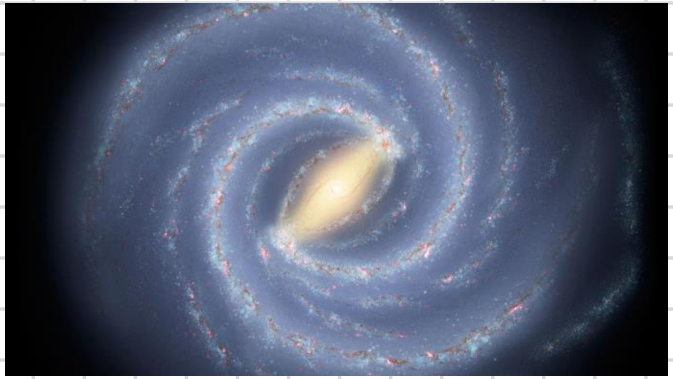
IHP (Dec 4 , 2023)

What is beyond D-Finite?

Igor Pak, UCLA

Workshop: ***Computer Algebra for Functional Equations in Combinatorics and Physics***

Deep Questions:



First Question: *What is a **formula**?*

Second Question: *Does* $\mathcal{A}(t) = \sum_n a_n t^n$ *have* *a formula?*

Third Question: *What is a **combinatorial object**?*

Fourth Question: *Does* $a_n = [t^n]\mathcal{A}(t)$ *count any combinatorial objects?*

Classes of combinatorial sequences:

(1) ***rational*** GF $\mathcal{A}(t) = P(t)/Q(t), \;\; P, Q \in \mathbb{Z}[t].$

E.g. $\;\; a_n := \mathrm{Fib}(n), \;\; \mathcal{A}(t) = 1/(1 - t - t^2).$

(2) ***algebraic*** GF $c_0\mathcal{A}^k + c_1\mathcal{A}^{k-1} + \ldots + c_k = 0, \; c_i \in \mathbb{Z}[t].$

E.g. $\;\; a_n := \mathrm{Cat}(n), \;\; \mathcal{A}(t) = (1 - \sqrt{1 - 4t})/2t.$

Classes of combinatorial sequences:

(3) ***diagonals*** $\mathcal{A}(t) = \operatorname{diag} P/Q, \quad P, Q \in \mathbb{Z}[x_1, \ldots, x_k].$

$$\mathcal{B} = \sum_{(i_1,\ldots,i_k)} b(i_1, \ldots, i_k) x_1^{i_1} \cdots x_n^{i_k} \quad \Longrightarrow \quad \operatorname{diag} \mathcal{B} := \sum_{n=0}^{\infty} b(n, \ldots, n) t^n$$

E.g. *Delannoy numbers* $\{D_n\}$ and *Apéry numbers* $\{A_n\}$

$$D_n := \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} \quad \text{and} \quad A_n := \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3$$

Classes of combinatorial sequences:

(4) ***D-finite*** GF $c_0\mathcal{A} + c_1\mathcal{A}' + \ldots + c_k\mathcal{A}^{(k)} = 0$, $c_i \in \mathbb{Z}[t]$.

E.g. $a_n :=$ # involutions in S_n, $a_n = a_{n-1} + (n-1)a_{n-2}$.

The sequences $\{a_n\}$ are called ***P-recursive***

(5) ***D-algebraic*** GF $Q\big(t, \mathcal{A}, \mathcal{A}', \ldots, \mathcal{A}^{(k)}\big) = 0$, $Q \in \mathbb{Z}[t, x_0, x_1, \ldots, x_k]$

E.g. $a_n = \#\{\sigma(1) < \sigma(2) > \sigma(3) < \ldots \in S_n\}$, $\mathcal{A}'' = \mathcal{A} \cdot \mathcal{A}'$.

