COMBINATORICS AND TRANSCENDENCE

Applications of inhomogeneous order 1 iterative functional equations

SFU

Marni Mishna

Simon Fraser University

with Lucia Di VIZIO and Gwladys FERNANDES

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Computer Algebra for Functional Equations in Combinatorics and Physics

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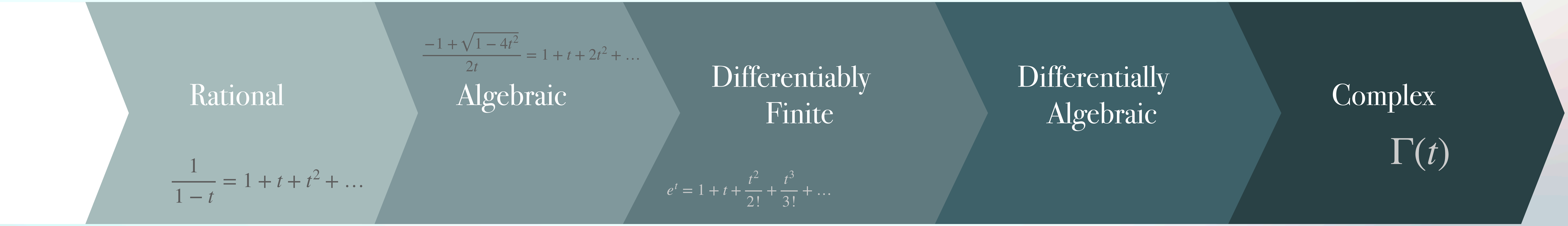
I work with gratitude on the unceded Traditional Coast Salish Lands including the Tsleil-Waututh (səl̓ilw̓ətaʔɬ), Squamish (Sḵwx̱wú7mesh Úxwumixw) and Musqueam (xʷməθkʷəy̓əm) Nations.

Motivation

Classification



D-finite : Satisfies a linear DE with polynomial coefficients. AKA Holonomic

D-Algebraic: Satisfies a polynomial DE.

D-Transcendental : NOT differentially algebraic

Combinatorial classes

A *combinatorial class* is a set equipped with a size function. *Ordinary Generating Functions (OGF)* encode enumerative data as integer coefficients of formal power series.

$$\mathscr{C} \implies C(t) := \sum_{n=0}^{\infty} \left|\mathscr{C}_n\right| t^n$$

TYPE OF CLASS	TYPICAL EXAMPLES	NATURE OF OGF
Finite class		Polynomial
Iterative grammar specification	Recognizable by a finite automaton Regular language, eg. Fibonacci	Rational function
Recursively grammar specification	Trees, Catalan classes, Maps	Algebraic function
?	Shuffles of Dyck Paths k-regular labelled graphs SYT of bounded height	D-finite
?	Families of decorated maps	D-algebraic

Rational

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots$$

Unconstrained simple walks

Regular languages

Fibonacci numbers

Algebraic

$$\frac{-1+\sqrt{1-4t^2}}{2t} = 1 + t + 2t^2 + \ldots$$

Walks in half plane

Excursions on Cayley graphs of free products of finite groups

Context free languages

132- avoiding permutations

2-3 Trees

Catalan numbers

Differentiably Finite

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$$

Simple walks in quarter plane

Constrained regular languages

Baxter permutations

K-regular graphs

Differentially Algebraic

Tree decorated maps

Bell numbers (EGF)

Complex

Simple walks in “transcendental” region

Excursions on Sierpinski gasket

Complete 2-3 Trees

Bell numbers (OGF)

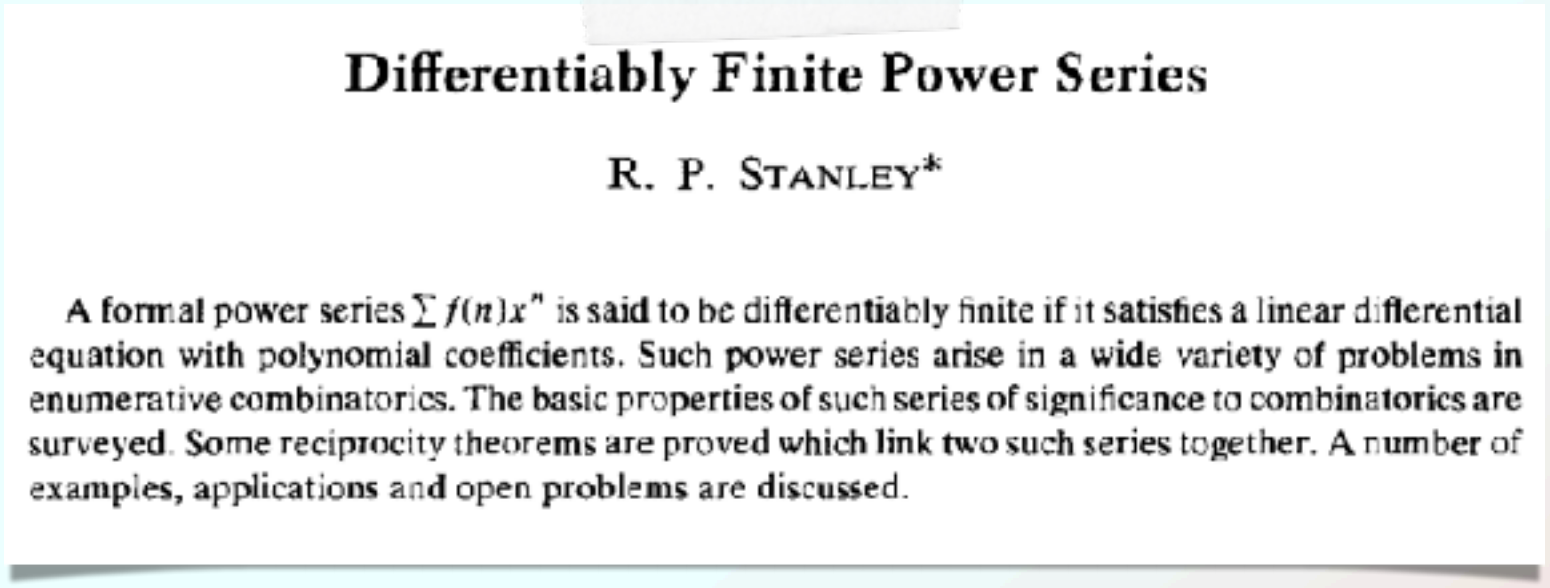
Applications of classification

- **Theoretical Computer Science**
  The following language is *not unambiguously context free*:
  $\mathscr{C} = \{w \in \{a,b,c\}^* \mid |w|_a \neq |w|_b \text{ or } |w|_a \neq |w|_c\}$ **because its generating function** $C(t) = \sum_n c_n t^n$ **is not algebraic.** *(Flajolet 1988)*

