Differential elimination ideals and spectral curves

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Recent Trends in Computer Algebra 2023
Elimination for Functional Equations
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I will present recent and ongoing joint work with M.A. Zurro

Algorithmic Differential Algebra and Integrability (ADAI)

Contents

The theory of commuting ODOs
- Algebro-geometric Schrödinger operators
- Centralizers
- Computing Commuting Operators

Burchnall-Chaundy ideal of a pair

Differential resultant of two ODOs

Elimination ideals

Factorization
- BC ideal of an operator
- Coefficient field for factorization

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The theory of commuting ODOs

[Zheglov, 2020] The theory of commuting ODOs has broad connections with many branches of modern mathematics:

- Non-linear partial differential equations (find new exact solutions).
- Algebra (the Dixmier or Jacobian or Poisson conjectures, highly non-trivial and still open).
- Complex analysis. Deformation quantisation. ...

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

Korteweg-de Vries equation modeled the solitary waves (solitons) in shallow water.

$\Updownarrow$

COMMUTING ODOs $\rightleftarrows$ ALGEBRAIC CURVES

[Burchnall-Chaundy 1923] Baker, Krichever, Mumford ...

Spectral problem

Schrödinger equation

$$\Psi_{xx} - u(x)\Psi = \lambda\Psi \tag{1}$$

with $u(x)$ satisfying a Korteweg de Vries (KdV) equation of the celebrated KdV hierarchy. For instance, the classical stationary KdV equation

$$u_{xxx} - 6uu_x = 0.$$

λ spectral parameter

(Drach's Ideology, 1919) Brehznev 2008, 2012, 2013.
Integrate (1) as an ODE to obtain a parametric solution $\Psi(x;\lambda)$

(Σ, ∂) ordinary differential field
field of constants $C = \overline{C}$, characteristic 0.

$u \in \Sigma$

λ algebraic variable over C, $\partial(\lambda) = 0$

Algebro-geometric Schrödinger operators

(with Morales-Ruiz, Zurro)
Given $u_s \in \Sigma$ and the Schrödinger operator $L_s = -\partial^2 + u_s$.

Using [Goodearl, 1983]

(MRZ 2021) The following are equivalent.

1. $\exists!$ monic operator A_{2s+1} of minimal order $2s+1$ such that

   $$\mathcal{C}(L_s) = C[L_s, A_{2s+1}] = \{p_0(L_s) + p_1(L_s)A_{2s+1} \mid p_0, p_1 \in C[L_s]\}$$

   and $A_{2s+1}^2 + R_{2s+1}(L_s) = f_s(L_s, A_{2s+1}) = 0$, with $f_s \in C[\lambda, \mu]$.
2. u_s is a *KdV*-potential of KdV level s.

L_s is called **algebro-geometric.**

Stationary KdV hierarchy

$\{kdv_n(u)\}_{n\geq 1}$ differential polynomials in $C\{u\} = C[u, u', u'', \ldots]$

$$kdv_0 := u', \ \ kdv_n := \mathcal{R}(kdv_{n-1}), \ \text{for } n \geq 1.$$

Recursion operator $\mathcal{R} = -\frac{1}{4}\partial^2 + u + \frac{1}{2}u'\partial^{-1}$

$$kdv_1 = -\frac{1}{4}u''' + \frac{3}{2}uu', \ \ kdv_2 = \frac{1}{16}u^{(5)} - \frac{5}{8}uu''' - \frac{5}{4}u'u'' + \frac{15}{8}u^2u'$$

Conditions on u for the existence of A_{2n+1} commuting with $-\partial^2 + u$

$$[P_{2n+1}, -\partial^2 + u] = kdv_n + c_1 kdv_{n-1} + \cdots + c_n kdv_0.$$

KdV solitons

Families of solutions of $\mathrm{KdV}_s(u, \bar{c}^s) = 0$.

Rational	Rosen-Morse	Elliptic
$u_s = \frac{s(s+1)}{x^2}$	$u_s = \frac{-s(s+1)}{\cosh^2(x)}$	$u_s = s(s+1)\wp(x; g_2, g_3)$

[Veselov, A.P., 2011. **On Darboux-Treibich-Verdier Potentials**. Letters in Mathematical Physics, 96(1), 209-216.]

Centralizers

Schur, Flanders, Krichever, Amitsur, Carlson, Ore....

Given $L \in \mathcal{D} = \Sigma[\partial]$

$$\mathcal{Z}(L) = \{A \in \Sigma[\partial] \mid [L, A] = 0\}$$

[Goodearl, 1983]

Σ differential field $\Rightarrow$ $\mathcal{Z}(L)$ commutative domain.