Also $p(n) =$ # integer partitions of n. Then $F(t) = \sum_n p(n)t^n$ satisfies:

$$4F^3F'' + 5tF^3F''' + t^2F^3F^{(4)} - 16F^2(F')^2 - 15tF^2F'F'' - 39t^2F^2(F'')^2 \\ + 20t^2F^2F'F''' + 10tF(F')^3 + 12t^2F(F')^2F'' + 6t^2(F')^4 = 0.$$

(Jacobi, Ramanujan)

Classes of combinatorial sequences:

$$\textit{Rational} \subsetneq \textit{Algebraic} \subsetneq \textit{Diagonal} \subsetneq \textit{D-finite} \subsetneq \textit{D-algebraic}$$

Question: What are the *positive analogues* of these classes? What classes are we missing?

$\mathbb{N}$-Rational

Definition 1. Let $\mathcal{R}_1$ be the smallest set of functions $F(t)$ which satisfies

(1) $0, t \in \mathcal{R}_1$,

(2) $F, G \in \mathcal{R}_1 \implies F + G, F \cdot G \in \mathcal{R}_1$,

(3) $F \in \mathcal{R}_1,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_1$.

Note that all $F \in \mathcal{R}_1$ satisfy: $F \in \mathbb{N}[[x]]$, and $F = P/Q$, for some $P, Q \in \mathbb{Z}[t]$.

Definition 1′. Let $\mathcal{R}_1$ be the class of $\mathcal{A} = \sum_n a_n t^n$ where a_n is the number of $s - t$ paths of length n in graph G.

For example,

$$\frac{1}{1 - x - x^2} \quad \text{and} \quad \frac{x^3}{(1 - x)^4} \in \mathcal{R}_1$$

Theorem [Schützenberger, 1962]
Def. 1 $\Leftrightarrow$ Def. 1′

$\mathbb{N}$-Rational (complete success)

Theorem [Berstel'71, Soittola'76]
Complete analytic characterization.

Corollary: $\mathit{Rational} = \mathcal{R}_1 - \mathcal{R}_1$

Note: Koutschan (2008) has a code.

$$\frac{t+5t^2}{1+t-5t^2-125t^3} \notin \mathcal{R}_1 \quad , \quad \frac{1+t}{1+t-2t^2-3t^3} \in \mathcal{R}_1$$

[Gessel, 2003]

Theorem [Berstel–Reutenauer, 2008]
Every $A(t) \in \mathcal{R}_1$ has *star height* at most two.

$\mathbb{N}$-Algebraic

Definition 2. Class of $A_0 = \sum_n a_n t^n$, where

$$\begin{cases} A_0(t) &= f_0(A_0, \ldots, A_k, t) \\ \vdots & \\ A_k(t) &= f_k(A_0, \ldots, A_k, t) \end{cases} \quad \text{and} \quad f_i \in \mathbb{N}[x_0, \ldots, x_k, t] \quad \text{are well-posed}$$

Definition 2′. Class of GFs for the number of accepting paths of PDA

Example: $A(t) = 1 + t\,A(t)^2$

$A(t) = \mathrm{Cat}(t) \in \mathbb{N}$-Algebraic

Example: The following $A(t) = \sum_n a_n t^n$ counts number of certain bilabeled trees:

$$\begin{cases} A(t) &= 1 + t\,A^2 B + t^2 B^3 \\ B(t) &= 1 + t\,AB + t^2 A^2 \end{cases} \quad \text{and} \quad A(0) = B(0) = 1$$

$\mathbb{N}$-Algebraic

Theorem [Banderier–Drmota, 2015]

Asymptotic characterization: $A(t) \in \mathbb{N}\text{-Algebraic}, \; A(t) = \sum_n a_n t^n$

$\implies a_n \sim C n^{\alpha} \lambda^n$, where $\alpha \in \left\{\frac{m}{2^k}\right\}$ and $\alpha > -1$ or $\alpha = -1 - \frac{1}{2^k}$ for some $k \geq 1$

Corollary: *Tutte's GFs for the number of rooted triangulations is not* $\mathbb{N}$*-Algebraic:*

$$T_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \sim C n^{-5/2} \left(\frac{64}{27}\right)^n$$

Proposition [Banderier–Drmota, 2015]: Algebraic = $\mathbb{N}$-Algebraic − $\mathbb{N}$-Algebraic

Question: Can we give a better description of this class?