- **Group Theory**
  Let G be a *finitely generated amenable group* that is not nilpotent-by-finite and let S be a finite symmetric generating set for G. **The OGF for walks starting and ending at the origin on the Cayley Graph X(G;S) is not D-finite**. *(Bell, M. 2021)*
  Gives a strategy to determine if Thompson's Group F is an amenable group. *(Elvey-Price, Guttmann 2019)*

D-finite series in combinatorics

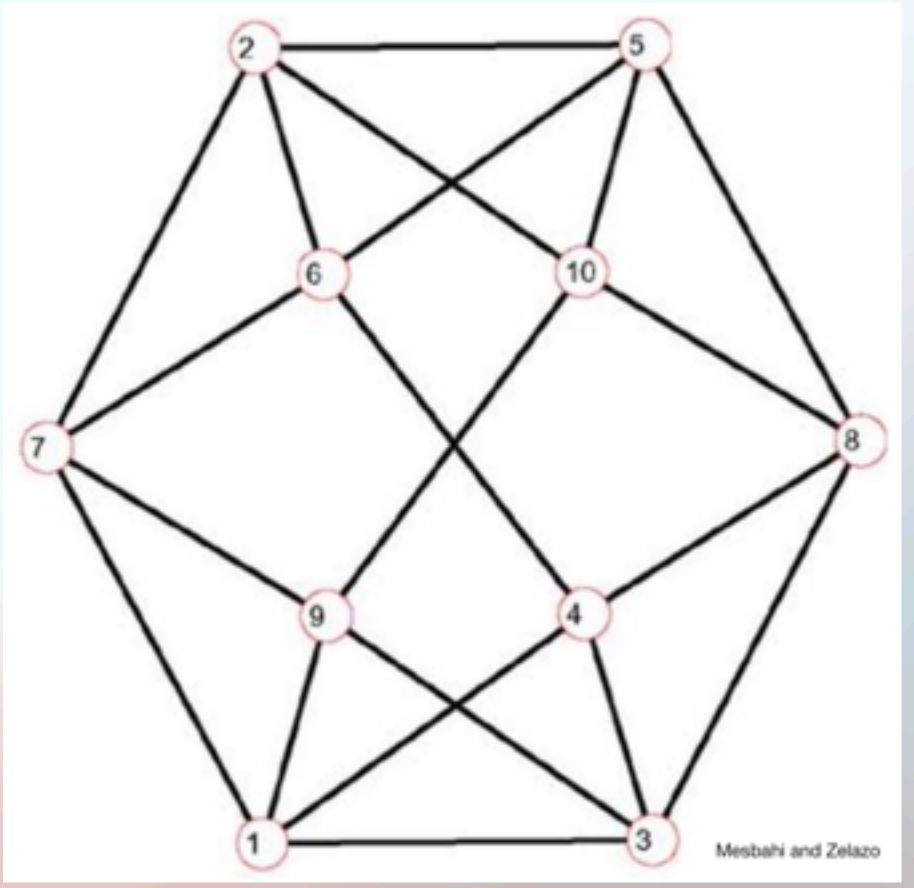
- Richard Stanley's 1980 article plants several seeds, many of which were considered by Gessel (1990):
  - Baxter permutations
  - Young Tableaux of bounded height
  - *k*-regular graphs
- Cited by > 500
  > 12 000 hits to {Holonomic | D-finite }+combinatorics
- Most D-finite classes are in some bijection with a class of *lattice walks*



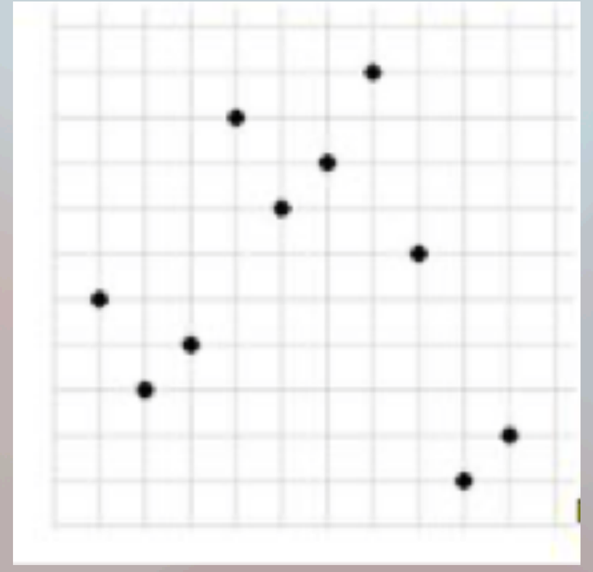
Differentiably Finite Power Series

R. P. STANLEY*

A formal power series $\sum f(n)x^n$ is said to be differentiably finite if it satisfies a linear differential equation with polynomial coefficients. Such power series arise in a wide variety of problems in enumerative combinatorics. The basic properties of such series of significance to combinatorics are surveyed. Some reciprocity theorems are proved which link two such series together. A number of examples, applications and open problems are discussed.



1	3	5
2	4	8
6	9	10
7		



Why the interest in D-finite series?

“Almost anything is non-holonomic unless it is holonomic by design.”

- Flajolet, Gerhold & Salvy, 2005

- Closure properties mirror combinatorial actions
- The differential equation is a useful data structure for both reasoning and computation
- Clear proof strategies
- **Conjecture** *(Christol, 1990):* If a series with non-negative integer coefficients and a positive, finite, radius of convergence is furthermore D-finite, then it can be written as the diagonal of a multivariate rational function.
- D-algebraic series are much more difficult to manipulate and characterize.

