Solving differential elimination problems with Thomas decomposition

Daniel Robertz

Lehrstuhl für Algebra und Zahlentheorie

RWTH AACHEN UNIVERSITY

1. Introduction

Theorem (Cauchy-Kovalevskaya, 1875)

The Cauchy problem

$$\begin{cases} \dfrac{\partial u_1}{\partial z_1} = \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{1,j,k}(z_2,\ldots,z_n,u_1,\ldots,u_m)\,\dfrac{\partial u_k}{\partial z_j} + b_1(z_2,\ldots,z_n,u_1,\ldots,u_m), \\ \quad\vdots \\ \dfrac{\partial u_m}{\partial z_1} = \displaystyle\sum_{j=2}^{n}\sum_{k=1}^{m} a_{m,j,k}(z_2,\ldots,z_n,u_1,\ldots,u_m)\,\dfrac{\partial u_k}{\partial z_j} + b_m(z_2,\ldots,z_n,u_1,\ldots,u_m), \\ u_1(0,z_2,\ldots,z_n) = 0 \qquad \text{for all } z_2,\ldots,z_n, \\ \quad\vdots \\ u_m(0,z_2,\ldots,z_n) = 0 \qquad \text{for all } z_2,\ldots,z_n, \end{cases}$$

where $a_{i,j,k}$ and b_i are real analytic functions around the origin of $\mathbb{R}^{m+n-1}$, has a unique real analytic solution $(u_1,...,u_m)$ in a neighborhood of $(z_1,...,z_n)=(0,...,0)$.

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M. Janet,
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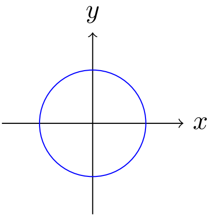
Algebraic Geometry

$$\begin{cases} \mathbb{R} \to \mathbb{R}^2 \\ t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \end{cases}$$



$$x^2 + y^2 - 1 = 0$$

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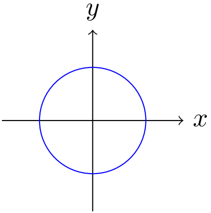

$$x^2 + y^2 - 1 = 0$$

Eliminate t in $\quad x = \dfrac{2t}{t^2+1}, \quad y = \dfrac{t^2-1}{t^2+1} \quad \ldots$

Special Solutions

$$\frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \frac{1}{\rho} \nabla p = 0 \qquad \text{(Navier-Stokes)}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

cylindrical coordinates r, θ, z, $\rho \equiv 1$ (incompressible flow)

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cylindrical coordinates $r,\ \theta,\ z,$ $\rho \equiv 1$ (incompressible flow)

Ansatz: $v_i(r, \theta, z) = f_i(r) g_i(\theta) h_i(z),$ $i = 1,\ 2,\ 3$

PDE: $u u_{x,y} - u_x u_y = 0,$ $u \in \{v_1, v_2, v_3\},$ $(x, y) \in \{(r, \theta), (r, z), (\theta, z)\}$

Special Solutions

$$\begin{aligned} \frac{\partial v}{\partial t} + v \cdot \nabla v - \nu \Delta v + \frac{1}{\rho} \nabla p &= 0 \qquad \text{(Navier-Stokes)} \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) &= 0 \end{aligned}$$

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Ansatz: $v_i(r, \theta, z) = f_i(r) g_i(\theta) h_i(z)$, $i = 1, 2, 3$

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one of the many simple systems of the Thomas decomposition:

$$\begin{aligned} v(t, r, \theta, z) &= \left(-\frac{(t+c_2) F_1(t)}{r} - \frac{r}{2(t+c_2)}, \ \frac{(\theta+c_1) r}{t+c_2}, \ 0 \right), \\ p(t, r, \theta, z) &= (t + c_2) \ln(r) \dot{F}_1(t) - \frac{(t+c_2)^2 F_1(t)^2}{2r^2} + (\ln(r) + (\theta + c_1)^2) F_1(t) \\ &\quad + F_2(t) - \frac{2\nu \ln(r)}{t+c_2} + \frac{((\theta+c_1)^2 - \frac{3}{4}) r^2}{2(t+c_2)^2}. \end{aligned}$$

Outline

1. Introduction
2. Janet bases
3. Thomas decomposition of differential systems
4. Nonlinear control theory

2. Janet bases

Janet's algorithm for linear PDEs

$$\left\{ \begin{array}{rcl} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{array} \right.$$

find: $u = u(x, y)$ analytic

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find: $u = u(x, y)$ analytic

$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \tfrac{x^2}{2!} + a_{1,1}\, \tfrac{xy}{1!1!} + a_{0,2}\, \tfrac{y^2}{2!} + \ldots$$

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Janet's algorithm computes a vector space basis for power series solutions

(Maurice Janet, $\sim$ 1920)

Decomposition into disjoint cones

Strategy of Janet's algorithm:

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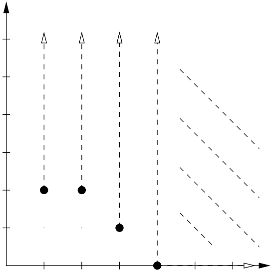

$$\langle\, \partial_1 \partial_2^2,\, \partial_1^3 \partial_2,\, \partial_1^4 \,\rangle$$

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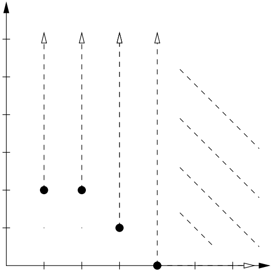

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decomposition:

$\partial_1\partial_2^2$	$\{\, *\,, \partial_2 \,\}$
$\partial_1^2\partial_2^2$	$\{\, *\,, \partial_2 \,\}$
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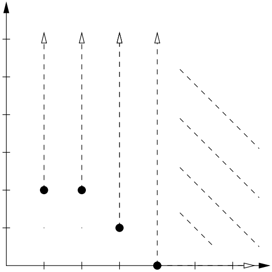

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This can also be done for $\mathrm{Mon}(\partial_1,\ldots,\partial_n) - S$.

