# Persistence probabilities for AR(1) processes

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# Probability, combinatorics, experiments



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## Persistence for a class of order-one autoregressive processes and Mallows-Riordan polynomials ☆

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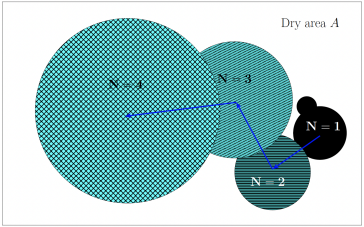
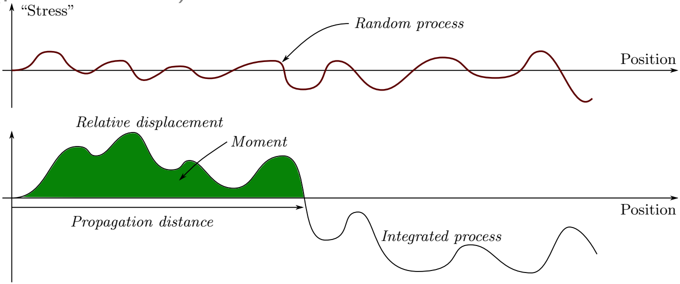
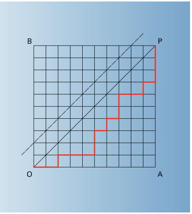
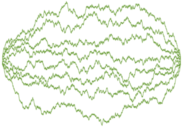
- Experimental physics
  (dew)
- Ising model
- Disordored systems
  (Sinaï model)
- Polymer chains
- Interface fluctuations
  (plate tectonics)



- Ballot problem
  (enumerative aspects)
- Fluctuation of random walks
  (survival, first passage times)
- Processes in cones
- Gaussian processes
  (Brownian motion between two barriers)
- Occupation time



**Persistence exponent**

$$P(n) \sim n^{-\alpha}$$

# Sparre Andersen theorem

## Random walk

$$S(n) = X(1) + \cdots + X(n),$$

with $X(i)$ iid and real

## Persistence (or survival) probability

$$\mathbb{P}(S(1) \geqslant 0, \ldots, S(n) \geqslant 0) = \mathbb{P}(\tau > n),$$

with $\tau = \inf\{p \geqslant 0 : S(p) < 0\}$

## Sparre Andersen universality result

If $X(i)$ is symmetric and without any atom,

$$\mathbb{P}(S(1) \geqslant 0, \ldots, S(n) \geqslant 0) = \frac{1}{4^n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$$

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$$\sum_{n=0}^{\infty} \mathbb{P}(S(1) \geqslant 0, \ldots, S(n) \geqslant 0) z^n = \exp\left(\sum_{n=1}^{\infty} \mathbb{P}(S(n) \leqslant 0) \frac{z^n}{n}\right)$$

# Auto-regressive processes: definition

HENRI LEBESGUE

## A classical model in statistics (ARMA)

Recursive definition

$$A(n) = \sum_{i=1}^{p} \theta(i) A(n-i) + X(n)$$

## In this talk

$$A(n) = \theta A(n-1) + X(n)$$

One-parameter generalisation of random walks

$$A(n) = \theta^{n-1} X(1) + \theta^{n-2} X(2) + \cdots + \theta X(n-1) + X(n)$$

## Perpetuity and duality

$$\widetilde{A}(n) = X(1) + \theta X(2) + \cdots + \theta^{n-2} X(n-1) + \theta^{n-1} X(n)$$

# AR(1) processes: questions...

Describing the persistence probability (function of $\theta$ and $n$)

$$p_n(\theta) = \mathbb{P}(A(1) \geqslant 0, \dots, A(n) \geqslant 0)$$

- Sparre Andersen theorem in the AR(1) case?
- Asymptotics as $n \to \infty$?
- Closed-form expressions?
- Symmetries in $\theta$?

**Some particular cases**

$$\theta \in \{-\infty, -1, 0, 1, \infty\}$$

# Sparre Andersen AR(1) version

**A duality $\theta \leftrightarrow \frac{1}{\theta}$**

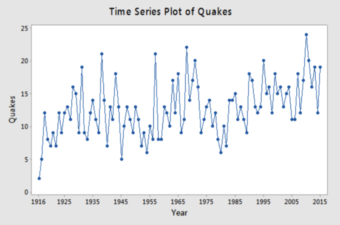
If $\theta > 0$ and $n \geqslant 0$,

$$\sum_{k=0}^{n} p_k(\theta) p_{n-k}(1/\theta) = 1$$

$$\left(\sum_{n=0}^{\infty} p_n(\theta) z^n\right) \left(\sum_{n=0}^{\infty} p_n(1/\theta) z^n\right) = \frac{1}{1-z}$$