$\mathbb{N}$-Diagonals = Positive Binomial Sums

Definition 3. Let $\mathcal{R}_k$ be the smallest set of functions $F(x_1, \ldots, x_k)$ which satisfies

(1) $0, x_i \in \mathcal{R}_k$,

(2) $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k$,

(3) $F \in \mathcal{R}_1,\ F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_k$.

Let $\mathcal{F}$ be the set of diagonals of $\mathcal{R}_k$

Definition 3′. Let $\mathcal{F}$ be the class of $\mathcal{A} = \sum_n a_n t^n$ where a_n is the number of $s - t$ paths of length kn in k-edge colored graph G with equal numbers of each color.

Definition 3″. Let $\mathcal{F}$ be the class of GFs of *binomial sums* with coefficients in $\mathbb{N}$

$$D_n = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k}, \qquad A_n = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3$$

Theorem [Garrabrant-P., 2014]

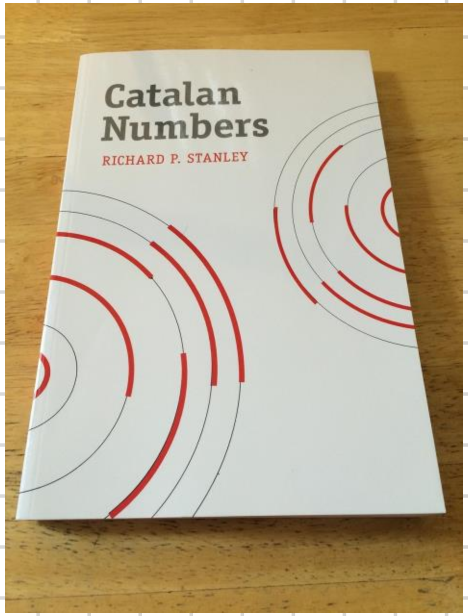
Def. 3 $\Leftrightarrow$ Def. 3′ $\Leftrightarrow$ Def. 3″

$\mathbb{N}$-Diagonals = Positive Binomial Sums

Corollary [Bostan–Lairez–Salvy'16]: Diagonals = $\mathbb{N}$-Diagonals − $\mathbb{N}$-Diagonals

Bad news: *Open whether* $\mathbb{N}$-*Algebraic* $\subseteq$ $\mathbb{N}$-*Diagonals*

Conjecture [Garrabrant-P., 2014]: $\mathrm{Cat}(t) \notin \mathbb{N}$-Diagonal



Note: One of only two open problems in Stanley's *Catalan Numbers* book.

Note: $\dfrac{1-(1-2z)^{1/4}\sqrt{2z\sqrt{1+2z}+\sqrt{1-2z}}}{2z} \notin \mathbb{N}$-Diagonal is potentially easier.

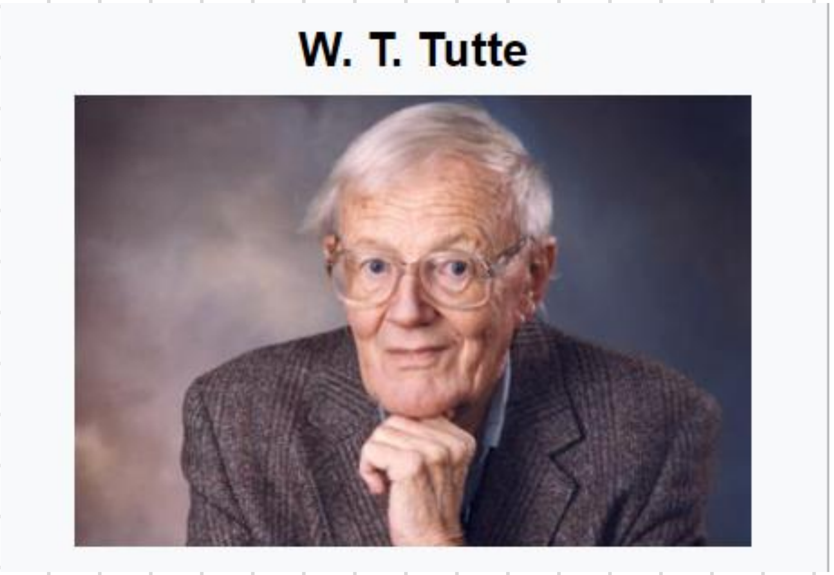
Question: Can we give an asymptotic characterization of this class?

