + \cdots + c_0(x)$$

Local analysis: for $\alpha \in \overline{\mathbb{K}}, s \in \mathbb{Z}$, indicial polynomial at α

$$M((x-\alpha)^s) = \mathrm{ind}_\alpha(s)(x-\alpha)^{s+\sigma_\alpha}(1+o(1)), \quad x \to \alpha\,.$$

$$M(x^s) = \mathrm{ind}_\infty(s)x^{s+\sigma_\infty}(1+o(1)), \quad x \to \infty\,.$$

1st step: weak reduction of $f = \dfrac{f_k}{(x-\alpha)^k}(1 + O(x-\alpha))$:

$$H_\alpha(f) := \begin{cases} H_\alpha\left(f - \frac{f_k}{\mathrm{ind}_\alpha(-k-\sigma_\alpha)} M((x-\alpha)^{-k-\sigma_\alpha})\right) & \text{if } \mathrm{ind}_\alpha(-k-\sigma_\alpha) \neq 0, \\ \frac{f_k}{(x-\alpha)^k} + H_\alpha\left(f - \frac{f_k}{(x-\alpha)^k}\right) & \text{otherwise.} \end{cases}$$

Similar $H_\infty(f)$

2nd step: also use those that have been skipped, ie,

$$H_\alpha(M((x-\alpha)^{-k})), \quad c_m(\alpha) = 0 \text{ and } \mathrm{ind}_\alpha(-k) = 0 \text{ or } 0 < k \leq \sigma_\alpha\,.$$

Plus analogous set at ∞

$$\boxed{\text{Prop. } f \mapsto g + M(h), \quad g = 0 \Leftrightarrow f \in \operatorname{Im} M\,.}$$

Reduction in the Recurrence Case

$$M = c_0(n) + \cdots + c_m(n) S_n^{-m}$$

maximal integer difference

Method:

1. reduce dispersion of the poles to at most $m - 1$;
2. reduce by special cases coming from the roots of c_0, c_m
3. reduce polynomial part

$$\boxed{\text{Prop. } f \mapsto g + M(h), \quad g = 0 \Leftrightarrow f \in \operatorname{Im} M\,.}$$

Demo & a Word on Certificates

If we have 5 min. left

Conclusions

1. Complete algorithms for D-finite integration & summation
2. Implementation available in Maple

   https://github.com/HBrochet/CreativeTelescoping

3. Certificates can be computed in a compact way
4. Efficiency can be improved further:

   . apparent singularities play a role, to be understood
   . need to save computation in the reductions
   . intermediate reductions can be too large

The End