“Classic” Strategies

To show a series is D-finite:

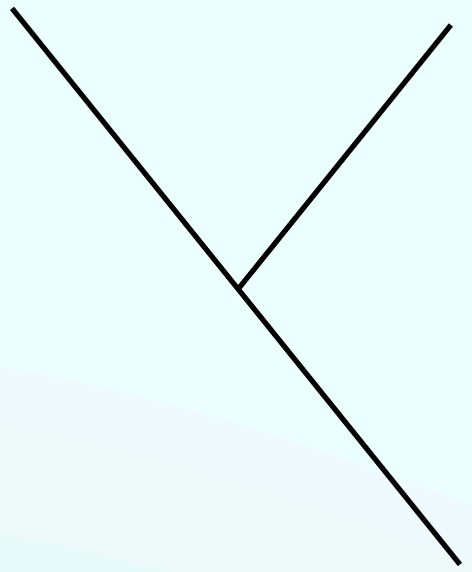
- Build it from other D-finite series
- Show the coefficients satisfy a linear recurrence
- Write it as the constant term (with respect to auxiliary variables) of a multivariable D-finite series (essentially, a Cauchy integral)

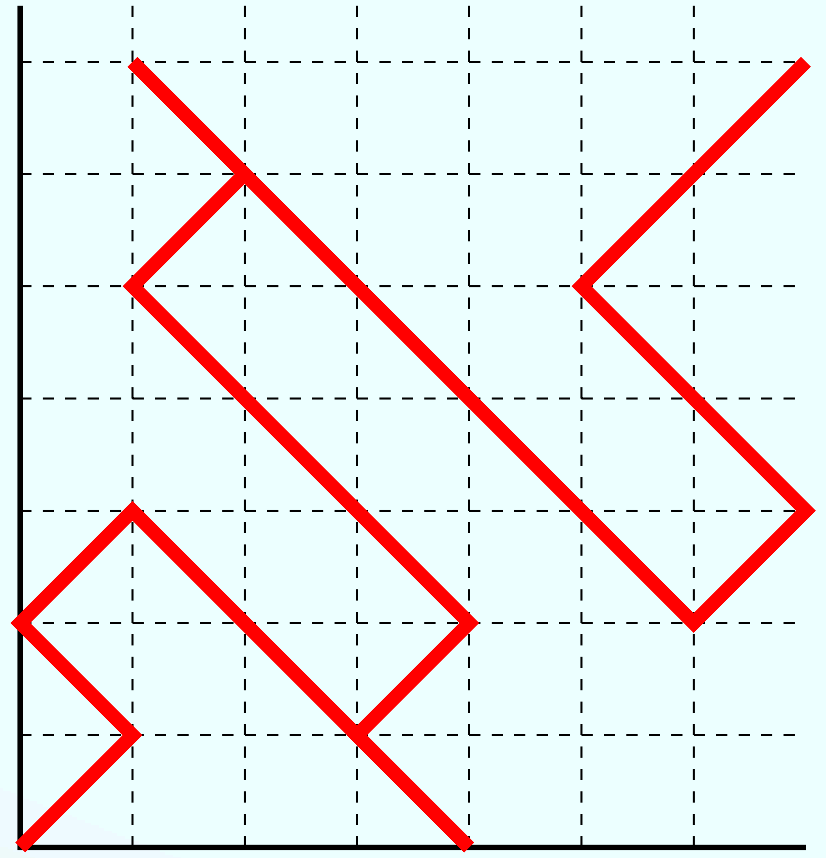
To show a series is NOT D-finite

- Show asymptotic growth of the coefficients is not of the correct form
- Show that it comes from a function with an infinite number of singularities
- *It is sufficient to show it is D-Transcendendal*

Differential Transcendence

A non-D-finite lattice model





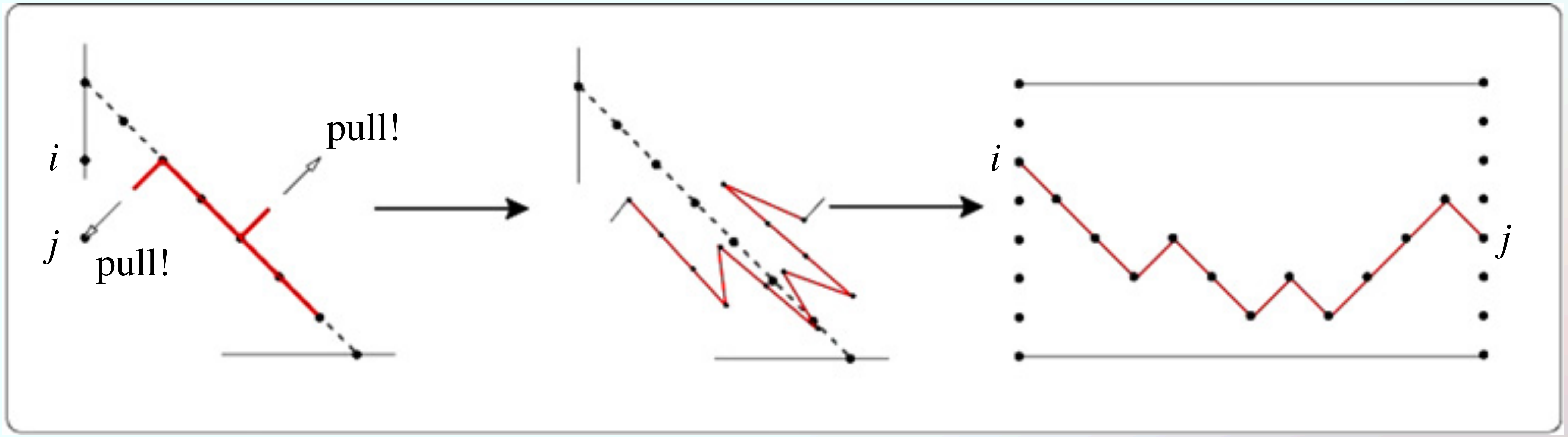
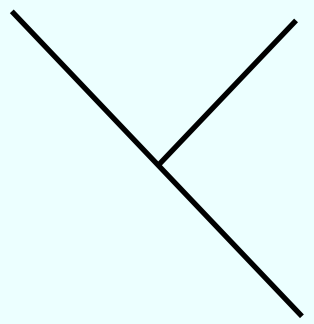


Fig. 5. Stretching the walk to find a directed path in a strip.