$\mathbb{N}$-D-Finite = ???

$\mathbb{N}$-D-Algebraic = $\mathbb{T}$-Algebraic

Definition 4 [Drmota–P, 2023+] Class of $A_0 = \sum_n a_n t^n$, where

$$\begin{cases} A_0' = f_0(A_0, \ldots, A_k, t) \\ \vdots \\ A_k' = f_k(A_0, \ldots, A_k, t) \end{cases} \quad \text{and} \quad f_i \in \mathcal{R}_{k+2}$$



$\mathbb{N}$-D-Algebraic $=$ $\mathbb{T}$-Algebraic

Definition 4 [Drmota–P, 2023+] Class of $A_0 = \sum_n a_n t^n$, where

$$\begin{cases} A_0' = f_0(A_0, \ldots, A_k, t) \\ \vdots \\ A_k' = f_k(A_0, \ldots, A_k, t) \end{cases} \quad \text{and} \quad f_i \in \mathcal{R}_{k+2}$$

Example: $A(t) = \tan(t) + \sec(t)$ is $\mathbb{T}$-algebraic since it can be written as

$$\begin{cases} A' = B \\ B' = A \cdot B \end{cases}$$

Motivating Example: Tutte's GFs for the numbers T_n of rooted plane triangulations and numbers M_n of rooted plane maps are $\mathbb{T}$-Algebraic. Here

$$M_n = \frac{2(2n!)3^n}{n!(n+2)!} \qquad \text{and} \qquad \sum_{n=0}^{\infty} M_n t^n = -\frac{1 - 18t - (1-12t)^{3/2}}{54t^2} \notin \mathbb{N}\text{-}Algebraic$$

$\mathbb{N}$-D-Algebraic = $\mathbb{T}$-Algebraic

Theorem [Drmota–P, 2023+] *Let $A(t)$ be a $\mathbb{T}$-Algebraic function. Then $A(t)$ is either an entire function or has finitely many singularities on the circle of convergence $|t| = \rho$, where $0 < \rho < \infty$.*

Corollary: *Partition and theta functions are D-Algebraic, but not $\mathbb{T}$-algebraic*

$$P(t) = \prod_{k=1}^{\infty} \frac{1}{1-t^k} = \sum_{n=0}^{\infty} p(n) t^n \qquad \text{and} \qquad \Theta(t) = \sum_{n=-\infty}^{\infty} t^{n^2}$$

Corollary: *D-Algebraic $\neq$ $\mathbb{T}$-Algebraic $-$ $\mathbb{T}$-Algebraic*

Open Problem: Are there any other necessary conditions?

$\mathbb{N}$-D-Algebraic = $\mathbb{T}$-Algebraic

Theorem [Kauers–Koutschan–Zeilberger'09] and [Banderier–Drmota'15]

The GF $\mathcal{G}(t) = \sum_{n=0}^{\infty} G_n t^n$ *for* Gessel numbers *is Algebraic, but not* $\mathbb{N}$*-Algebraic*

$$\text{where} \quad G_n = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}, \quad (z)_m := z(z+1)\cdots(z+m-1).$$

Open Problem: *Decide if* $\mathcal{G}(t)$ *is* $\mathbb{T}$*-Algebraic.*

A-Recursive

Definition:

$\{a_n\}$ is ***P-recursive*** if $c_0 a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0,\ c_i \in \mathbb{Z}[n]$, for all $n \in \mathbb{N}$.