Example

Let $I := \langle g_1, g_2 \rangle \trianglelefteq K[\partial_1, \partial_2]$, $\quad g_1 := \partial_1^2 - \partial_2, \quad g_2 := \partial_1\partial_2 - \partial_2.$

Let $>$ be degrevlex, $\partial_1 > \partial_2$.

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Decomposition into disjoint cones of $\quad \langle\, \mathrm{lm}(g_1),\, \mathrm{lm}(g_2) \,\rangle$:

$\{\, (\partial_1^2,\, \{\partial_1, \partial_2\}),\, (\partial_1 \partial_2, \{\partial_2\}) \,\}$

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$\{\, (g_1, \{\partial_1, \partial_2\}),\, (g_2, \{\partial_2\}),\, (g_3, \{\partial_2\}) \,\}$ $\qquad$ (minimal) Janet basis for I

Janet's algorithm for linear PDEs

$$\begin{cases} u_{y,y} = 0 \\ u_{x,x} - y u_{z,z} = 0 \end{cases}$$

is equivalent to

$$\begin{cases} u_{y,y} = 0 \\ u_{x,x} - y u_{z,z} = 0 \\ u_{y,z,z} = 0 \\ u_{x,y,y} = 0 \\ u_{z,z,z,z} = 0 \\ u_{x,y,z,z} = 0 \\ u_{x,z,z,z,z} = 0 \end{cases}$$

Janet's algorithm for linear PDEs

$$\begin{cases} u_{y,y} = 0 & A \\ u_{x,x} - y u_{z,z} = 0 & B \end{cases}$$

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$$\begin{cases} u_{y,y} = 0 & A \\ u_{x,x} - y u_{z,z} = 0 & B \\ u_{y,z,z} = 0 & \frac{1}{2}(\partial_x^2 - y\partial_z^2)A - \frac{1}{2}\partial_y^2 B \\ u_{x,y,y} = 0 & \partial_x A \\ u_{z,z,z,z} = 0 & \frac{1}{2}(\partial_x^4 - 2y\partial_x^2\partial_z^2 + y^2\partial_z^4)A - \frac{1}{2}(\partial_x^2\partial_y^2 - y\partial_y^2\partial_z^2 + 2\partial_y\partial_z^2)B \\ u_{x,y,z,z} = 0 & \frac{1}{2}(\partial_x^3 - y\partial_x\partial_z^2)A - \frac{1}{2}\partial_x\partial_y^2 B \\ u_{x,z,z,z,z} = 0 & \frac{1}{2}(\partial_x^5 - 2y\partial_x^3\partial_z^2 + y^2\partial_x\partial_z^4)A - \frac{1}{2}(\partial_x^3\partial_y^2 + y\partial_x\partial_y^2\partial_z^2 - 2\partial_x\partial_y\partial_z^2)B \end{cases}$$

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Taylor coeff's for $1, z, y, x, z^2, yz, xz, xy, z^3, xz^2, xyz, xz^3$ *arbitrary*,

all other coeff's determined by linear equations

Power series solutions

$$\frac{\partial^2 u}{\partial x\,\partial y} = 0, \quad \{\ *\ ,\partial_y\,,\partial_z\ \},$$
$$\frac{\partial^3 u}{\partial x^2\,\partial y} = 0, \quad \{\ *\ ,\partial_y\,,\partial_z\ \},$$
$$\frac{\partial^4 u}{\partial x^3\,\partial z} = 0, \quad \{\ \partial_x\,,\ *\ ,\partial_z\ \},$$
$$\frac{\partial^4 u}{\partial x^3\,\partial y} = 0, \quad \{\ \partial_x\,,\partial_y\,,\partial_z\ \}.$$

Janet decomposition of the set of parametric derivatives / generalized Hilbert series:

$$\begin{array}{rl} 1, & \{\ *\ ,\partial_y\,,\partial_z\ \},\\ \partial_x, & \{\ *\ ,\ *\ ,\partial_z\ \},\\ \partial_x^2, & \{\ *\ ,\ *\ ,\partial_z\ \},\\ \partial_x^3, & \{\ \partial_x\,,\ *\ ,\ *\ \}. \end{array} \qquad \frac{1}{(1-\partial_y)(1-\partial_z)} + \frac{\partial_x}{1-\partial_z} + \frac{\partial_x^2}{1-\partial_z} + \frac{\partial_x^3}{1-\partial_x}.$$

Accordingly, a formal power series solution u is uniquely determined as

$$u(x,y,z) = f_0(y,z) + x\,f_1(z) + x^2\,f_2(z) + x^3\,f_3(x)$$

by any choice of formal power series f_0, f_1, f_2, f_3 of the indicated variables.

3. Thomas decomposition of differential systems

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Systems of PDEs

A *differential system* S is given by

$$p_1 = 0, \quad p_2 = 0, \quad \ldots, \quad p_s = 0, \quad q_1 \neq 0, \quad q_2 \neq 0, \quad \ldots, \quad q_t \neq 0,$$

where p_1, ..., p_s and q_1, ..., q_t are polynomials in u_1, ..., u_m of z_1, ..., z_n and their partial derivatives.

Ω open and connected subset of $\mathbb{C}^n$ with coordinates $z_1, \ldots, z_n$

The *solution set* of S on Ω is

$$\mathrm{Sol}_\Omega(S) \;:=\; \{\, f = (f_1, \ldots, f_m) \mid f_k \colon \Omega \to \mathbb{C} \text{ analytic}, \, k = 1, \ldots, m, \\ p_i(f) = 0, \, q_j(f) \neq 0, \, i = 1, \ldots, s, \, j = 1, \ldots, t \,\} \, .$$

Appropriate choice of Ω is possible only *after* formal treatment.

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Consequences of the system obtained in a finite number of steps from:

- $p_1 = 0$, $p_2 = 0$, ..., $p_s = 0$ are consequences,
- if $p = 0$ is consequence, then any partial derivative of $p = 0$ is,
- if $p \cdot q = 0$ is consequence and q a factor of some q_i, then $p = 0$ is consequence,
- if $p = 0$, $r = 0$ are consequences, then $a\,p + b\,r = 0$ is (a, b differential polynomials)

Differential algebraic geometry

Differential algebra (Ritt, Kolchin, Seidenberg, ...)

