A generating function method for the determination of differentially algebraic integer sequences modulo prime powers, II

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Counting subgroups

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$$f_n = 6nf_{n-1} + \sum_{m=1}^{n-2} f_m f_{n-m-1}, \quad f_1 = 5.$$

```
f[n_] := f[n] = 6 n f[n - 1] +
   Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[f[n], {n, 1, 20}]
{5, 60, 1105, 27 120, 828 250, 30 220 800, 1 282 031 525,
 61 999 046 400, 3 366 961 243 750, 202 903 221 120 000,
 13 437 880 555 850 250, 970 217 083 619 328 000,
 75 849 500 508 999 712 500, 6 383 483 988 812 390 400 000,
 575 440 151 532 675 686 278 125,
 55 318 762 960 656 722 780 160 000,
 5 649 301 494 178 851 172 304 968 750,
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The recurrence is equivalent to the differential equation

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0$$

for the generating function $F(z) = 1 + \sum_{n=1}^{\infty} f_n z^n$.

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We are going to analyse the sequence $(f_n)_{n\ge1}$ modulo prime powers p^α.

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We are going to analyse the sequence $(f_n)_{n\geq 1}$ modulo prime powers p^α.

For the primes $p = 2$ and $p = 3$ there exists a method based on generating function calculus which is able to solve the problem modulo any power of 2 and of 3, respectively.

f_n modulo 2 and 3

Theorem

The number f_n is odd if and only if n is of the form $n = 2^k - 1$ for some positive integer k.

Theorem

We have:

1. $f_n \equiv -1 \pmod 3$ *if, and only if, the* 3*-adic expansion of n is an element of* $\{0,2\}^*1$;
2. $f_n \equiv 1 \pmod 3$ *if, and only if, the* 3*-adic expansion of n is an element of*
$$\{0,2\}^*100^* \cup \{0,2\}^*122^*;$$
3. *for all other n, we have* $f_n \equiv 0 \pmod 3$.

f_n modulo powers of 5

```
f[n_] := f[n] = 6 n f[n - 1] +
   Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[f[n], {n, 1, 20}]
{5, 60, 1105, 27 120, 828 250, 30 220 800, 1 282 031 525,
 61 999 046 400, 3 366 961 243 750, 202 903 221 120 000,
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```

f_n modulo powers of 5

It is easy to convince oneself that, for fixed α, we have $f_n \equiv 0 \pmod{5^\alpha}$ for large enough n.

f_n modulo powers of 7

```
f[n_] := f[n] =
   6 n f[n - 1] + Sum[f[m] f[n - m - 1],
     {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 7], {n, 1, 40}]
{5, 4, 6, 2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4,
 6, 2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4, 6,
 2, 3, 1, 5, 4, 6, 2, 3, 1, 5, 4, 6, 2}
(1 + 5 z + 4 z^2 + 6 z^3 + 2 z^4 + 3 z^5) /
 (1 - z^6)
```

$$\frac{1+5z+4z^2+6z^3+2z^4+3z^5}{1-z^6}$$

```
Factor[%, Modulus → 7]
```

$$\frac{1}{2(4+z)}$$

```
f[n_] := f[n] = 6 n f[n - 1] +
   Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 49], {n, 1, 100}]
```

```
{5, 11, 27, 23, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13,
 37, 45, 15, 5, 18, 6, 2, 17, 22, 40, 46, 48, 16, 38,
 29, 26, 25, 41, 30, 10, 36, 12, 4, 34, 44, 31, 43, 47,
 32, 27, 9, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13, 37,
 45, 15, 5, 18, 6, 2, 17, 22, 40, 46, 48, 16, 38, 29,
 26, 25, 41, 30, 10, 36, 12, 4, 34, 44, 31, 43, 47, 32,
 27, 9, 3, 1, 33, 11, 20, 23, 24, 8, 19, 39, 13, 37}
```

$$1 + 5z + 11z^2 + 27z^3 + 23z^4 + z^4 \big(3z + z^2 + 33z^3 + 11z^4 + 20z^5 + 23z^6 + 24z^7 + 8z^8 + 19z^9 + 39z^{10} + 13z^{11} + 37z^{12} + 45z^{13} + 15z^{14} + 5z^{15} + 18z^{16} + 6z^{17} + 2z^{18} + 17z^{19} + 22z^{20} + 40z^{21} + 46z^{22} + 48z^{23} + 16z^{24} + 38z^{25} + 29z^{26} + 26z^{27} + 25z^{28} + 41z^{29} + 30z^{30} + 10z^{31} + 36z^{32} + 12z^{33} + 4z^{34} + 34z^{35} + 44z^{36} + 31z^{37} +$$