$\{a_n\}$ is ***A-recursive*** if $Q(a_n, a_{n-1}, \ldots, a_{n-k}, n) = 0,\ Q \in \mathbb{Z}[x_0, \ldots, x_n, n]$, for all $n \in \mathbb{N}$.

Two Open Problems:

- Build a theory of such sequences
- Find a nice positive integer subclass of such sequences

A-Recursive

A006720 Somos-4 sequence: a(0)=a(1)=a(2)=a(3)=1; for n >= 4, a(n) = (a(n-1) * a(n-3) + a(n-2)^2) / a(n-4). [88]
(Formerly M0857)

1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, 126742987, 1687054711, 47301104551, 1123424582771, 32606721084786, 1662315215971057, 61958046554226593, 42579988448335457, 334806306946199122193, 23385756731869683322514, 3416372868727801226636179 (list; graph; refs; listen; history; text; internal format)

Theorem [Gale'91] $a_n \in \mathbb{N}$

Theorem [Fomin–Zelevinsky'02]

Advanced generalization via *Laurent phenomenon*

Theorem [Speyer'06]

$\{a_n\}$ has a combinatorial interpretation

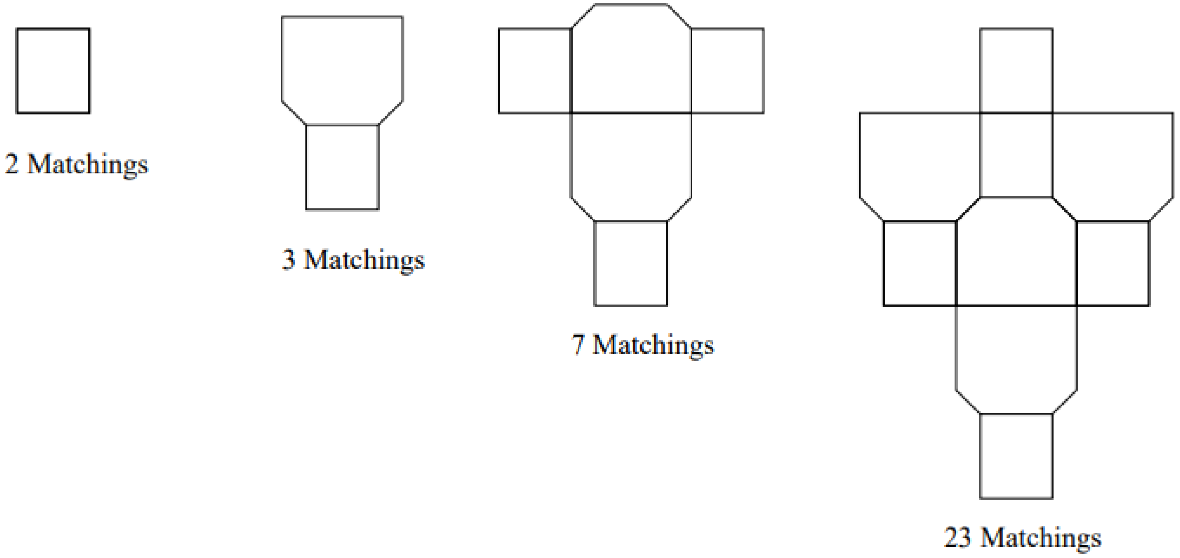
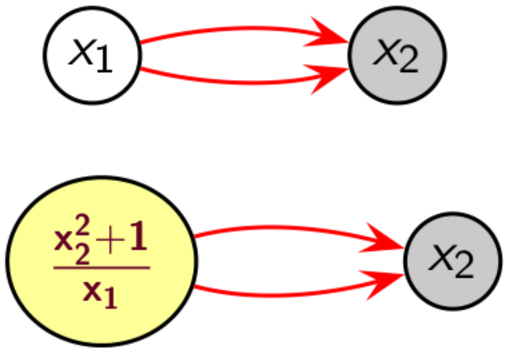
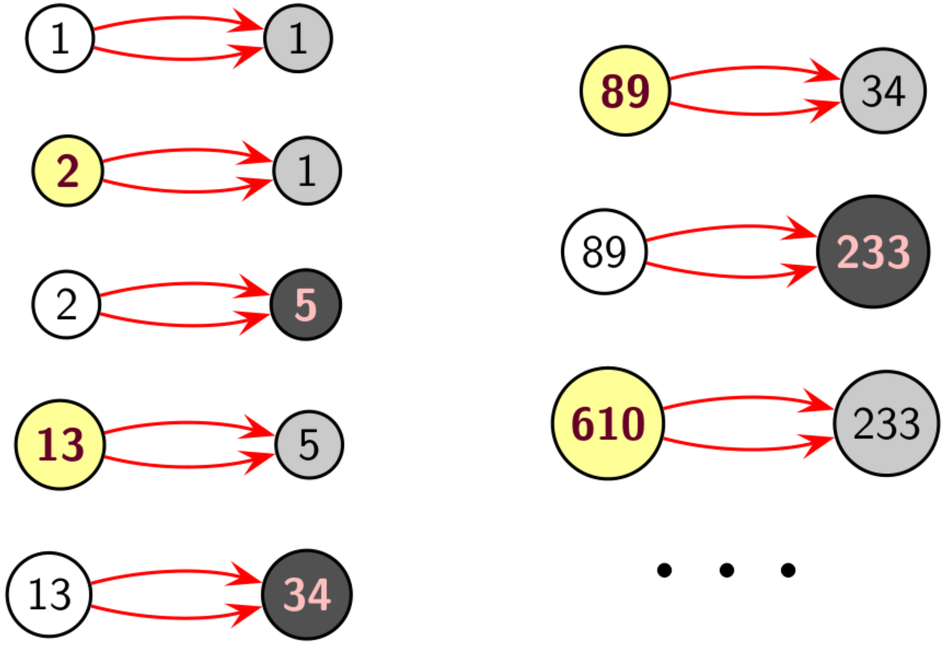