$\mathbb{Q} \subseteq K$ a differential field with commuting derivations ∂_1, ..., ∂_n

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Differential polynomial ring with derivations $\Delta = \{\partial_1, ..., \partial_n\}$

$$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$$

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$K\{u\}$ not Noetherian (e.g., $[u'u'', u''u''', \ldots] \subseteq K\{u\}$ not fin. gen.)

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Differential polynomial ring with derivations $\Delta = \{\partial_1, ..., \partial_n\}$

$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$

$K\{u\}$ not Noetherian (e.g., $[u'u'', u''u''', \ldots] \subseteq K\{u\}$ not fin. gen.)

Thm. (Ritt-Raudenbush).
Every radical differential ideal of $K\{u_1, \ldots, u_m\}$ is finitely generated and is intersection of finitely many prime differential ideals.

Differential algebraic geometry

Differential algebra (Ritt, Kolchin, Seidenberg, ...)

$\mathbb{Q} \subseteq K$ a differential field with commuting derivations ∂_1, ..., ∂_n

Differential polynomial ring with derivations $\Delta = \{\partial_1, ..., \partial_n\}$

$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$

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Thm. (Ritt-Raudenbush).
Every radical differential ideal of $K\{u_1, \ldots, u_m\}$ is finitely generated and is intersection of finitely many prime differential ideals.

Thm. (Differential Nullstellensatz).
Every radical diff. ideal $I \subsetneq K\{u_1, ..., u_m\}$ has a zero in a diff. field ext. of K. If $f \in K\{u_1, ..., u_m\}$ vanishes for all zeros of I, then $f \in I$.

Thomas Decomposition

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

Thomas Decomposition

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

algebraic reduction:

$$p = u_{x,x,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$p \;\rightarrow\; r = c \cdot p - u_{x,x,y} \cdot q$$

Thomas Decomposition

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

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algebraic reduction:

$$p = \; u_{x,x,y}^3 + \ldots$$
$$q = c\, u_{x,x,y}^2 + \ldots$$
$$p \;\to\; r = c \cdot p - u_{x,x,y} \cdot q$$

differential reduction:

$$p = \; u_{x,x,y,y}^3 + \ldots$$
$$q = c\, u_{x,x,y}^2 + \ldots$$
$$\partial_y\, q = \tfrac{\partial q}{\partial u_{x,x,y}}\, u_{x,x,y,y} + \ldots$$
$$p \;\to\; r = \tfrac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

Thomas Decomposition

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

algebraic reduction:

$$p = u_{x,x,y}^3 + \ldots$$

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differential reduction:

$$p = u_{x,x,y,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

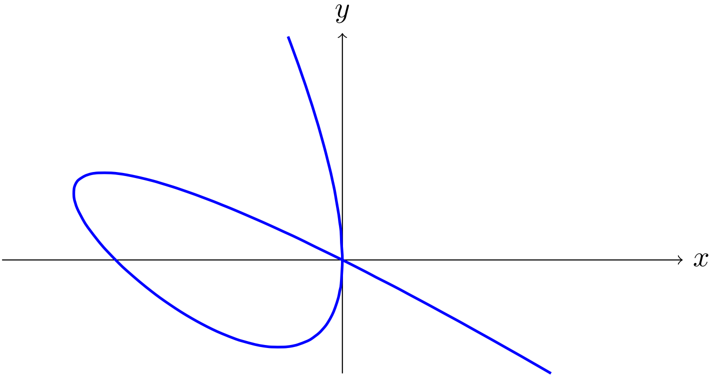
$$\partial_y\, q = \frac{\partial q}{\partial u_{x,x,y}}\, u_{x,x,y,y} + \ldots$$

$$p \;\to\; r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

reduction requires: initial $c \neq 0$ and separant $\frac{\partial q}{\partial u_{x,x,y}} \neq 0$

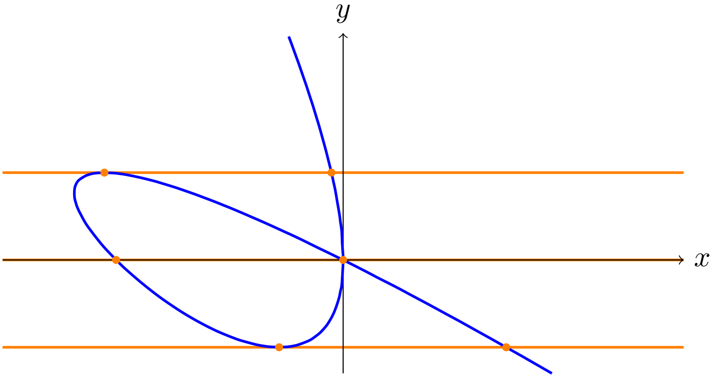
Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



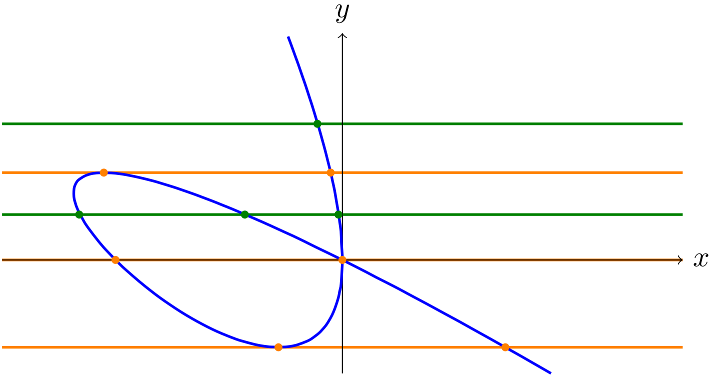
Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