- Trivial

$$\mathcal{Z}(L) = C[L] = \left\{ \sum_{i=1}^{s} a_i L^i \mid a_i \in C \right\}$$

- Non-trivial

$\mathcal{Z}(L)$ is a free $C[L]$-module
the cardinal of a basis divides $\mathrm{ord}(L)$.

True rank pairs

(with Previato, Zurro)
Ring of differential operators $\mathcal{D} = \Sigma[\partial]$. The rank of a subset $\mathcal{S} \subseteq \mathcal{D}$ is

$$\text{rk}\,(\mathcal{S}) = \text{gcd}\{\mathrm{ord}(L) \mid L \in \mathcal{S}\}.$$

Given a pair $P, Q \in \mathcal{D}$ then $\text{rk}(P, Q) \leq \text{rk}\, C[P, Q]$.

$$C[P, Q] = \left\{ \sum_{i,j} \sigma_{i,j} P^i Q^j \mid \sigma_{i,j} \in C \right\}.$$

P and Q is a true rank pair if equality holds, a fake rank pair otherwise.

Rank of L (PRZ 2019)

Maximal commutative subalgebras in $\Sigma[\partial]$ are centralizer.

If $\mathcal{Z}(L) \neq C[L]$, we define the rank of L to be $\operatorname{rk}\mathcal{Z}(L)$.

Given $P, Q \in \mathcal{Z}(L)$ then

$$C[P, Q] \subseteq \mathcal{Z}(L).$$

If $\mathcal{Z}(L) = C[L, B]$ we call L, B a Burchnall-Chaundy (BC) pair.

If L, B is a BC pair then L, B is a true rank pair

$$\operatorname{rk}\mathcal{Z}(L) = \operatorname{rk} C[L, B] = \operatorname{rk}(L, B)$$

More than Algebro-geometric ODOs

Given

$$f(\lambda, \mu) = \mu^2 - \lambda^3$$

is the BC polynomial of

- Rank 1 pair,

$$L_2 = -\partial^2 + \frac{2}{x^2} \quad \text{and} \quad P_3 = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3},$$

$$P_3^2 - L_2^3 = 0$$

- Rank 2 pair, [Dixmier, 1968].

$$L_4 = H^2 + 2x \quad \text{and} \quad P_6 = H^2 + \frac{3}{2}(xH + Hx), \quad \text{with} \quad H = \partial^2 + x^2.$$

$$P_6^2 - L_4^3 = 0$$

$\Sigma = C(x)$ with $\partial = d/dx$

Centralizer $\mathrm{ord}(L) = p$ prime

Using [Goodearl, 1983]

$\mathcal{Z}(L) = C[L]$ or
$\mathcal{Z}(L)$ equals the free $C[L]$-module with basis $\{1, A_1, \ldots, A_{p-1}\}$,

$$\mathcal{Z}(L) = \{q_0(L) + q_1(L)A_1 + \cdots + q_{p-1}(L)A_{p-1} \mid q_i \in C[L]\}$$

with A_i of minimal order $\equiv i (mod\ p)$

Computing centralizers

Given $L_n \in \Sigma[\partial]$ of order n.

If n prime then rk $\mathcal{Z}(L) = 1$

- (MRZ 2020,2021) $\mathcal{Z}(L_2) = C[L_2, A_1]$, with $\mathrm{ord}(A_1) = 2s + 1$.
- (RZ 2021) $\mathcal{Z}(L_3) = C[L_3, A_1, A_2]$,
  In some cases $\mathcal{Z}(L_3) = C[L_3, A_i]$, for instance if $\mathrm{ord}(A_2) = 2$.

If n not prime then rk $\mathcal{Z}(L) \geq 1$

- (PRZ 2019) $\mathcal{Z}(L_4) = C[L_4, A_2]$, with $\mathrm{ord}(A_2) = 4g + 2$.
  with L_4 in the first Weyl algebra.

Computing Commuting Operators

L algebro-geometric $\Leftrightarrow \mathcal{Z}(L) \neq C[L]$

$L = \partial^n + u_{n-2}\partial^{n-2} \cdots + u_1\partial + u_0$ in $\Sigma[\partial]$

- $n = 2$: u_0 solutions of KdV hierarchy
- $n = 3$: u_0, u_1 solutions of Boussinesq (systems) hierarchy
- $n = 4$: u_0, u_1, u_2 solutions of Krichever-Novikov (KN) hierarchy
- ...
- $u_0, u_1, \ldots, u_{n-2}$ solutions of the Gelfand-Dikii hierarchies.