Figure 11: The First Four Nontrivial Somos-4 Graphs

A-Recursive

Example:





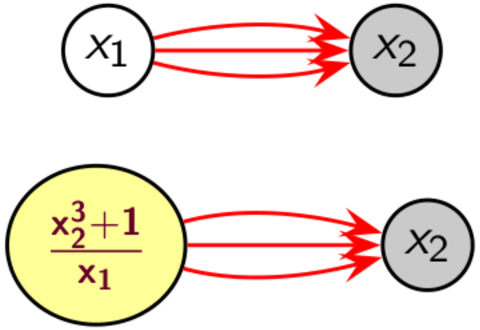
Zamolodchikov periodicity and integrability

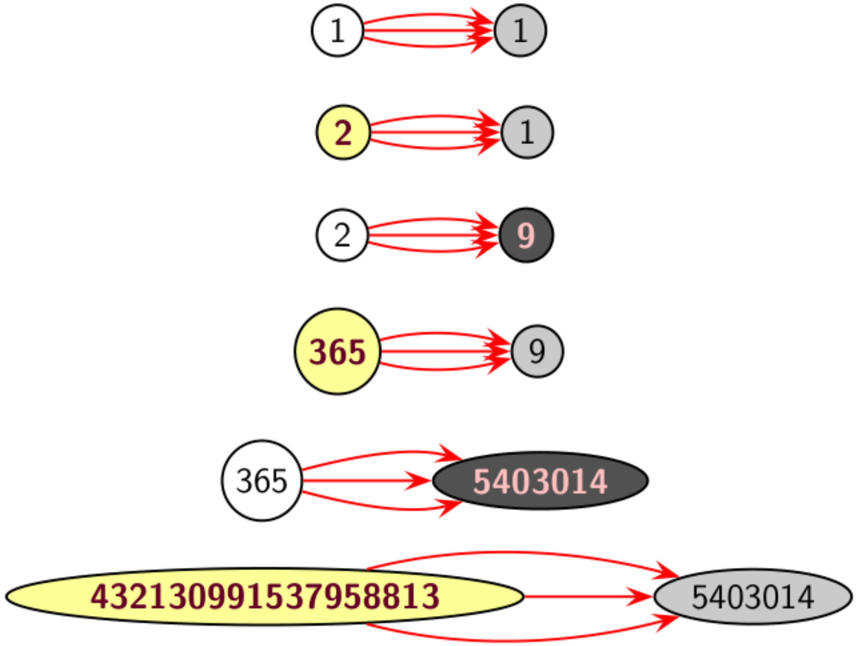
Pavel Galashin

Odd Fibonacci numbers $\mathrm{Fib}(2n-1)$

A-Recursive

Example:





Zamolodchikov periodicity and integrability

Pavel Galashin

A003818 a(1)=a(2)=1, a(n+1) = (a(n)^3 +1)/a(n-1).

1, 1, 2, 9, 365, 5403014, 43213099153795881 3,

graph; refs; listen; history; text; internal format)

OFFSET 1,3

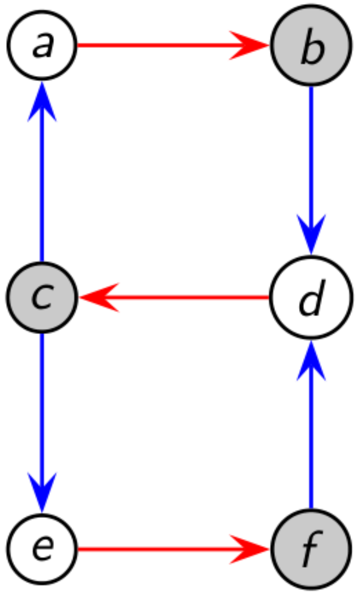
COMMENTS The term a(9) has 121 digits. - Harvey P. Dale,

A-Recursive

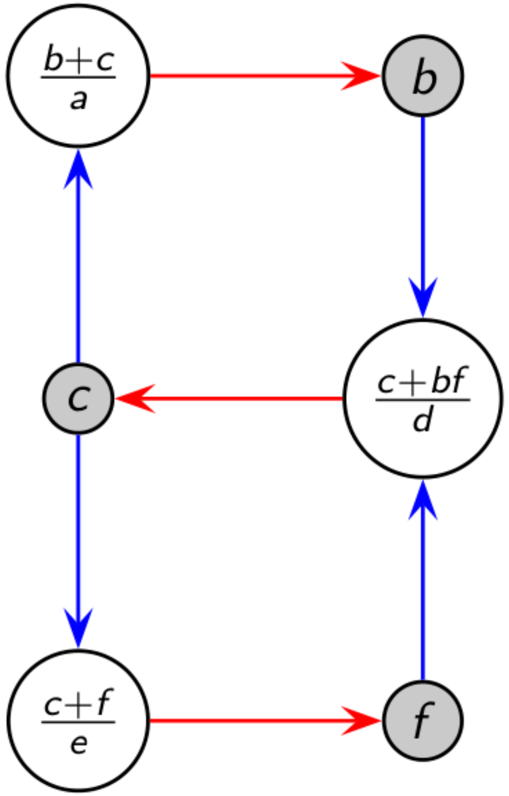
Zamolodchikov periodicity and integrability

Pavel Galashin

Bipartite T-system



$\longrightarrow$



A-Recursive

Zamolodchikov periodicity and integrability

Bipartite T-system

Pavel Galashin

Conjecture [Galashin–Pylyavskyy'19]

Except for finite number of (known) examples and series, $\{a_n\}$ grow doubly exponentially.

Note: Conjecture implies no combinatorial interpretation in #P. But maybe $\#EXP$?

Open Problem: Build a theory to show that $\{a_n\}$ *do not* have a combinatorial interpretation for general A-Recursive functions.

Thank you!