Thomas Decomposition

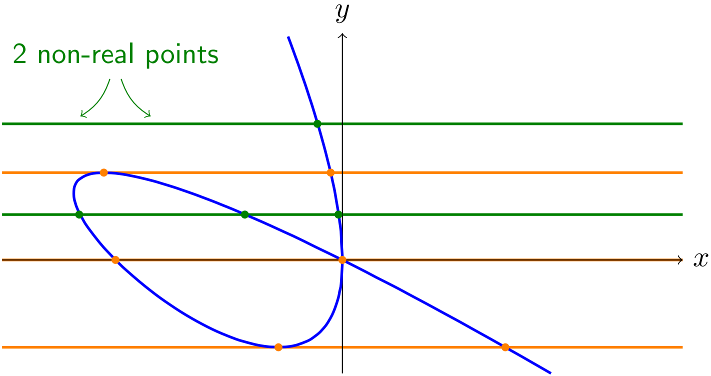
$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas Decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



$$\mathrm{disc}_x(p) = y^2(4 - 27y^2)$$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$$a\underline{x}^2 + b\underline{x} + c \ = \ 0$$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c = 0$	$b\underline{x} + c = 0$
$a \neq 0$	$a = 0$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$$\begin{array}{rcl} a\underline{x}^2 + b\underline{x} + c & = & 0 \\ 4a\underline{c} - b^2 & \neq & 0 \\ a & \neq & 0 \end{array}$$

$$\begin{array}{rcl} 2a\underline{x} + b & = & 0 \\ 4a\underline{c} - b^2 & = & 0 \\ a & \neq & 0 \end{array}$$

$$\begin{array}{rcl} b\underline{x} + c & = & 0 \\ a & = & 0 \end{array}$$

Thomas Decomposition

$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$a\underline{x}^2 + b\underline{x} + c = 0$	$2a\underline{x} + b = 0$	$b\underline{x} + c = 0$	
$4a\underline{c} - b^2 \neq 0$	$4a\underline{c} - b^2 = 0$		$c = 0$
		$b \neq 0$	$b = 0$
$a \neq 0$	$a \neq 0$	$a = 0$	$a = 0$

Thomas Decomposition

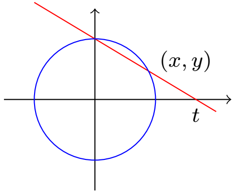
$$p = ax^2 + bx + c = 0, \qquad p \in \mathbb{Q}[x, c, b, a], \qquad x > c > b > a$$

$x_1 \neq x_2$	$x_1 = x_2$	x_1	all $x \in \overline{\mathbb{Q}}$
$a\underline{x}^2 + b\underline{x} + c = 0$	$2a\underline{x} + b = 0$	$b\underline{x} + c = 0$	
$4a\underline{c} - b^2 \neq 0$	$4a\underline{c} - b^2 = 0$		$c = 0$
		$b \neq 0$	$b = 0$
$a \neq 0$	$a \neq 0$	$a = 0$	$a = 0$

solve $p(x) = 0$ for fixed choice of a, b, c

Thomas Decomposition

$$\begin{cases} x^2 + y^2 - 1 & = & 0 \\ (1-y)\,t - x & = & 0 \end{cases}$$



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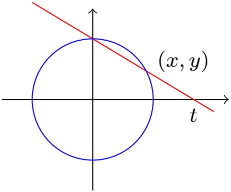
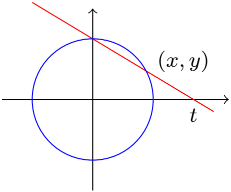


$$p_1 := x^2 + y^2 - 1, \qquad p_2 := x + t\,y - t \quad \in \quad \mathbb{Q}[x, y, t], \qquad x > y > t$$

Thomas Decomposition

$$\begin{cases} x^2 + y^2 - 1 & = & 0 \\ (1-y)\,t - x & = & 0 \end{cases}$$

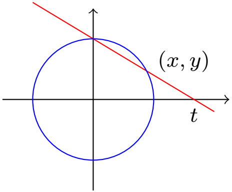


$$p_1 := x^2 + y^2 - 1, \qquad p_2 := x + t\,y - t \quad \in \quad \mathbb{Q}[x, y, t], \qquad x > y > t$$

$$p_1 - (x - ty + t)\,p_2 \;=\; (y-1)\,((t^2+1)y - t^2 + 1)$$

Thomas Decomposition

$$\begin{cases} x^2 + y^2 - 1 & = & 0 \\ (1 - y)\, t - x & = & 0 \end{cases}$$



$$p_1 := x^2 + y^2 - 1, \qquad p_2 := x + t\, y - t \quad \in \quad \mathbb{Q}[x, y, t], \qquad x > y > t$$

$$p_1 - (x - ty + t)\, p_2 \;=\; (y - 1)\,((t^2 + 1)y - t^2 + 1)$$

Thomas decomposition:

$$\boxed{\begin{array}{rcl} (t^2 + 1)\, \underline{x} - 2t & = & 0 \\ (t^2 + 1)\, \underline{y} - t^2 + 1 & = & 0 \\ \underline{t}^2 + 1 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} \underline{x} & = & 0 \\ \underline{y} - 1 & = & 0 \end{array}}$$

Thomas Decomposition

$$S = \{\, p_1 = 0, \ldots, p_s = 0, \, q_1 \neq 0, \ldots, q_t \neq 0 \,\}$$

Def. *Thomas decomposition* of differential system S (or $\mathsf{Sol}(S)$):

$\mathsf{Sol}(S) = \mathsf{Sol}(S_1) \uplus \ldots \uplus \mathsf{Sol}(S_r)$, S_i simple differential system

Thomas Decomposition

$$S = \{ p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \}$$

Def. *Thomas decomposition* of differential system S (or Sol(S)):

Sol(S) = Sol$(S_1) \uplus \ldots \uplus$ Sol(S_r), S_i simple differential system

Def. S is *simple* if

(a) $p_1, \ldots, p_s, q_1, \ldots, q_t$ have pairwise distinct leaders,

(b) leading coefficients and discriminants of p_i and q_j do not vanish,

(c) $p_1, \ldots, p_s$ form a passive PDE system,

(d) $q_1, \ldots, q_t$ are reduced modulo $p_1, \ldots, p_s$.

set of *admissible derivations* $\mu_i \subseteq \{\partial_1, \ldots, \partial_n\}$ for p_i, $i = 1, \ldots, s$

Thomas Decomposition

$R = K\{u_1, \ldots, u_m\}$

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Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E differential ideal generated by $p_1, \ldots, p_s$

q product of initials and separants of all p_i

Thomas Decomposition

$R = K\{u_1, \ldots, u_m\}$

Def. *Thomas decomposition* of differential system S (or $\mathsf{Sol}(S)$):

$\mathsf{Sol}(S) = \mathsf{Sol}(S_1) \uplus \ldots \uplus \mathsf{Sol}(S_r)$, S_i simple differential system

Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E differential ideal generated by p_1, ..., p_s

q product of initials and separants of all p_i

Then

$$E : q^\infty := \{\, p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\} = \mathcal{I}_R(\mathsf{Sol}(S))$$

consists of all differential polynomials in R vanishing on $\mathsf{Sol}(S)$.