- **Theorem** *(M., Rechnitzer, 2009)*
  The univariate OGF has an *infinite number of singularities* and is not D-finite.

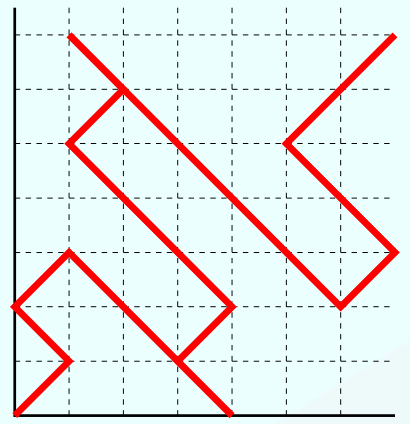
- A possible combinatorial explanation: A sequence of directed paths in strips of increasing height

- Similar models proved in an *ad hoc* manner.



In fact.. D-transcendental

Dreyfus+Hardouin+Roques+Singer 17 • Bostan 19



Combinatorial recurrence	*A walk is either the* **empty walk**, *or it is* **a shorter walk with a step appended**, *but you must* **exclude** *those walks that then* **step out** *of the quarter plane*
Functional equation for $Q_{\mathcal{S}}(x, y)$	$Q(x,y) = 1 + z(x/y + y/x + xy)Q(x,y) - z(x/y)Q(x,0) - z(y/x)Q(0,y)$
Rewrite so LHS is $K_{\mathcal{S}}(x,y)Q_{\mathcal{S}}(x,y)$	$K(x,y)Q(x,y) = xy - R(x) - R(y)$
Find rational parametrization for $E_{\mathcal{S}}$	$x(s) = \frac{v(1-v^2)s}{(s^2+1)}, y(s) = \frac{(1-v^2)s}{v^2s^2+1}, z = \frac{v}{v^2+1} \implies 0 = x(s)y(s) - R(x(s)) - R(y(s))$
Deduce an Ishizaki/Ogawara style equation for R(x(s))	$f(qt) = a(t)f(t) + b(t)$
Conclude D-transcendance	

Solution dichotomy

Lemma *(Ishizaki 1998; Ogawara 2015)*

Given a Laurent series $f(t)$,and Taylor expansions of rational functions $a(t), b(t) \in \mathbb{C}(t)$, and q, a complex number that is not a root of unity such that

$$f(qt) = a(t)f(t) + b(t)$$

then $f(t)$ is **EITHER rational or D-transcendental.**

Strategy

Rough principle: *(ref. Adamczewski, Dreyfus, Hardouin 2021)*

A Laurent series solution *f(t)* of a linear [shift | Mahler | q-shift] equation is EITHER *rational*, or D-*transcendental*

- q-shift: $f(t) \mapsto f(qt)$ (q not a root of unity)
  Example: Genus 0 quarter plane walks
- Shift operator: $f(t) \mapsto f(t+h)$
  Example: $\Gamma(t+1) = t\Gamma(t)$
- Mahler operator: $f(t) \mapsto f(t^k)$
  Example:
  $f(t) = \sum t^{2^n}$ satisfies $f(t) = t + f(t^2)$
- $f(t) \mapsto f\left(\frac{t}{1+t}\right)$
  Example: Bell numbers
  $B(t) = \sum B_n t^n \implies B\left(\frac{t}{t+1}\right) = tB(t) + 1$
  (Klazar 2003; Bostan, DiVizio, Raschel 2020+)

...not rational, hence it must be D-transcendental.

Order 1 Iterative Equations

$$f(R(t)) = a(t)f(t)+b(t)$$

NEW!

Extending the strategy

Theorem I. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = f(t) + b(t)$ with $R(t)$, $b(t)$ rational, and furthermore $R(0) = 0$, $R'(0) \in \{0, 1, \text{roots of unity}\}$ no iterate of R is the identity , then $f(t)$ is either rational or D-transcendental.

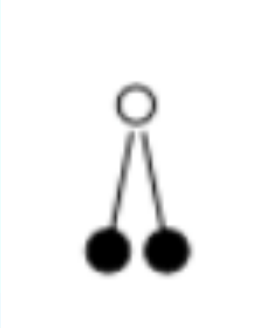
Theorem II. *(Di Vizio, Fernandes, M. 2023+)*

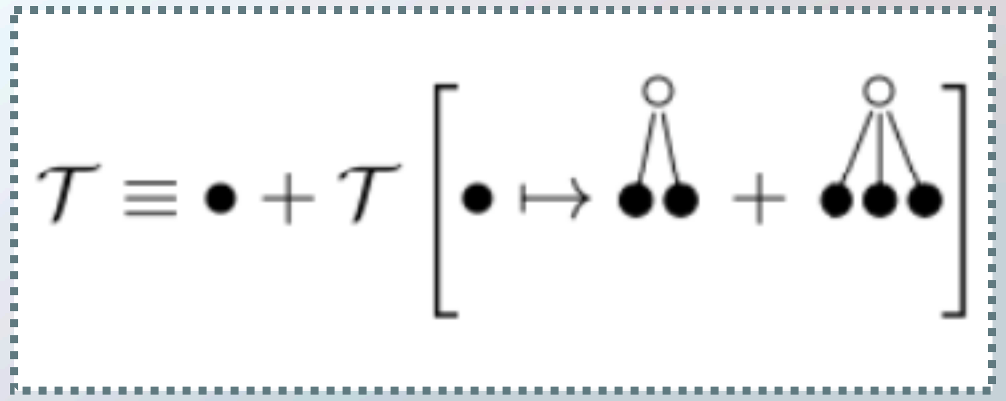
If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t)$ with $R(t)$, $a(t)$, $b(t)$ rational, and furthermore $R(0) = 0$, $R'(0) \in \{0, 1, \text{roots of unity}\}$ no iterate of R is the identity ,then $f(t)$ is either algebraic or D- transcendental.