Centralizers in ring of pseudo-differential operators

Commutative ring of differential operators (R, ∂), whose ring of constants is a field of zero characteristic C

$$R((\partial^{-1})) = \left\{ \sum_{i=-\infty}^{n} a_i \partial^i \mid a_i \in R, n \in \mathbb{Z} \right\}$$

$L \in R[\partial]$, centralizer in the ring of pseudo-differential operators

$$\mathcal{Z}(L) \subset \mathcal{Z}((L)) = \{A \in R((\partial^{-1})) \mid [L, A] = 0\}$$

$\operatorname{ord}(L) = n$, $\exists!$ monic pseudo-differential operator $Q = L^{1/n}$.
Generalized Schur's Theorem [Goodearl, 1983]

$$\mathcal{Z}((L)) = \left\{ \sum_{j=-\infty}^{m} c_j Q^j \mid c_j \in C, m \in \mathbb{Z} \right\}$$

Almost commuting basis

(with Jimenez-Pastor, Delgado, Hernandez-Heredero, Zurro)

Given $A = \sum_{i=-\infty}^{n} a_i \partial^i$ in $R((\partial^{-1}))$ then $A_+ = \sum_{i=0}^{n} a_i \partial^i$.

Based on [Wilson 1985]

$L \in R[\partial]$, $\mathrm{ord}(L) = n$,

$\mathcal{Z}((L))_+ := \{A_+ \mid A \in \mathcal{Z}((L))\} = \{B \in R[\partial] \mid \mathrm{ord}([L, B]) \leq n - 2\}$.

C-vector space of almost commuting operators with basis

$$\mathcal{B}(L) := \{P_m := (Q^m)_+ \mid m \in \mathbb{N}, Q = L^{1/n}\}$$

Formal differential operators

$U=\{u_0,\dots,u_{n-2}\}$ differential variables over C.
Formal differential operator

$$L=\partial^n+u_{n-2}\partial^{n-2}\cdots+u_1\partial+u_0\in C\{U\}[\partial]$$

Linear algorithm based on assigning weights to U:

- **Almost commuting basis** of homogeneous operators

$$\{P_m:=(Q^m)_+ \mid M\geq m\geq 0, Q=L^{1/n}\}.$$

- **Hierarchy** $H_{m,j}\in C\{U\}$

$$[L,P_m]=H_{m,0}+H_{m,1}\partial+\ldots+H_{m,n-2}\partial^{n-2}.$$

Implementation in SAGE. Workstation MOUNTAIN, 1TB Ram.

Spectral problem

(Σ, ∂) ordinary differential field
field of constants $C = \overline{C}$, characteristic 0.

Given

$$L \text{ in } \Sigma[\partial] \backslash C[\partial]$$

assuming

NON-TRIVIAL CENTRALIZER $\mathcal{Z}(L)$

Integrate to obtain a parametric solution $\Psi(x; \lambda, \mu)$

$$L(\Psi) = \lambda\Psi, \quad B(\Psi) = \mu\Psi$$

for $B \in \mathcal{Z}(L)$.

The theory of commuting ODOs

Non-linear differential equations (KdV, Boussinesq, KN...KP)

$\Updownarrow$

COMMUTING ODOs $\rightleftarrows$ ALGEBRAIC CURVES

[Burchnall-Chaundy 1923] Baker, Krichever, Mumford ...

DIRECT PROBLEM $\longrightarrow$

Implicitization

Inverse problem $\longleftarrow$

Parametrization

Beret's conjecture [Guo, Zheglov 2022].

ADAI Goals

`Algorithmic Differential Algebra and Integrability (ADAI)`

Develop Picard-Vessiot (PV) theory for spectral problems.
Use effective differential algebra to develop symbolic algorithms:

- Parametric factorization of algebro-geometric ODOs.
- Compute integrable hierarchies and almost commuting basis.
- Compute new algebro-geometric ODOs, order ≥ 3.

(with M.A. Zurro)

Computing defining ideals of space spectral curves for algebro-geometric third order ODOs. arXiv:2311.09988, 2023.

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BC Ideal of a pair

Commuting P and Q in $\Sigma[\partial]$

$$e_{P,Q} : C[\lambda, \mu] \rightarrow \Sigma[\partial]$$

homomorphism of C-algebras defined by

$$g(P, Q) := e_{P,Q}(g) = e_{P,Q}(\sigma_{i,j}\lambda^i\mu^j) = \sigma_{i,j}P^iQ^j.$$

Define the Burchnall-Chaundy BC ideal of the pair P and Q as

$$\mathrm{BC}(P, Q) := \mathrm{Ker}(e_{P,Q}) = \{g \in C[\lambda, \mu] \mid g(P, Q) = 0\}.$$

Its elements are BC polynomials

Spectral curve of a pair

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$

$\mathcal{Z}(P)$ finitely generated $C[P]$-module $\Rightarrow \mathtt{BC}(P,Q)$ non zero ideal.