Thomas Decomposition

$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad\qquad p \in \mathbb{Q}(t)\{u\}$$

Separant of p: $\frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

Thomas Decomposition

$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad p \in \mathbb{Q}(t)\{u\}$$

Separant of p: $\frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{u}}, \dot{u}) = -16u + 16t^2$

Thomas decomposition:

$$\boxed{\begin{array}{rcl} p & = & 0 \\ u - t^2 & \neq & 0 \end{array}} \qquad \boxed{\begin{array}{rcl} & & \\ u - t^2 & = & 0 \end{array}}$$

Thomas Decomposition

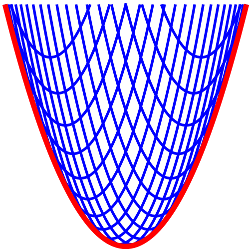
$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad\qquad p \in \mathbb{Q}(t)\{u\}$$

Separant of p: $\frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{u}}, \dot{u}) = -16u + 16t^2$

Thomas decomposition:

$p = 0$	
$u - t^2 \neq 0$	$u - t^2 = 0$

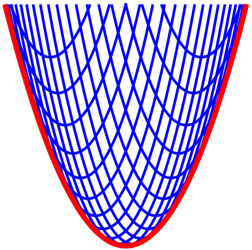


Thomas Decomposition

$$p = \dot{u}^2 - 4t\dot{u} - 4u + 8t^2 = 0 \qquad p \in \mathbb{Q}(t)\{u\}$$

Separant of p: $\frac{\partial p}{\partial \dot{u}} = 2\dot{u} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{u}}, \dot{u}) = -16u + 16t^2$



Thomas decomposition:

$p = 0$
$u - t^2 \neq 0$

$u - t^2 = 0$

general solution: $u(t) = 2((t+c)^2 + c^2), \qquad c \in \mathbb{R}$

essential singular solution: $u(t) = t^2$

Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\ \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{cases}$$

Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\ \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{cases}$$

Define

$$p_1 \ := \ u_{x,x} - u_{y,y}, \qquad p_2 \ := \ u_x - u^2$$

$R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y.

degree-reverse lexicographical ranking $>$ on R with $\partial_x u > \partial_y u$

Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\ \qquad \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{cases}$$

Define

$$p_1 \ := \ u_{x,x} - u_{y,y}, \qquad p_2 \ := \ u_x - u^2$$

$R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y.

degree-reverse lexicographical ranking $>$ on R with $\partial_x\, u > \partial_y\, u$

$$p_3 \ := \ p_1 - \partial_x\, p_2 - 2\, u\, p_2 \ = \ -u_{y,y} + 2\, u^3\,.$$

Example

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} & = & 0, \\ \dfrac{\partial u}{\partial x} - u^2 & = & 0 \end{cases}$$

Define

$$p_1 \ := \ u_{x,x} - u_{y,y}, \qquad p_2 \ := \ u_x - u^2$$

$R = \mathbb{Q}\{u\}$ with commuting derivations ∂_x, ∂_y.

degree-reverse lexicographical ranking $>$ on R with $\partial_x u > \partial_y u$

$$p_3 \ := \ p_1 - \partial_x p_2 - 2\,u\,p_2 \ = \ -u_{y,y} + 2\,u^3\,.$$

Janet division associates the sets of admissible derivations:

$$\begin{cases} \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_{y,y}} - 2\,u^3 & = & 0, & \{\,*\,, \partial_y\} \end{cases}$$

Example

passivity check:

$$\begin{aligned} \partial_x p_3 + \partial_y^2 p_2 - 6\,u^2\,p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4) \\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,. \end{aligned}$$

factorization $\rightsquigarrow$ splitting possible

Example

passivity check:

$$\begin{aligned}\partial_x p_3 + \partial_y^2 p_2 - 6\,u^2\,p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4)\\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,.\end{aligned}$$

factorization $\rightsquigarrow$ splitting possible

If the above factorization is ignored, then the discriminant of $p_4 := u_y^2 - u^4$ needs to be considered, which implies vanishing or non-vanishing of the separant $2\,u_y$. This case distinction leads to the

Thomas decomposition

$\underline{u_x} - u^2 = 0, \quad \{\partial_x, \partial_y\}$ $\underline{u_y}^2 - u^4 = 0, \quad \{\,*\,, \partial_y\}$ $\underline{u} \neq 0$	$\underline{u} = 0$

Example

passivity check:

$$\begin{aligned} \partial_x p_3 + \partial_y^2 p_2 - 6\,u^2\,p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4) \\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,. \end{aligned}$$

factorization $\rightsquigarrow$ splitting possible

Example

passivity check:

$$\begin{aligned} \partial_x p_3 + \partial_y^2 p_2 - 6\,u^2\,p_2 - 2\,u\,p_3 &= -2\,(\underline{u_y}^2 - u^4) \\ &= -2\,(\underline{u_y} + u^2)\,(\underline{u_y} - u^2)\,. \end{aligned}$$

factorization $\rightsquigarrow$ splitting possible

For both systems a differential reduction of p_3 modulo the chosen factor is applied because the monomial ∂_y defining the new leader divides the monomial $\partial_{y,y}$ defining $\mathrm{ld}(p_3)$. We obtain the

Thomas decomposition

$$\begin{array}{|rcll|} \hline \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_y} + u^2 & = & 0, & \{\,*\,, \partial_y\} \\ & & & \\ \hline \end{array} \qquad \begin{array}{|rcll|} \hline \underline{u_x} - u^2 & = & 0, & \{\partial_x, \partial_y\} \\ \underline{u_y} - u^2 & = & 0, & \{\,*\,, \partial_y\} \\ \underline{u} & \neq & 0. & \\ \hline \end{array}$$