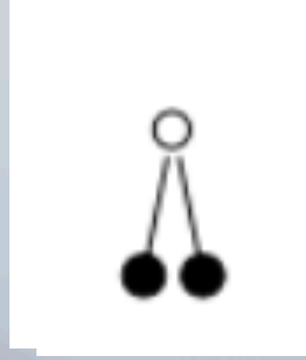
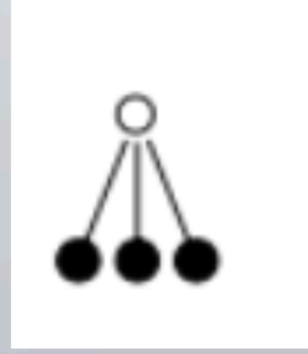
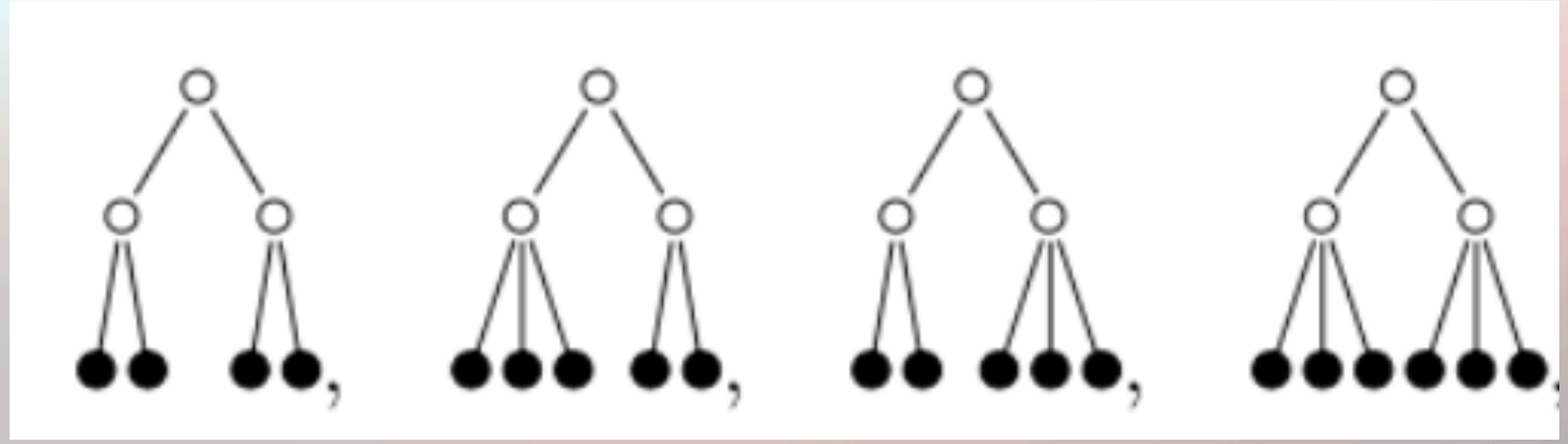
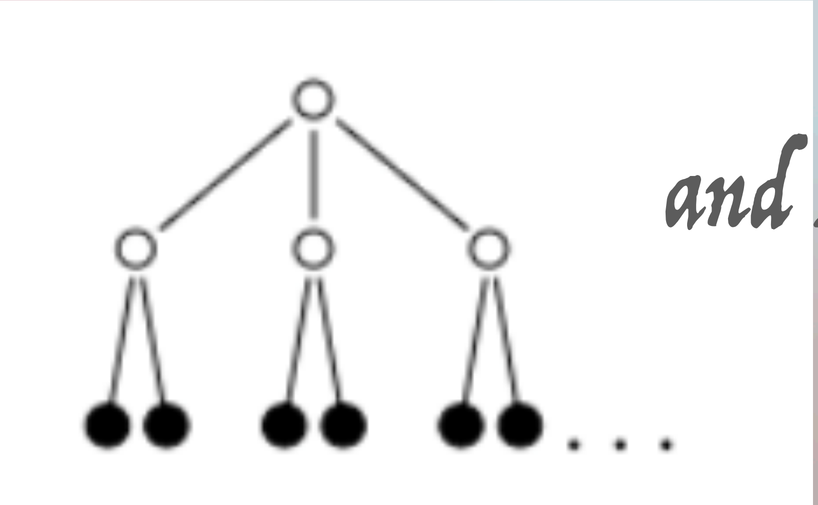
New Examples: $R(t) = t^2 + t^3, R(t) = \dfrac{t}{1 + t^2}$

Complete 2-3 Trees

Complete trees have all leaves at the same level. Size is given by # of leaves

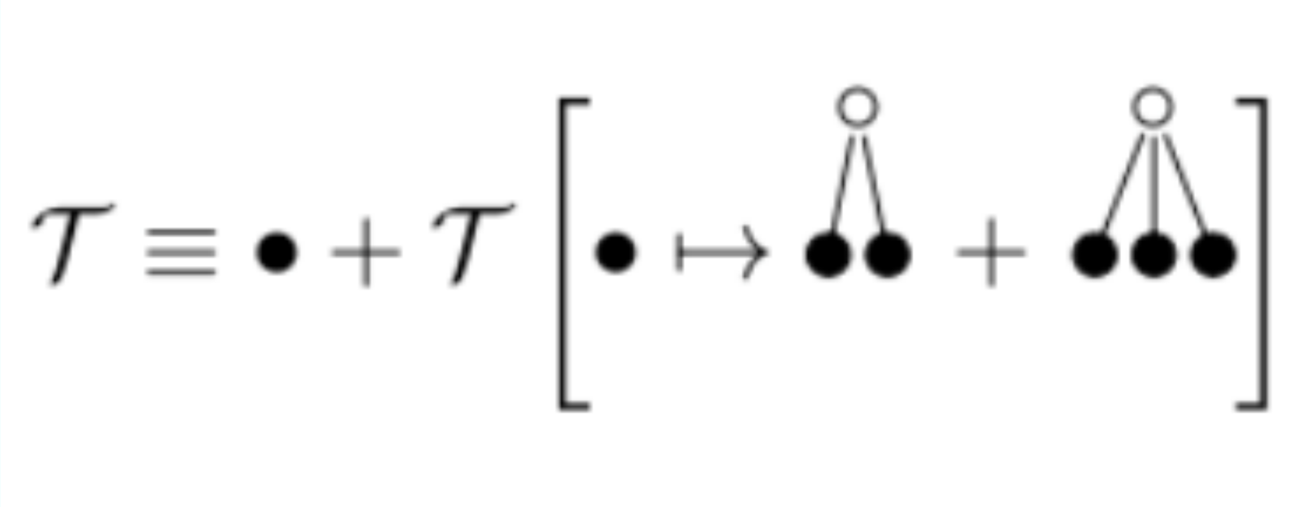
Iterative description: Start with single leaf. Each iteration generates trees of depth one more by replacing a leaf with either  or  in all possible ways.