$\Sigma[\partial]$ domain $\Rightarrow \mathtt{BC}(P,Q)$ prime ideal

Spectral curve:

$$\Gamma_{P,Q} := V(\mathtt{BC}(P,Q))$$

Coordinate ring of $\Gamma_{P,Q}$

$$\frac{C[\lambda,\mu]}{\mathtt{BC}(P,Q)} \simeq C[P,Q].$$

Spectral curve of a pair

There exists an irreducible polynomial $f \in C[\lambda, \mu]$ such that

$$\mathtt{BC}(P, Q) = (f)$$

$$\Gamma_{P,Q} = \{ (\lambda_0, \mu_0) \in C^2 \mid f(\lambda_0, \mu_0) = 0 \} .$$

How do we compute f?

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Differential resultant of 2 ODOs

Defined by Ritt (1932), Berkovich and Tsirulik (1986) and studied by Chardin (1991), Li (1998), McCallum and Winkler (2018).

$(\mathbb{D}, \partial)$ differential domain, with fraction field $\mathbb{K}$.

$P, Q \in \mathbb{D}[\partial]$, $\mathrm{ord}(P) = n$, $\mathrm{ord}(Q) = m$

$\mathcal{D} := \mathbb{K}[\partial]$ is a (left and right) Euclidean domain. $\mathbb{K}$-linear map

$$\begin{aligned} S : \mathcal{D}_n \oplus \mathcal{D}_m &\to \mathcal{D}_{n+m} \\ (C, D) &\mapsto CP + DQ. \end{aligned}$$

Fix $\mathbb{K}$-basis $\{\partial^\ell, \ldots, \partial, 1\}$ for $\mathcal{D}_\ell$

Differential resultant of 2 ODOs

Sylvester matrix $S(P,Q)$, coefficient matrix of

$$\{\partial^{m-1}P,\dots,\partial P,P,\partial^{n-1}Q,\dots,\partial Q,Q\},$$

squared matrix of size $n+m$ and entries in $\mathbb{D}$.

Differential (Sylvester) resultant of P and Q,

$$\partial\mathrm{Res}(P,Q) := \det(S(P,Q)) = C_0P + D_0Q \in \mathrm{Im}(S)\cap\mathbb{D}$$

with $\mathrm{ord}(C_0) = m-1$ and $\mathrm{ord}(D_0) = n-1$.

$$\partial\mathrm{Res}(P,Q) \in (P,Q)\cap\mathbb{D}$$

$\mathrm{Im}(S) \subseteq (P,Q) = \mathbb{K}[\partial]P + \mathbb{K}[\partial]Q$.

Example

$$P = a_2\partial^2 + a_1\partial + a_0, \quad Q = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0$$

$$\partial\mathrm{Res}(P, Q) =$$

$$\begin{vmatrix} a_2 & a_1 + 2\partial(a_2) & a_0 + 2\partial(a_1) + \partial^2(a_2) & 2\partial(a_0) + \partial^2(a_1) & \partial^2(a_0) \\ 0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) & \partial(a_0) \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) & \partial(b_0) \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}$$

Differential Resultant Theorem I

Let us consider $P, Q \in \mathbb{D}[\partial] \subset \mathbb{K}[\partial]$.

If $\mathrm{gcrd}(P, Q) \in \mathbb{K}$, we call P and Q right coprime.

The following statements are equivalent:

1. $\partial\mathrm{Res}(P, Q) \neq 0$.
2. $\mathrm{Im}(S) \cap \mathbb{D} \neq 0$.
3. P and Q are right coprime in $\mathbb{K}[\partial]$.

If $\partial\mathrm{Res}(P, Q) \neq 0$ then the elimination ideal $(P, Q) \cap \mathbb{D}$ is nonzero.

Poisson's Formula

$(\mathbb{K}, \partial)$ differential field, field of constants $C = \overline{C}$ and of zero characteristic.

$P, Q \in \mathbb{D}[\partial] \subset \mathbb{K}[\partial]$.
Picard-Vessiot extensions $(\mathcal{E}_P, \partial_P)$ and $(\mathcal{E}_Q, \partial_Q)$ of $\mathbb{K}$ for $P(y) = 0$ and $Q(y) = 0$ resp., whose field of constants is C.

$\mathrm{ord}(P) = n$, $\mathrm{ord}(Q) = m$, leading coefficients a_n and b_m.
Given fundamental systems of solutions $\psi_1, \ldots, \psi_n$ of $P(y) = 0$ in $\mathcal{E}_P$ and $\phi_1, \ldots, \phi_m$ of $Q(y) = 0$ in $\mathcal{E}_Q$ then

$$\partial\mathrm{Res}(P, Q) = a_n^m \frac{\det W(Q(\psi_i))}{\det W(\psi_i)} = (-1)^{mn} b_m^n \frac{\det W(P(\phi_i))}{\det W(\phi_i)}.$$

Differential Resultant Theorem II

Let $\mathcal{E}$ be a Picard-Vessiot extension of $\mathbb{K}$ for $P(y) = 0$ (or $Q(y) = 0$). Then the system

$$P(y) = 0\,,\ Q(y) = 0$$

has a nontrivial solution in $\mathcal{E}$ if and only if $\partial\mathrm{Res}(P, Q) = 0$.