Implementation

Maple package `DifferentialThomas` (M. Lange-Hegermann)

`https://www.art.rwth-aachen.de/go/id/rnab`

GNU LPGL license

V. P. Gerdt, M. Lange-Hegermann, D. R.

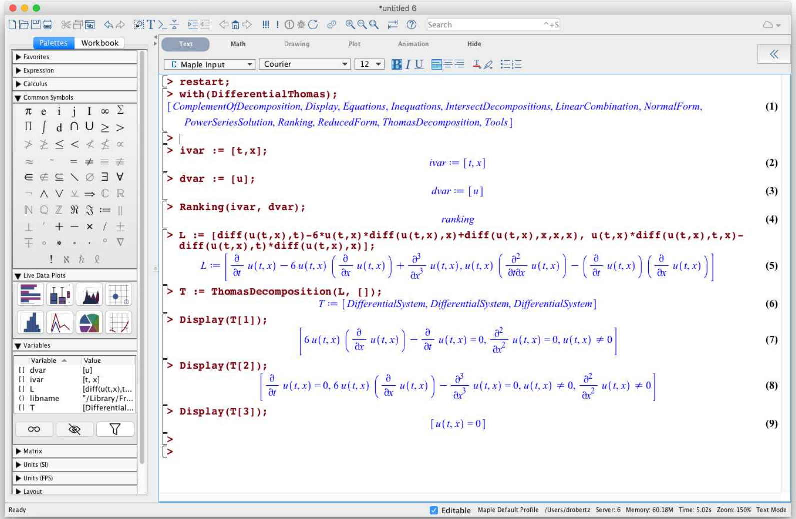
The MAPLE package TDDS for computing Thomas decompositions of systems of nonlinear PDEs

Computer Physics Communications 234:202–215, 2019

`arXiv:1801.09942`

`DifferentialThomas` in Maple 2018 (interface by E. S. Cheb-Terrab)

Maple 2018



*untitled 6
Palettes
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Live Data Plots
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ivar
[t, x]
L
[diff(u(t,x),t...
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"/Library/Fr...
T
[Differential...
Matrix
Units (SI)
Units (FPS)
Layout
Text
Math
Drawing
Plot
Animation
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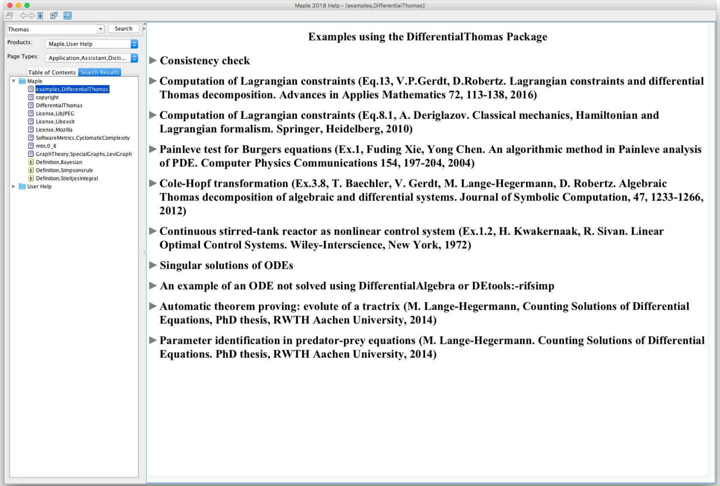
Courier
12
> restart;
> with(DifferentialThomas);
[ComplementOfDecomposition, Display, Equations, Inequations, IntersectDecompositions, LinearCombination, NormalForm, PowerSeriesSolution, Ranking, ReducedForm, ThomasDecomposition, Tools]
(1)
> ivar := [t,x];
ivar := [t, x]
(2)
> dvar := [u];
dvar := [u]
(3)
> Ranking(ivar, dvar);
ranking
(4)
> L := [diff(u(t,x),t)-6*u(t,x)*diff(u(t,x),x)+diff(u(t,x),x,x,x), u(t,x)*diff(u(t,x),t,x)-diff(u(t,x),t)*diff(u(t,x),x)];
(5)
> T := ThomasDecomposition(L, []);
T := [DifferentialSystem, DifferentialSystem, DifferentialSystem]
(6)
> Display(T[1]);
(7)
> Display(T[2]);
(8)
> Display(T[3]);
[u(t, x) = 0]
(9)
Ready
Editable
Maple Default Profile
/Users/drobertz
Server: 6
Memory: 60.18M
Time: 5.02s
Zoom: 150%
Text Mode

Maple 2018


Maple 2018 Help - [examples,DifferentialThomas]
Thomas
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License,Libexslt
License,Mozilla
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mtn,0_8
GraphTheory,SpecialGraphs,LeviGraph
Definition,Bayesian
Definition,Simpsonsrule
Definition,Stieltjesintegral
User Help
Examples using the DifferentialThomas Package
Consistency check
Computation of Lagrangian constraints (Eq.13, V.P.Gerdt, D.Robertz. Lagrangian constraints and differential Thomas decomposition. Advances in Applies Mathematics 72, 113-138, 2016)
Computation of Lagrangian constraints (Eq.8.1, A. Deriglazov. Classical mechanics, Hamiltonian and Lagrangian formalism. Springer, Heidelberg, 2010)
Painleve test for Burgers equations (Ex.1, Fuding Xie, Yong Chen. An algorithmic method in Painleve analysis of PDE. Computer Physics Communications 154, 197-204, 2004)
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An example of an ODE not solved using DifferentialAlgebra or DEtools:-rifsimp
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Differential Elimination

$R = K\{u_1, \ldots, u_m\}, \quad B_1 \uplus \ldots \uplus B_k = U := \{u_1, \ldots, u_m\}$ partition

Block ranking: $u_{i_1} \in B_{j_1}, \quad u_{i_2} \in B_{j_2}, \quad J_1, J_2 \in (\mathbb{Z}_{\geq 0})^n$

$$\partial^{J_1} u_{i_1} > \partial^{J_2} u_{i_2} \iff \begin{cases} j_1 < j_2 \text{ or } (j_1 = j_2 \text{ and } (\partial^{J_1} > \partial^{J_2} \\ \text{ or } (J_1 = J_2 \text{ and } i_1 < i_2)) \end{cases}$$

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Thm. $\quad S$ simple diff. system, $1 \leq i \leq k$

E_i diff. ideal of $K\{B_i, \ldots, B_k\}$ gen. by $\{p_1, \ldots, p_s\} \cap K\{B_i, \ldots, B_k\}$

q_i product of initials and separants of all p_j in intersection

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Then, $\quad (E : q^\infty) \cap K\{B_i, \ldots, B_k\} = E_i : q_i^\infty$.