and seven others

Complete 2-3 Trees



$$T(z) = z + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + O(z^7)$$

$$T(z) = z + T(z^2 + z^3).$$

$T(z)$ is not rational …

Theorem 1. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = f(t) + b(t)$
with $R(t), b(t)$ rational, and furthermore $R(0) = 0$, $R'(0) \in \{0, 1, \text{roots of unity}\}$
no iterate of R is the identity , then $f(t)$ is either rational or D-transcendental.

$$T(z) = z + T(z^2 + z^3).$$

- *(Odlyzko 82)* $T(z) \sim -c \log(\phi^{-1} - z)$ as $z \to \phi^{-1}$, $z \in (0, \phi^{-1})$.

$c = (\phi \log(4 - \phi))^{-1}$. $\phi = (1 + 5^{1/2})/2 = 1.618\ldots$ is the "golden ratio."

$T(z)$ is D-transcendental

Complete 2-3 Trees

$$T(z) = z + T(z^2 + z^3).$$

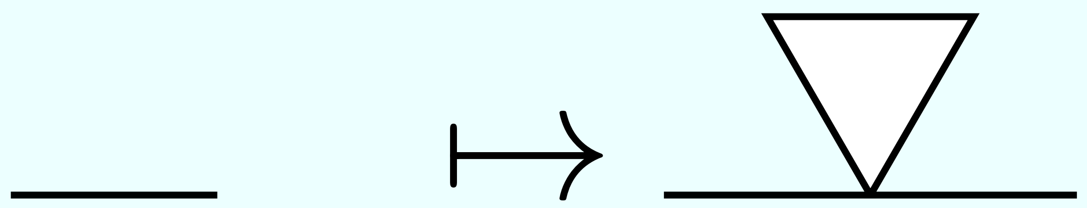
When *R(t)* is a polynomial we have a stronger result that we can apply here.

Corollary 1.2. *In the notation and under the assumptions of Theorem 1.1, we suppose moreover that* $R \in t^2\mathbb{C}[t]$, *and that* $b \in t\mathbb{C}[t]$, *with* $b \neq 0$ *and* $\deg_t b \leq \deg_t R$. *Then* f *is differentially transcendental over* $\mathbb{C}(t)$.

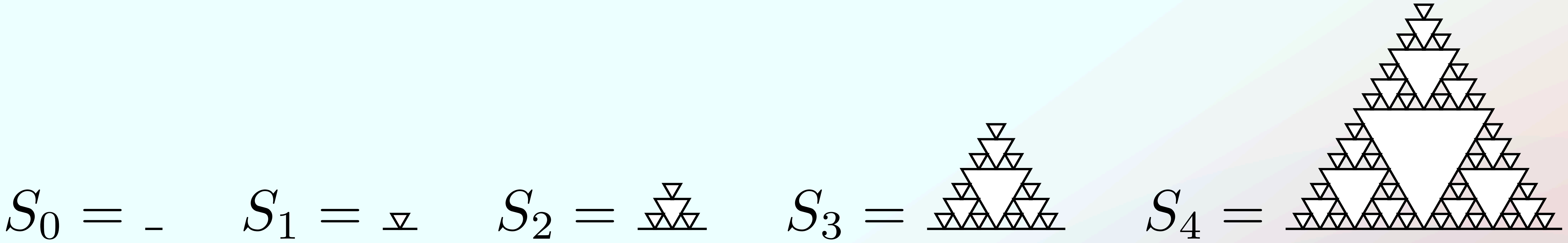
T(z) is D-transcendental (Indeed, most complete tree classes are similar)