By Poisson's formula,

$$\partial\mathrm{Res}(P, Q) = 0 \text{ if and only if } \det(W(Q(\psi_i))) = 0$$

Equivalent to the existence of a nonzero $\psi = \sum_i c_i\psi_i$ in $V = \oplus_i C\psi_i \subset \mathcal{E}_P$ such that $Q(\psi) = 0$.

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Computing BC ideals

Given (monic) P, $Q \in \Sigma[\partial]$, then $P-\lambda$, $Q-\mu$ in $\mathbb{D}=\Sigma[\lambda,\mu]$.

$\mathrm{ord}(P)=n$, $\mathrm{ord}(Q)=m$

$$h(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) = \mu^n - \lambda^m + \ldots$$

a non trivial polynomial in $\Sigma[\lambda,\mu]$

Generalize [Wilson, 1985], [Previato, 1991].

(RZ 2023) Arbitrary (Σ,∂), $Const(\Sigma)=C=\overline{C}$.

If $[P,Q]=0$ then $h(\lambda,\mu)\in \mathtt{BC}(P,Q)$.

1. Proof by Poisson's Formula $h(\lambda,\mu)\in C[\lambda,\mu]$.
2. Proof by elimination ideals $h(P,Q)=0$.

Rosen-Morse potential $u_1 = \frac{-2}{\cosh^2(x)}$

$$L_1 = -\partial^2 + u_1,\ [L_1, A_3] = \mathrm{KdV}_0(u_1) + \mathrm{KdV}_1(u_1) = 0$$

$$f_1(\lambda,\mu) = -\mu^2 - \lambda(\lambda-1)^2 =$$
$$= \partial\mathrm{Res}(L_1-\lambda, A_3-\mu) =$$

$$\begin{vmatrix} -1 & 0 & \frac{-2}{(\cosh(x))^2}-\lambda & 8\frac{\sinh(x)}{(\cosh(x))^3} & \frac{4}{(\cosh(x))^2} - 12\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2}-\lambda & 4\frac{\sinh(x)}{(\cosh(x))^3} \\ 0 & 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2}-\lambda \\ -1 & 0 & \frac{-3}{(\cosh(x))^2}+1 & 9\frac{\sinh(x)}{(\cosh(x))^3}-\mu & \frac{3}{(\cosh(x))^2} - 9\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-3}{(\cosh(x))^2}+1 & 3\frac{\sinh(x)}{(\cosh(x))^3}-\mu \end{vmatrix}$$

Elimination ideals

Left ideal

$$(P-\lambda, Q-\mu) = \{C(P-\lambda) + D(Q-\mu) \mid C, D \in \Sigma[\lambda,\mu][\partial]\}$$

Two sided ideals

$$\mathcal{E}(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap \Sigma[\lambda,\mu].$$

and

$$\mathcal{E}_C(P-\lambda, Q-\mu) := (P-\lambda, Q-\mu) \cap C[\lambda,\mu].$$

By definition of the differential resultant

$$h(\lambda,\mu) = \partial\mathrm{Res}(P-\lambda, Q-\mu) \in \mathcal{E}_C(P-\lambda, Q-\mu).$$

Thus both elimination ideals are nonzero.

Elimination ideals

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$, both of positive order,

$$f = \sqrt{h}, \text{ with } h = \partial\mathrm{Res}(P-\lambda, Q-\mu).$$

(RZ 2023)

1. The radical of the elimination ideal $\mathcal{E}_C(P-\lambda, Q-\mu)$ equals
$$\mathrm{BC}(P,Q) = (f).$$
2. The radical of the elimination ideal $\mathcal{E}(P-\lambda, Q-\mu)$ equals $[f]$.

Recall $f \in C[\lambda,\mu]$,

$$(f) = C[\lambda,\mu]f \text{ and } [f] = \Sigma[\lambda,\mu]f \text{ differential ideal.}$$

Σ-linear evaluation map

$\Sigma[\lambda, \mu]$ as a Σ-vector space with basis $\{\lambda^i \mu^j\}$

$$\varepsilon_{P,Q} : \Sigma[\lambda, \mu] \to \Sigma[\partial], \text{ defined by}$$

$$\varepsilon_{P,Q}\left(\sum_{i,j} \sigma_{i,j} \lambda^i \mu^j\right) = \sum_{i,j} \sigma_{i,j} e_{P,Q}\left(\lambda^i \mu^j\right).$$

given $g \in \Sigma[\lambda, \mu]$ denote $g(P, Q) := \varepsilon_{P,Q}(g)$

$$\mathrm{Ker}(\varepsilon_{P,Q}) = \{g \in \Sigma[\lambda, \mu] \mid g(P, Q) = 0\}.$$