Differential Elimination

Lemma

Let S be simple, w.r.t. any ranking $>$, E diff. ideal generated by

$S^{=} = \{ p_1, \ldots, p_s \}$, q prod. init. sep. of all p_i, $V \subset \{ u_1, \ldots, u_m \}$

If $P := \{ p \in S^{=} \mid p \in K\{V\} \} = \{ p \in S^{=} \mid \mathrm{ld}(p) \in \mathrm{Mon}(\Delta)\, V \}$,

then $(E : q^{\infty}) \cap K\{V\} = E' : (q')^{\infty}$,

E' diff. ideal of $K\{V\}$ gen. by P, q' prod. of init. and sep. of $p \in P$.

Differential Elimination

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Let S be simple, w.r.t. any ranking $>$, E diff. ideal generated by $S^{=} = \{ p_1, \ldots, p_s \}$, q prod. init. sep. of all p_i, $V \subset \{ u_1, \ldots, u_m \}$

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E' diff. ideal of $K\{V\}$ gen. by P, q' prod. of init. and sep. of $p \in P$.

Proof. Let $0 \neq p \in (E : q^{\infty}) \cap K\{V\}$. Since S is simple,

$b \, p = R$-linear comb. of p_1, ..., p_s and their derivatives

By assumption, Janet divisor of $b \, p$ is in $K\{V\}$.

Pseudo-reduction $p \to 0$ in $K\{V\}$. □

4. Nonlinear Control Theory

Control Theory

$$R = K\{u_1, \ldots, u_m\}, \quad U := \{u_1, \ldots, u_m\}, \quad S \text{ simple diff. system}$$

Control Theory

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Def.

$x \in U$ is *observable w.r.t.* $Y \subseteq U - \{x\}$

$$\iff \begin{cases} \exists\, p \in (E : q^\infty) - \{0\} \quad \text{s.t.} \\ p \in K\{Y\}[x] \qquad \text{(without derivatives of } x\text{)} \\ \text{initial of } p \notin (E : q^\infty), \quad \dfrac{\partial p}{\partial x} \notin (E : q^\infty) \end{cases}$$

Control Theory

$R = K\{u_1, \ldots, u_m\}, \quad U := \{u_1, \ldots, u_m\}, \quad S$ simple diff. system

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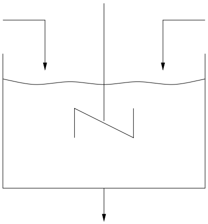
Def.

$Y \subseteq U$ is a *flat output*

$$\iff \begin{cases} (E : q^\infty) \cap K\{Y\} = \{0\} \\ \text{every} \quad x \in U - Y \quad \text{is observable w.r.t.} \quad Y \end{cases}$$

Example

Stirred tank:



$$\begin{cases} \dot{V}(t) & = & F_1(t) + F_2(t) - k\sqrt{V(t)} \\ \dot{\overline{c(t)\,V(t)}} & = & c_1\,F_1(t) + c_2\,F_2(t) - c(t)\,k\sqrt{V(t)} \end{cases}$$

H. Kwakernaak, R. Sivan, *Linear Optimal Control Systems*, John Wiley & Sons, 1972.

Example

$R = \mathbb{Q}\{F_1, F_2, sV, c, c_1, c_2\}$, ranking $>$ s.t. $\{F_2, F_2\} \gg \{sV, c\} \gg \{c_1, c_2\}$

```
> with(DifferentialThomas):

> ivar := [t]:  dvar := [F1,F2,sV,c,c1,c2]:

> ComputeRanking(ivar, [[F1,F2],[sV,c],[c1,c2]]):

> L := [2*sV[t]*sV-F1-F2+k*sV,
c[t]*sV^2-c2*F2+c*k*sV-c1*F1+2*c*sV[t]*sV, c1[t], c2[t]]:

> LL := Diff2JetList(Ind2Diff(L, ivar, dvar));
```

$$LL := [2\, sV_1 sV_0 - F1_0 - F2_0 + k sV_0,$$
$$c_1 sV_0{}^2 - c2_0 F2_0 + c_0 k sV_0 - c1_0 F1_0 + 2\, c_0 sV_1 sV_0, \quad c1_1, \quad c2_1]$$

```
> TD := DifferentialThomasDecomposition(LL,
[sV[0],c1[0],c2[0]]);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

Example

```
> Print(TD[1]);
```

$$[c2\,\underline{F1} - c1\,\underline{F1} + 2\,csV\,sV_t - 2\,c2\,sV\,sV_t + c_t sV^2 + cksV - c2\,ksV = 0,$$
$$c1\,\underline{F2} - c2\,\underline{F2} + 2\,csV\,sV_t - 2\,c1\,sV\,sV_t + c_t sV^2 + cksV - c1\,ksV = 0,$$
$$\underline{c1_t} = 0, \quad \underline{c2_t} = 0, \quad \underline{c2} \neq 0, \quad \underline{c1} \neq 0, \quad \underline{c1} - c2 \neq 0, \quad \underline{sV} \neq 0]$$

```
> collect(%[1], F1);
```

$$(c2 - c1)\,F1 + 2\,csV\,sV_t - 2\,c2\,sV\,sV_t + c_t sV^2 + cksV - c2\,ksV = 0$$

```
> collect(%%[2], F2);
```

$$(c1 - c2)\,F2 + 2\,csV\,sV_t - 2\,c1\,sV\,sV_t + c_t sV^2 + cksV - c1\,ksV = 0$$

$\Rightarrow$ F_1, F_2 observable with respect to $\{c, sV\}$

$(E : q^\infty) \cap \mathbb{Q}\{sV, c\} = \{0\}$ $\Rightarrow$ $\{c, sV\}$ is flat output

Example