Walks on self-similar graphs

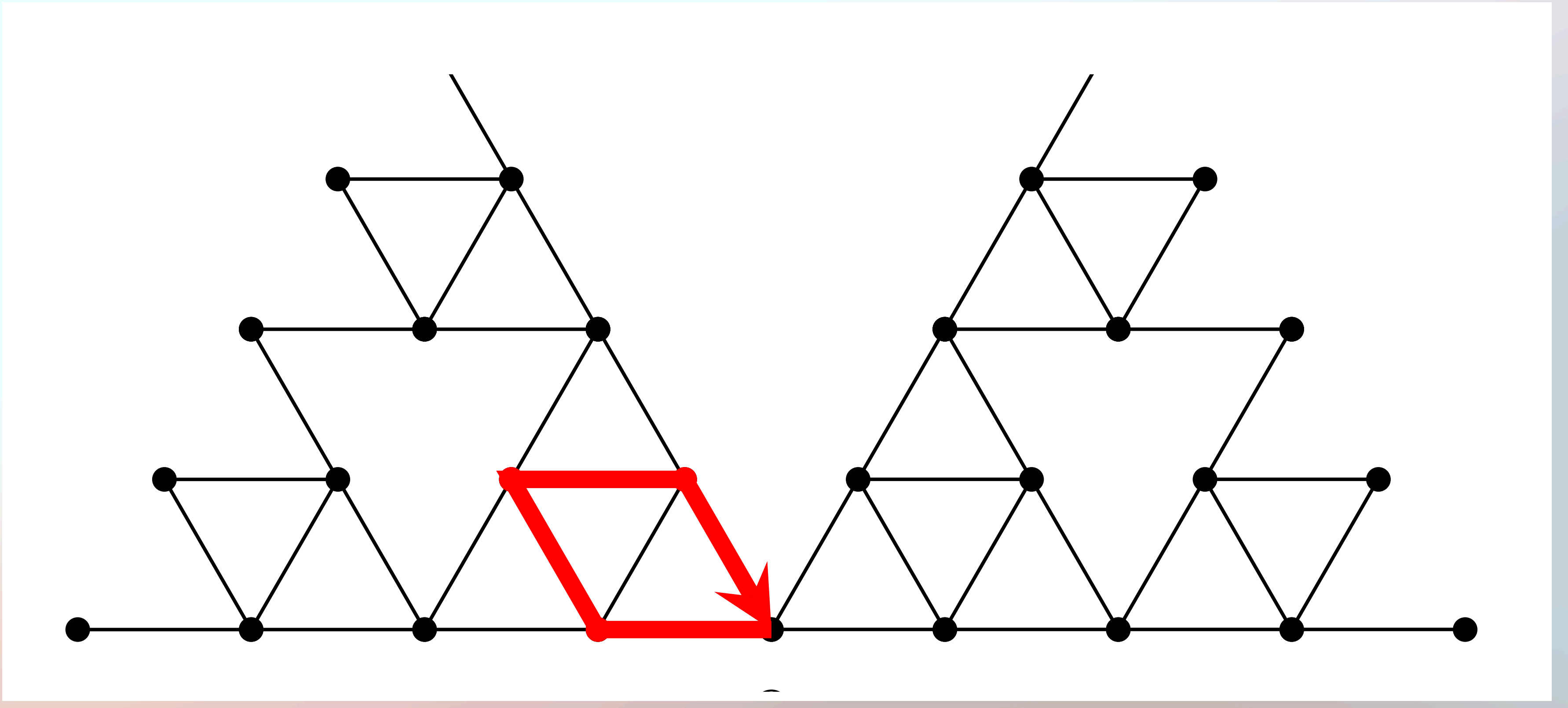
Generate a fractal with the following rule:



First few iterations:



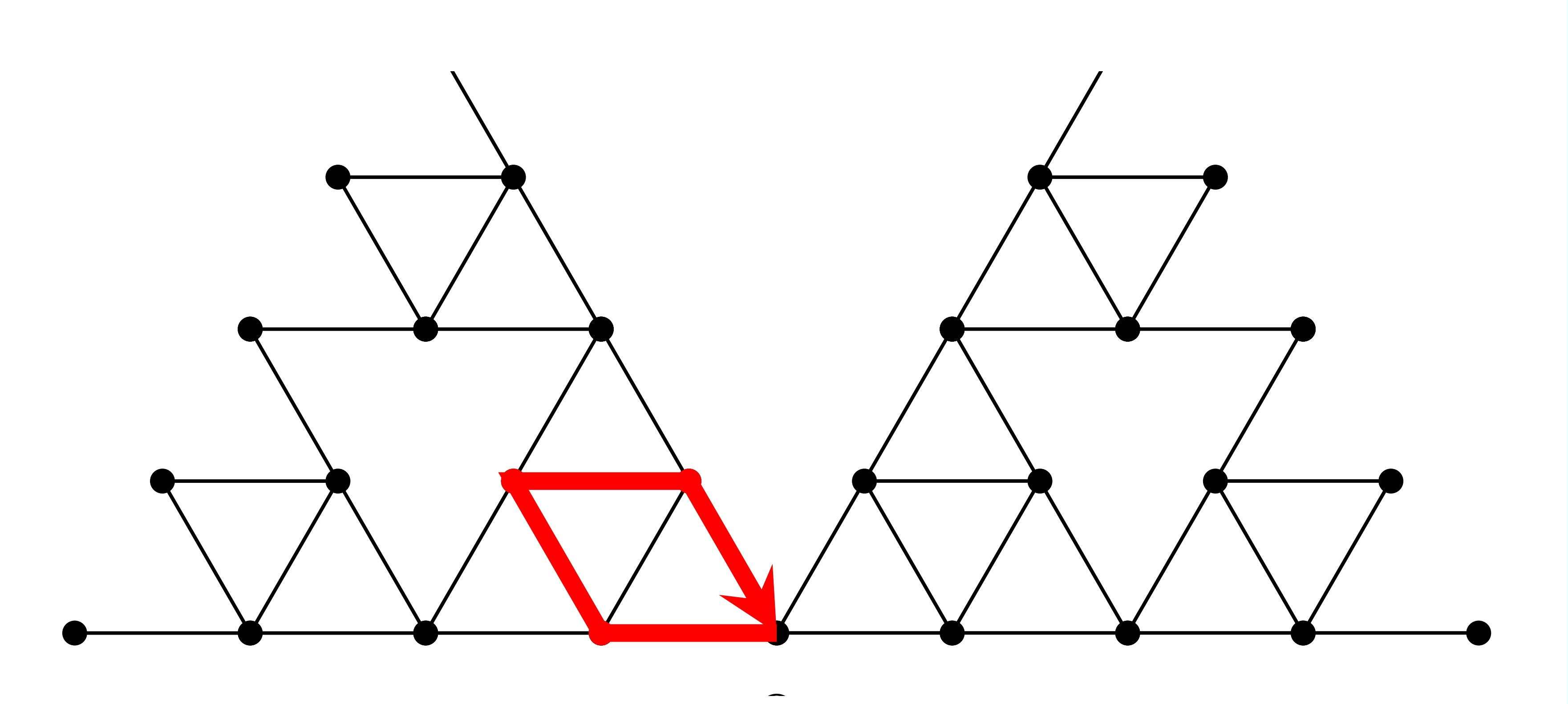
Consider a walk starting and ending at the same point on the limit of this process.