Restriction of $\varepsilon_{P,Q}$ to $C[\lambda, \mu]$ is the ring homomorphism $e_{P,Q}$, and

$$\mathtt{BC}(P, Q) = \mathrm{Ker}(e_{P,Q}) = \mathrm{Ker}(\varepsilon_{P,Q}) \cap C[\lambda, \mu].$$

Σ-linear evaluation map

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$, both of positive order

$$\mathcal{E}(P-\lambda, Q-\mu) \subseteq \mathrm{Ker}(\varepsilon_{P,Q})$$

$$\Downarrow$$

$$h(P,Q)=0$$

that is $h(\lambda,\mu)=\partial\mathrm{Res}(P-\lambda, Q-\mu)\in \mathtt{BC}(P,Q)$

Σ-linear evaluation map

Given g in $\mathcal{E}(P-\lambda, Q-\mu)$

$$g(\lambda,\mu) = C(P-\lambda) + D(Q-\mu), \quad C, D \in \Sigma[\lambda,\mu][\partial].$$

Given $\lambda_0 \in C$, $\exists \mu_0 \in C$ such that $h(\lambda_0,\mu_0) = 0$.
By the Differential Resultant Theorem $\exists \psi_{\lambda_0}$ such that

$$P(\psi_{\lambda_0}) = \lambda_0 \psi_{\lambda_0}, Q(\psi_{\lambda_0}) = \mu_0 \psi_{\lambda_0}$$

$$\Psi = \{\psi_{\lambda_0} \mid \lambda_0 \in C\} \text{ infinite set of eigenfunctions}$$

$$g(P,Q)(\psi_{\lambda_0}) = g(\lambda_0,\mu_0)\cdot\psi_{\lambda_0} = C^0(P-\lambda_0)(\psi_{\lambda_0}) + D^0(Q-\mu_0)(\psi_{\lambda_0}) = 0$$

Ψ included in the C-linear space of solutions of $g(P,Q)(y) = 0$.
Then $g(P,Q)$ is the zero operator.

Σ-linear evaluation map

P and Q in $\Sigma[\partial]\backslash C[\partial]$, both of positive order

$$\mathcal{E}(P-\lambda, Q-\mu) \subseteq \mathrm{Ker}(\varepsilon_{P,Q})$$

As a consequence $h = \partial\mathrm{Res}(P-\lambda, Q-\mu) \in \mathcal{E}_C(P-\lambda, Q-\mu)$ belongs to

$$\mathtt{BC}(P,Q) = \mathrm{Ker}(\varepsilon_{P,Q}) \cap C[\lambda,\mu]$$

$f = \sqrt{h}$

$\mathrm{Ker}(\varepsilon_{P,Q}) = [f]$ is a prime differential ideal in $\Sigma[\lambda,\mu]$

Elimination ideals

Commuting P and Q in $\Sigma[\partial]\backslash C[\partial]$, both of positive order,

$$f=\sqrt{h}, \text{ with } h=\partial\mathrm{Res}(P-\lambda, Q-\mu).$$

(RZ 2023)

1. The radical of the elimination ideal $\mathcal{E}_C(P-\lambda, Q-\mu)$ equals
$$\mathtt{BC}(P,Q)=(f).$$
2. The radical of the elimination ideal $\mathcal{E}(P-\lambda, Q-\mu)$ equals $[f]$.

Recall $f\in C[\lambda,\mu]$,

$$(f)=C[\lambda,\mu]f \text{ and } [f]=\Sigma[\lambda,\mu]f \text{ differential ideal.}$$

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Spectral curve of L

Generalized Schur's Theorem [Goodearl, 1983]

$$\mathcal{Z}((L)) = \left\{ \sum_{j=-\infty}^{m} c_j Q^j \mid c_j \in C, m \in \mathbb{Z} \right\}$$

Commutative differential domain

$$\mathcal{Z}(L) = \mathcal{Z}((L)) \cap \Sigma[\partial]$$

$Spec(\mathcal{Z}(L))$ is an abstract algebraic curve Γ

Compute the defining ideal of Γ

BC ideal of L. CASE I

Given $L \in \Sigma[\partial] \backslash C[\partial]$, with $\mathcal{Z}(L) \neq C[L]$.

CASE I: $\mathcal{Z}(L) = C[L, A]$.

Define then **Burchnall-Chaundy ideal of L** to be the

$$\mathtt{BC}(L) := \mathtt{BC}(L, A).$$

Examples

- (MRZ2020) If $\operatorname{ord}(L) = 2$ then $\mathcal{Z}(L) = C[L, A]$ with A of odd order.
- (PRZ2019) If $\operatorname{ord}(L) = 4$ and L belongs to the first Weyl algebra $\mathcal{Z}(L) = C[L, A]$ with A of even order $\equiv 2$ $(mod\ 4)$.