```
> Print(TD[2]);
```

$$[c\underline{F1} - c2\,\underline{F1} + cF2 - c2\,F2 + c_t sV^2 = 0,$$
$$2\,c\underline{sV_t} - 2\,c2\,\underline{sV_t} + c_t sV + ck - c2\,k = 0, \quad \underline{c1} - c2 = 0, \quad \underline{c2_t} = 0,$$
$$\underline{c2} \neq 0, \quad \underline{c} - c2 \neq 0, \quad \underline{sV} \neq 0]$$

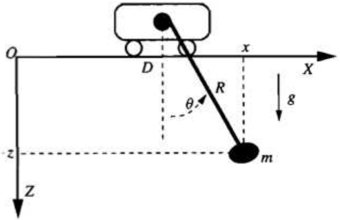
```
> Print(TD[3]);
```

$$[\underline{F1} + F2 - 2\,sV\,sV_t - k\,sV = 0, \quad \underline{c} - c2 = 0, \quad \underline{c1} - c2 = 0, \quad \underline{c2_t} = 0,$$
$$\underline{c2} \neq 0, \quad \underline{sV} \neq 0]$$

conditions $c_1 = c_2$ and $(c_1)_t = (c_2)_t = 0$ preclude control of the concentration in the tank

Example

2-D crane:



$$\left\{\begin{array}{rcl} m\,\ddot{x} & = & -T\,\sin\theta \\ m\,\ddot{z} & = & -T\,\cos\theta + m\,g \\ x & = & R\,\sin\theta + d \\ z & = & R\,\cos\theta \end{array}\right.$$

M. Fliess, J. Lévine, P. Martin, P. Rouchon, *Flatness and defect of non-linear systems: introductory theory and examples*, Internat. J. Control 61(6), 1327–1361, 1995.

Example

$\mathbb{Q}(m, g)\{T, s, c, d, R, x, z\}$

block ranking $>$ satisfying $\{T, s, c, d, R\} \gg \{x, z\}$

```
> with(DifferentialThomas):

> ivar := [t]:  dvar := [T,s,c,d,R,x,z]:

> ComputeRanking(ivar, [[T,s,c,d,R],[x,z]]):

> TD := DifferentialThomasDecomposition(
[m*x[2]+T[0]*s[0], m*z[2]+T[0]*c[0]-m*g,
x[0]-R[0]*s[0]-d[0], z[0]-R[0]*c[0], c[0]^2+s[0]^2-1],
[]);
```

$$TD := [\mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}, \mathit{DifferentialSystem}]$$

Example

```
> Print(TD[2]);
```

$$[z\underline{T} + m z_{t,t} R - m g R = 0, \quad z_{t,t} R \underline{s} - g R \underline{s} - z x_{t,t} = 0, \quad R\underline{c} - z = 0,$$
$$z_{t,t}\underline{d} - g\underline{d} + z x_{t,t} - x z_{t,t} + g x = 0,$$
$$z_{t,t}{}^2 \underline{R}^2 - 2\, g z_{t,t} \underline{R}^2 + g^2 \underline{R}^2 - z^2 x_{t,t}{}^2 - z^2 z_{t,t}{}^2 + 2\, g z^2 z_{t,t} - g^2 z^2 = 0,$$
$$\underline{z} \neq 0, \quad \underline{z_{t,t}} - g \neq 0, \quad \underline{x_{t,t}} \neq 0, \quad \underline{x_{t,t}}{}^2 + z_{t,t}{}^2 - 2\, g z_{t,t} + g^2 \neq 0]$$

```
> collect(%[5], R, factor);
```

$$(z_{t,t} - g)^2 R^2 - z^2 \left(x_{t,t}{}^2 + z_{t,t}{}^2 - 2\, g z_{t,t} + g^2\right) = 0$$

$\Rightarrow$ T, s, c, d, R observable with respect to $\{x, z\}$

$(E : q^\infty) \cap \mathbb{Q}\{x, z\} = \{0\}$ $\Rightarrow$ $\{x, z\}$ is flat output

Example

```
> Print(TD[1]);
```

$$[\underline{T}=0,\quad R\underline{s}+d-x=0,\quad R\underline{c}-z=0,\quad \underline{d}^2-2\,x\underline{d}+x^2-R^2+z^2=0,$$
$$\underline{x_{t,t}}=0,\quad \underline{z_{t,t}}-g=0,\quad \underline{z}\neq 0,\quad \underline{R}\neq 0,\quad \underline{R}+z\neq 0,\quad \underline{R}-z\neq 0]$$

```
> Print(TD[3]);
```

$$[\underline{T}-mz_{t,t}+mg=0,\quad \underline{s}=0,\quad \underline{c}+1=0,\quad \underline{d}-x=0,\quad \underline{R}+z=0,$$
$$\underline{x_{t,t}}=0,\quad \underline{z}\neq 0]$$

```
> Print(TD[4]);
```

$$[\underline{T}+mz_{t,t}-mg=0,\quad \underline{s}=0,\quad \underline{c}-1=0,\quad \underline{d}-x=0,\quad \underline{R}-z=0,$$
$$\underline{x_{t,t}}=0,\quad \underline{z}\neq 0]$$

```
> Print(TD[5]);
```

$$[c\underline{T}-mg=0,\quad g\underline{s}+x_{t,t}c=0,\quad g^2\underline{c}^2+{x_{t,t}}^2\underline{c}^2-g^2=0,\quad \underline{d}-x=0,\quad \underline{R}=0,$$
$$\underline{z}=0,\quad \underline{x_{t,t}}\neq 0,\quad \underline{x_{t,t}}^2+g^2\neq 0]$$

```
> Print(TD[6]);
```

$$[\underline{T}+mg=0,\quad \underline{s}=0,\quad \underline{c}+1=0,\quad \underline{d}-x=0,\quad \underline{R}=0,\quad \underline{x_{t,t}}=0,\quad \underline{z}=0]$$

```
> Print(TD[7]);
```

$$[\underline{T}-mg=0,\quad \underline{s}=0,\quad \underline{c}-1=0,\quad \underline{d}-x=0,\quad \underline{R}=0,\quad \underline{x_{t,t}}=0,\quad \underline{z}=0]$$

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