Generating function for walks that start and end at the same point:

$$G(t) = 1 + 4\,t^2 + 4\,t^3 + 32\,t^4 + 76t^5 + 348\,t^6 + 1112\,t^7 + O(t^8)\,.$$

Walks on self-similar graphs



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$$G(t) = 1 + 4\,t^2 + 4\,t^3 + 32\,t^4 + 76t^5 + 348\,t^6 + 1112\,t^7 + O(t^8)\,.$$

G(t) satisfies a recurrence *(Grabner + Woess)*

This factor comes from the self-similarity of the graph

$$G\left(\frac{4t^2}{1-3t}\right) = \frac{6t^2+t-1}{2t^2+t-1}\,G(t)\,.$$

The $G(t)$ is not algebraic …

Theorem II. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t)$
with $R(t), a(t), b(t)$ rational, and furthermore $R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$
no iterate of R is the identity, then $f(t)$ is either algebraic or D- transcendental.

Related to the fractal dimension

The asymptotics of the coefficients are incompatible with algebraicity. The coefficients of t^n grow like

$$n^{-\log 3/\log 5} F(\log n/\log 5)$$

for some non-constant period function *F. (Grabner and Woess 97)*

G(*t*) is D-transcendental (When are walks on other fractals similar?)

The result is best possible.

- We cannot hope for a stronger conclusion for Theorem II.
- Eg. The equation $y(t + t^2 + t^4 - t - t^3) = (1 + t + t^3)y(t)$ has an irrational, yet algebraic, solution: $y(t) = \sqrt{t-1}$
  - **Construction:** Consider $R(t) = 1 + (t-1)S(t)^2$, with *S(t)* a rational series so that the hypotheses on *R(t)* are satisfied.
  - Then $y = (t-1)^{1/2}$ is a non-rational, algebraic solution of $y(R(t)) = S(t)y(t)$, with coherent choice of square roots for $(t-1)$ and $S(t)^2$.

NEW!

Extending the strategy

Theorem III. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t)+b(t)$
with $R(t)$, $a(t)$, $b(t)$ rational and furthermore $R(0) = 0$, $R'(0) \in \{0,1,\text{roots of unity}\}$,
no iterate of R is the identity
then $f(t)$ is either D-finite or D- transcendental.

Permutations avoiding consecutive patterns

- A permutation σ of n avoids the consecutive pattern 1423 if there is no $0 \leq i \leq n-4$ so that $\sigma(i+1) < \sigma(i+4) < \sigma(i+2) < \sigma(i+3)$.

- The EGF $\hat{P}(t) = \sum \frac{p_n}{n!} t^n$ of 1423-avoiding permutations can be written using *S(t)* satisfying the following: *(Elizalde and Noy 2012)*

$$\widehat{P}(t) = \frac{1}{2 - \widehat{S}(t)} \quad \text{such that} \quad S(t) = S\left(\frac{t}{1+t^2}\right)\frac{t}{1+t} + 1.$$

- Similar situation for $1m23\ldots(m-1)$ avoiding permutations

$S(t)$ is not D-finite …

Theorem III. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = a(t)f(t) + b(t)$
with $R(t), a(t), b(t)$ rational and furthermore $R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$,
no iterate of R is the identity
then $f(t)$ is either D-finite or D- transcendental.

$$S(t) = 1 + \frac{1}{1+t} S\left(\frac{1}{1+t^2}\right)$$

- *S(t)* has an infinite number of singularities. *(Beaton, Conway and Guttmann 2018)*
- Since *S(t)* is not D-finite, by Theorem III, *S(t)* is D-transcendental.
- NOTE: We cannot conclude anything about $\widehat{P}(t) = \dfrac{1}{2 - \widehat{S}(t)}$

Concluding remarks

Rational

$$\frac{1}{1-t} = 1 + t + t^2 + \ldots$$

Unconstrained simple walks

Regular languages

Fibonacci numbers

Algebraic

$$\frac{-1+\sqrt{1-4t^2}}{2t} = 1 + t + 2t^2 + \ldots$$

Walks in half plane

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Differentiably Finite

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Simple walks in "transcendental" region

Complex

Excursions on Sierpinski gasket

Complete 2-3 Trees

Bell numbers (OGF)

Open questions & future work

- Identify combinatorial contexts that result in such functional equations.
- Simplify proofs of non-D-finiteness by proving D-transcendence.
- Higher order equations.
- Automated "guessing" tools for other kinds of functional equations.
- Can we improve Theorem III ?

Thank you for your attention!

Proof strategy

Theorem 1. *(Di Vizio, Fernandes, M. 2023+)*

If $f(t) \in \mathbb{C}[[t]]$ satisfies $f(R(t)) = f(t) + b(t)$ with $R(t), b(t)$ rational, and furthermore $R(0) = 0,\ R'(0) \in \{0, 1, \text{roots of unity}\}$ then $f(t)$ is either rational or D-transcendental.

- There exists a series solution τ to the equation $\tau(R(t)) = \tau(t)^d$. (Böttcher function)
- Hypothesis: R has a zero of order >1 => limited possibilities: Either R(t) is (roughly) t^k or a Chebyshev polynomial (and hence previous results apply) *or* τ is D-transcendental.
- Define $\Psi = \frac{\tau'}{\tau}\log(\tau)$. Ψ is D-transcendental over $\mathbb{C}(t)$ and *this is key*.
- Ψ is a formal solution to the associated Julia equation $y(R(t)) = R'(t)y(t)$
- $\partial := \Psi\frac{d}{dt}$ a derivation ∂ that commutes with $\Phi_R : \sum f_n t^n \mapsto \sum f_n (R(t))^n$
- *f(t)* D-algebraic wrt d/dt over over $\mathbb{C}(t)$ => f(t) D-algebraic wrt ∂ => key statement from differential Galois theory *(Hardouin 08)*
  1. *There exist* $n \geq 0$, $\lambda_0, \ldots, \lambda_n \in C$, *not all zero, and* $g \in \mathbf{K}$ *such that* $\lambda_0 b + \lambda_1 \partial(b) + \cdots + \lambda_n \partial^n(b) = \Phi_R(g) - g$.
- Can deduce from this statement that => $f(t) \in \mathbb{C}(t)$