Spectral curve of L. CASE I

Define the spectral curve of L to be

$$\Gamma := \Gamma_{L,A} = V(\mathtt{BC}(L, A))$$

whose coordinate ring is

$$\frac{C[\lambda, \mu]}{\mathtt{BC}(L)} \simeq \mathcal{Z}(L) = C[L, A]$$

isomorphic to the centralizer of L.

BC ideal of L. CASE II

Given $L \in \Sigma[\partial] \backslash C[\partial]$, with $\mathcal{Z}(L) \neq C[L]$.

CASE II: $\mathcal{Z}(L) \neq C[L, A]$.

$\mathrm{ord}(L) = 3$. $\mathcal{Z}(L)$ is a free $C[L]$-module of rank 3.

$\{1, A_1, A_2\}$ basis of $\mathcal{Z}(L)$ as a $C[L]$-module. Each A_i is a monic operator in $\mathcal{Z}(L) \backslash C[L]$ of minimal order

$$o_i := \mathrm{ord}(A_i) \equiv i \ (mod\ 3).$$

$$\mathcal{Z}(L) = C[L] \oplus C[L]A_1 \oplus C[L]A_2 = C[L, A_1, A_2]$$

BC ideal of L. CASE II

$\mathrm{ord}(L) = 3.$

$$e_L : C[\lambda, \mu_1, \mu_2] \to \Sigma[\partial]$$
$$e_{P,Q}(\lambda) = L, \ \ e_{P,Q}(\mu_1) = A_1, \ \ e_{P,Q}(\mu_2) = A_2.$$

Image of e_L,

$$\mathcal{Z}(L) = C[L, A_1, A_2]$$

Given $g \in C[\lambda, \mu_1, \mu_2]$ denote

$$g(L, A_1, A_2) := e_L(g).$$

BC-ideal of L

$$\mathrm{BC}(L) := \mathrm{Ker}(e_L) = \{g \in C[\lambda, \mu_1, \mu_2] \mid g(L, A_1, A_2) = 0\}.$$

Call the elements of the BC ideal BC-polynomials.

Spectral curve of L

$$\mathrm{ord}(L) = 3 \text{ in } \Sigma[\partial], \quad \mathcal{Z}(L) = C[L, A_1, A_2], \mathrm{ord}(A_i) = 3n_i + i$$

$$\begin{cases} f_i = \partial\mathrm{Res}(L - \lambda, A_i - \mu_i), \quad i = 1, 2 \\ f_3^r = \partial\mathrm{Res}(A_1 - \mu_1, A_2 - \mu_2) \end{cases}$$

are irreducible in $C[\lambda, \mu_1, \mu_2]$ since

$$\mathtt{BC}(L, A_i) = (f_i) \text{ and } BC(A_1, A_2) = (f_3)$$

(RZ 2023) $\mathtt{BC}(L)$ is a prime ideal, affine algebraic curve in C^3

$$\Gamma = V(\mathtt{BC}(L))$$

$$\mathcal{Z}(L) \simeq C[\Gamma] = \frac{C[\lambda, \mu_1, \mu_2]}{\mathtt{BC}(L)}$$

$$\mathtt{BC}(L) = (f_1, f_2, f_3)$$

Spectral curves for third order operators

If a non constant coefficient operator A_2 of order 2 belongs to $\mathcal{Z}(L)$ then

$$\mathcal{Z}(L) = \mathcal{C}(A_2) = C[L, A_2] \simeq \frac{C[\lambda, \mu]}{(f_2)}$$

which is isomorphic to the ring of the plane algebraic curve.

In this case the operator of minimal order $3n_1 + 1$ in $\mathcal{Z}(L)$ is A_2^2, implying that $f_3 = (\mu - \gamma^2)^2$.

Planar spectral curve

[Dickson, Gesztesy, Unterkofler, 1999] $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{15}{x^2}\partial + \frac{15}{x^3} + h \ .$$

$$\mathcal{Z}(L) = C[L, A_1, A_2] \ , \ \mathrm{ord}(A_1) = 4, \mathrm{ord}(A_2) = 8.$$

We compute the generators of the ideal $\mathtt{BC}(L) = (f_1, f_2, f_3)$ using differential resultants

$$f_1 = -\mu_1^3 + (\lambda - h)^4 \ , f_2 = -\mu_2^3 + (\lambda - h)^8, f_3^4 = (\mu_2 - \mu_1^2)^4.$$