**Idea of proof**

Pathwise decomposition at global minimum time



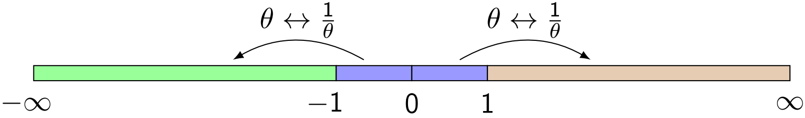
# A second duality

## Once again the duality $\theta \leftrightarrow \frac{1}{\theta} \cdots$

If $\theta < 0$ and $n \geqslant 1$,

$$\sum_{k=0}^{n} (-1)^k p_k(\theta) p_{n-k}(1/\theta) = 0$$

$$\left(\sum_{n=0}^{\infty} p_n(\theta) z^n\right)\left(\sum_{n=0}^{\infty} p_n(1/\theta)(-z)^n\right) = 1$$



# Central regime $\theta \in [-1, 1]$

Assume the $X(i)$ admit a density $f(x)$ over $\mathbb{R}$

## A naive formula

$$p_n(\theta) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1) \cdots f(x_n) \mathbf{1}_{x_1 \geqslant 0} \mathbf{1}_{x_2 + \theta x_1 \geqslant 0} \cdots dx_n \cdots dx_1$$

- Volume of polytopes
- Recurrence formulas
- Addition formulas (exponential function)
- Uniform distribution $f(x) = \frac{1}{2} \mathbf{1}_{[-1,1]}(x)$

# Uniform distribution on $[-1, 1]$

## Some particular cases...

$$p_1(\theta) = \mathbb{P}(X(1) \geqslant 0) = \frac{1}{2}$$

$$p_2(\theta) = \mathbb{P}(X(1) \geqslant 0, X(2) + \theta X(1) \geqslant 0)$$

$$= \frac{1}{4} \int_0^1 \int_{-(1 \wedge \theta x_1)}^1 dx_2 dx_1$$

$$= \frac{\theta + 2}{2^2 2!}$$

$$p_3(\theta) = \frac{\theta^3 + 3\theta^2 + 6\theta + 6}{2^3 3!}$$

# Uniform distribution on $[-1, 1]$

## ...and a theorem

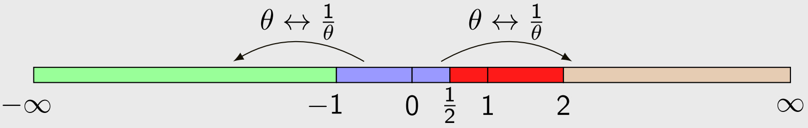
For $n \geqslant 0$ and $\theta \in [-1, \frac{1}{2}]$,

$$p_n(\theta) = \frac{J_{n+1}(\theta)}{2^n n!}$$

## Combinatorial polynomials

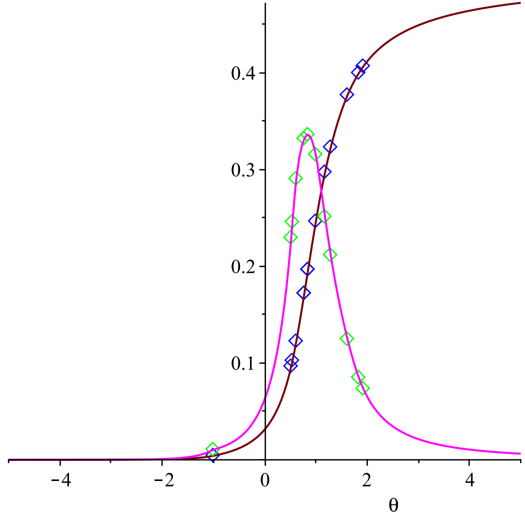
$$J_n(\theta) = n\text{-th Mallows-Riordan polynomials}$$

## Help



# Uniform distribution on $[-1, 1]$

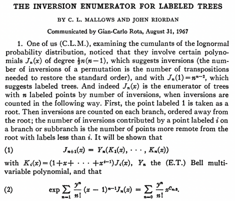
Graph of $p_5(\theta)$



# Mallows-Riordan by Mallows-Riordan

**Deformed exponential function**

$$\sum_{n=0}^{\infty} \theta^{\frac{n(n-1)}{2}} \frac{z^n}{n!} = 1 + z + \theta \frac{z^2}{2} + \cdots$$