Since f_3 is the BC polynomial of A_1 and A_2 we have $A_2 = A_1^2$, implying that

$$\mathcal{Z}(L) = C[L, A_1] \simeq \frac{C[\lambda, \mu_1]}{(f_1)}$$

Non-planar spectral curves

(RZ 2022) $\Sigma = \mathbb{C}(x), \partial = d/dx$

$$L = \partial^3 - \frac{6}{x^2}\partial + \frac{12}{x^3} + h\,,\ \ h \in \mathbb{C}.$$

$\mathcal{Z}(L) = \mathbb{C}[L, A_1, A_2]$ with $\mathrm{ord}(A_1) = 4$, $\mathrm{ord}(A_2) = 5$.
Using differential resultants we compute

$$f_1 = -\mu^3 + (\lambda - h)^4,\ \ f_2 = -\gamma^3 + (\lambda - h)^5,\ \ f_3 = \gamma^4 - \mu^5.$$

$\mathtt{BC}(L) = (f_1, f_2, f_3)$ is a prime ideal.

First explicit example of a non-planar spectral curve.

The curve defined by $\mathtt{BC}(L)$ is a non-planar curve Γ parametrized by

$$\aleph(\tau) = (h - \tau^3, \tau^4, -\tau^5), \tau \in \mathbb{C}.$$

$$\Sigma = \mathbb{C}(z = e^x), \partial = d/dx$$

$$L = \partial^3 + \frac{24z}{(z+1)^2}\partial + \frac{-48z(z-1)}{(z+1)^3}, \ \ \mathrm{ord}(A_1) = 4, \mathrm{ord}(A_2) = 5$$

Non-planar spectral curve Γ defined by the prime ideal

$$\mathtt{BC}(L) = (f_1, f_2, f_3)$$

$$\begin{aligned}
f_1 =& \partial\mathrm{Res}(L-\lambda, A_1-\mu_1) = 1 + \lambda^4 + \frac{44}{27}\lambda^2 - \mu_1^3 - 4\lambda^2\mu_1 + 3\mu_1^2 - 3\mu_1 \\
f_2 =& \partial\mathrm{Res}(L-\lambda, A_2-\mu_2) = \\
& \lambda^5 + 16(\mu_2-1)\lambda^2/3 + (4096\lambda)/729 - (\mu_2-1)^3 \\
f_3 =& \partial\mathrm{Res}(A_1-\mu_1, A_2-\mu_2) = \ldots
\end{aligned}$$

New coefficient field

$P, Q \in \Sigma[\partial]$

$$[P, Q] = 0 \Rightarrow \partial\mathrm{Res}(P-\lambda, Q-\mu) = f(\lambda,\mu)^r \in C[\lambda,\mu].$$

As differential operators in $\Sigma[\lambda,\mu][\partial]$,

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) \neq 0 \Rightarrow \mathrm{gcrd}(P-\lambda, Q-\mu) = 1.$$

$$\Gamma_{P,Q} := \{(\lambda,\mu) \in C^2 \mid f(\lambda,\mu) = 0\}$$

$$\Sigma(\Gamma_{P,Q}) = Fr\left(\frac{\Sigma[\lambda,\mu]}{[f]}\right)$$

As differential operators in $\Sigma(\Gamma_{P,Q})[\partial]$,

$$\partial\mathrm{Res}(P-\lambda, Q-\mu) = 0 \Rightarrow \mathrm{gcrd}(P-\lambda, Q-\mu) \neq 1.$$

New coefficient field

$[\mathrm{BC}(L)]$ is a prime differential ideal of $\Sigma[\lambda, \mu_1, \mu_2]$

Differential domain

$$\Sigma[\Gamma] = \frac{\Sigma[\lambda, \mu_1, \mu_2]}{[\mathrm{BC}(L)]}$$

Its fraction field

$$\Sigma(\Gamma)$$

is a differential field with the extended derivation.

Intrinsic right factor

$\mathrm{ord}(L) = 3$ in $\Sigma[\partial]$, $\mathcal{Z}(L) = C[L, A_1, A_2]$

The greatest common right divisor in $\Sigma(\Gamma)[\partial]$

$$\partial + \phi = \mathrm{gcrd}(L - \lambda, A_1 - \mu_1, A_2 - \mu_2)$$

equals

$$\mathrm{gcrd}(L - \lambda, A_1 - \mu_1) = \mathrm{gcrd}(L - \lambda, A_2 - \mu_2)$$

Spectral Picard-Vessiot fields

(MRZ 2021) for Schrödinger operators

Definition, existence and computation of spectral Picard-Vessiot fields

- Differential field extension of $\Sigma(\Gamma)$, the minimal extension containing all the solutions.
- Requires a full factorization of $L - \lambda$ over $\Sigma(\Gamma)$

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- (PRZ 2019) E. Previato, S.L. Rueda, and M.A. Zurro. **Commuting Ordinary Differential Operators and the Dixmier Test**. SIGMA Symmetry Integrability Geom. Methods Appl., 15(101):23 pp., 2019.
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