THE INVERSION ENUMERATOR FOR LABELED TREES

BY C. L. MALLOWS AND JOHN RIORDAN

Communicated by Gian-Carlo Rota, August 31, 1967

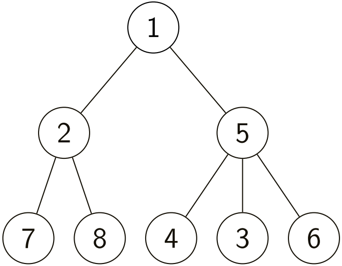
1. One of us (C.L.M.), examining the cumulants of the lognormal probability distribution, noticed that they involve certain polynomials $J_n(x)$ of degree $\frac{1}{2}n(n-1)$, which suggests inversions (the number of inversions of a permutation is the number of transpositions needed to restore the standard order), and with $J_n(1)=n^{n-2}$, which suggests labeled trees. And indeed $J_n(x)$ is the enumerator of trees with $n$ labeled points by number of inversions, when inversions are counted in the following way. First, the point labeled 1 is taken as a root. Then inversions are counted on each branch, ordered away from the root; the number of inversions contributed by a point labeled $i$ on a branch or subbranch is the number of points more remote from the root with labels less than $i$. It will be shown that

$$J_{n+1}(x) = Y_n(K_1(x), \cdots, K_n(x)) \tag{1}$$

with $K_i(x)=(1+x+\cdots+x^{i-1})J_i(x)$, $Y_n$ the (E.T.) Bell multivariable polynomial, and that

$$\exp \sum_{n=1} \frac{y^n}{n!}(x-1)^{n-1}J_n(x) = \sum_{n=0} \frac{y^n}{n!} x^{C_{n,2}}. \tag{2}$$

$$\log\left(\sum_{n=0}^{\infty} \theta^{\frac{n(n-1)}{2}} \frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} (\theta-1)^{n-1} J_n(\theta) \frac{z^n}{n!}$$




$$n^{n-2} = J_n(1) = \sum \cdots$$

An alternative definition...

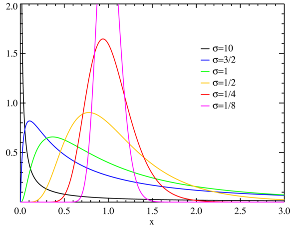
$$\sum_{n=0}^{\infty} J_{n+1}(\theta)\frac{z^n}{n!} = \frac{\sum_{n=0}^{\infty} \frac{(1+\theta+\cdots+\theta^n)^n}{\theta^{\frac{n(n+1)}{2}}} \frac{z^n}{n!}}{\sum_{n=0}^{\infty} \frac{(1+\theta+\cdots+\theta^{n-1})^n}{\theta^{\frac{n(n+1)}{2}}} \frac{z^n}{n!}}$$

...associated to a recurrence

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(\theta-1)^k}{\theta^{\frac{k(k-1)}{2}-kn}} \frac{(\theta^{n-k}-1)^{n-k}}{(\theta^{n+1}-1)^n} J_{k+1}(\theta) = 1$$

**The log-normal distribution**

If $X \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$, then $Y = e^X$ is log-normal




Cumulants generating function

$$\log \mathbb{E}(e^{tY}) = \log\left(\sum_{n=0}^{\infty} e^{n\mu + n^2\sigma^2/2} \frac{t^n}{n!}\right)$$

$$= \log\left(\sum_{n=0}^{\infty} \theta^{\frac{n(n-1)}{2}} \frac{z^n}{n!}\right)$$

## Cayley's polytope (and friends)

The volume of

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : 1 \leqslant x_1 \leqslant 2, 1 \leqslant x_2 \leqslant 2x_1, \ldots, 1 \leqslant x_n \leqslant 2x_{n-1}\}$$

is given by $\frac{J_{n+1}(2)}{n!}$

$$J_{n+1}(\theta) = \frac{n!}{(\theta - 1)^n} \int_1^{\theta} \int_1^{\theta x_1} \cdots \int_1^{\theta x_{n-1}} dx_n \cdots dx_1$$

# Idea of proof

Pour $\theta \in [-1, \frac{1}{2}]$,

$$p_n(\theta) = \frac{1}{2^n} \int_0^1 \cdots \int_{-(\theta x_{n-1} + \cdots + \theta^{n-1} x_1)}^1 dx_n \cdots dx_1$$

- Recurrence à la Gessel
- Order statistics

## Recall of the statement

For $n \geqslant 0$ and $\theta \in [-1, \frac{1}{2}]$,

$$p_n(\theta) = \mathbb{P}(X(1) \geqslant 0, X(2) + \theta X(1) \geqslant 0, \dots) = \frac{J_{n+1}(\theta)}{2^n n!}$$

- $\theta = 0 \longrightarrow J_{n+1}(0) = n!$
  (bijection between ordered labeled trees and permutations)
- $\theta = 1 \longrightarrow J_{n+1}(1) = (n+1)^{n-1}$
  (Cayley's formula for the number of labeled trees)
- $\theta = -1 \longrightarrow J_{n+1}(-1) = A_n$
  ($n$-th zig-zag number $\frac{1+\sin x}{\cos x} = 1 + \sum_{n\geqslant 1} \frac{A_n}{n!} x^n$)
- $\theta = \frac{1}{2} \longrightarrow 2^{\frac{n(n-1)}{2}} J_{n+1}(\frac{1}{2})$
  (acyclic initially connected digraphs with $n+1$ vertices)

## Cayley's polytope

The volume of

$$\{1 \leqslant x_1 \leqslant \theta, 1 \leqslant x_2 \leqslant \theta x_1, \ldots, 1 \leqslant x_n \leqslant \theta x_{n-1}\}$$

is given by $\frac{J_{n+1}(\theta)}{n!}$

## Tutte's polytope $\mathbf{T}(q,t)$

$$\{x_n \geqslant 1-q,\ qx_j \leqslant q(1+t)x_{j-1} - t(1-q)(1-x_{i-1}),\ 1 \leqslant i \leqslant j \leqslant n\}$$

with $q \in (0,1]$ and $t \geqslant 0$

Konvalinka-Pak formula for the volume of $\mathbf{T}(q,t)$ in terms of Tutte's polynomial of the complete graph (Mallows-Riordan like)

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## Zeros of the deformed exponential function

$z_\theta$ smallest real $> 0$ such that

$$\sum_{n=0}^{\infty} \theta^{\frac{n(n-1)}{2}} \frac{(-z)^n}{n!} = 0$$

**Big Conjecture #1.** All roots of $F(\cdot, y)$ are simple for $|y| < 1$. [and also for $|y| = 1$, I suspect]

**Consequence of Big Conjecture #1.** Each root $x_k(y)$ is analytic in $|y| < 1$.

**Big Conjecture #2.** The roots of $F(\cdot, y)$ are non-crossing *in modulus* for $|y| < 1$:

$$|x_0(y)| < |x_1(y)| < |x_2(y)| < \dots$$

[and also for $|y| = 1$, I suspect]

**Consequence of Big Conjecture #2.** The roots are actually separated in modulus by a factor at least $|y|$, i.e.

$$|x_k(y)| < |y|\,|x_{k+1}(y)| \qquad \text{for all } k \geq 0$$

## When $n \to \infty$

For $\theta \in [-1, \frac{1}{2}]$,

$$p_n(\theta) \sim \frac{1}{z_\theta (2(1-\theta) z_\theta)^n}$$

# Asymptotics $\theta \to \infty$

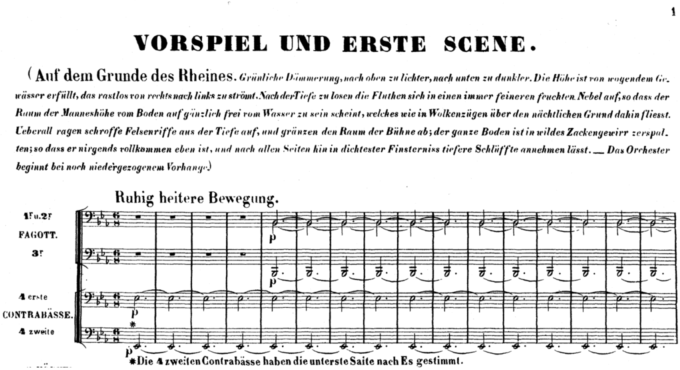
**A mysterious series**

$$\lim_{n\to\infty} p_n(\theta) = \frac{1}{2} - \left(\frac{1}{8\theta} + \frac{1}{16\theta^2} + \frac{5}{96\theta^3} + \frac{1}{24\theta^4} + \frac{5}{128\theta^5} + \cdots\right)$$

- Radius of convergence?
- Regularity of coefficients $\frac{1}{2}, \frac{5}{6}, \frac{4}{5}, \frac{15}{16}, \frac{14}{15}, \frac{27}{28}, \frac{107}{108}, \frac{641}{642}, \ldots$

# The Persistence of Memory

# Wagner's persistence Leitmotiv



# Sisyphus