$$39 z^{10} + 13 z^{11} + 37 z^{12} + 45 z^{13} + 15 z^{14} + 5 z^{15} + 18 z^{16} + 6 z^{17} + 2 z^{18} + 17 z^{19} + 22 z^{20} + 40 z^{21} + 46 z^{22} + 48 z^{23} + 16 z^{24} + 38 z^{25} + 29 z^{26} + 26 z^{27} + 25 z^{28} + 41 z^{29} + 30 z^{30} + 10 z^{31} + 36 z^{32} + 12 z^{33} + 4 z^{34} + 34 z^{35} + 44 z^{36} + 31 z^{37} + 43 z^{38} + 47 z^{39} + 32 z^{40} + 27 z^{41} + 9 z^{42}\Big) \Big/ (1 - z\,\hat{}\,42)$$

$$1 + 5 z + 11 z^2 + 27 z^3 + 23 z^4 + \frac{1}{1 - z^{42}} z^4 \Big(3 z + z^2 + 33 z^3 + 11 z^4 + 20 z^5 + 23 z^6 + 24 z^7 + 8 z^8 + 19 z^9 + 39 z^{10} + 13 z^{11} + 37 z^{12} + 45 z^{13} + 15 z^{14} + 5 z^{15} + 18 z^{16} + 6 z^{17} + 2 z^{18} + 17 z^{19} + 22 z^{20} + 40 z^{21} + 46 z^{22} + 48 z^{23} + 16 z^{24} + 38 z^{25} + 29 z^{26} + 26 z^{27} + 25 z^{28} + 41 z^{29} + 30 z^{30} + 10 z^{31} + 36 z^{32} + 12 z^{33} + 4 z^{34} + 34 z^{35} + 44 z^{36} + 31 z^{37} + 43 z^{38} + 47 z^{39} + 32 z^{40} + 27 z^{41} + 9 z^{42}\Big)$$

$$\Big(1 + 9 z + 35 z^2 + 42 z^3 + 28 z^4 + 7 z^5 + 7 z^6\Big) \Big/ (1 + 2 z)^2$$

f_n modulo powers of 7

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 7^α, is eventually periodic, with period length $6\cdot 7^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 7^α, equals

$$\frac{P_\alpha(z)}{(1+2z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 11

```
f[n_] := f[n] = 6 n f[n - 1] +
    Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 11], {n, 1, 40}]
{5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5,
 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5}
Table[Mod[f[n], 11^2], {n, 1, 40}]
{5, 60, 16, 16, 5, 82, 104, 27, 49, 82, 115, 27, 60, 93,
 5, 38, 71, 104, 16, 49, 82, 115, 27, 60, 93, 5, 38,
 71, 104, 16, 49, 82, 115, 27, 60, 93, 5, 38, 71, 104}
```

$$7 + 99 z + 44 z^3 + 11 z^4 + 88 z^5 + 11 z^6 + 11 z^8 + \frac{99}{(-1 + 12 z)^2} + \frac{105}{-1 + 12 z}$$

f_n modulo powers of 11

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 11^α, is eventually periodic, with period length $11^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 11^α, equals

$$\frac{P_\alpha(z)}{(1-z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 13

```
f[n_] := f[n] = 6 n f[n - 1] +
   Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 13], {n, 1, 50}]
```

{5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8, 0, 2, 7, 12, 8, 5, 0, 11, 6, 1, 5, 8}

$$\text{Factor}\left[1+\frac{1}{1-z^{12}}\left(5\,z+8\,z^2+2\,z^4+7\,z^5+12\,z^6+8\,z^7+5\,z^8+11\,z^{10}+6\,z^{11}+z^{12}\right),\ \text{Modulus}\to 13\right]$$

$$\frac{7\,(5+z)}{(6+z)\,(8+z)}$$

f_n modulo powers of 13

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 13^α, is eventually periodic, with period length $12\cdot 13^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 13^α, equals

$$\frac{P_\alpha(z)}{\big((1-2z)(1+5z)\big)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 17

```
f[n_] := f[n] = 6 n f[n - 1] +
   Sum[f[m] f[n - m - 1], {m, 1, n - 2}]; f[1] = 5;
Table[Mod[f[n], 17], {n, 1, 200}]
{5, 9, 0, 5, 10, 2, 2, 7, 0, 2, 4, 11, 11, 13, 0, 11,
 5, 1, 1, 12, 0, 1, 2, 14, 14, 15, 0, 14, 11, 9, 9,
 6, 0, 9, 1, 7, 7, 16, 0, 7, 14, 13, 13, 3, 0, 13, 9,
 12, 12, 8, 0, 12, 7, 15, 15, 10, 0, 15, 13, 6, 6, 4,
 0, 6, 12, 16, 16, 5, 0, 16, 15, 3, 3, 2, 0, 3, 6, 8,
 8, 11, 0, 8, 16, 10, 10, 1, 0, 10, 3, 4, 4, 14, 0,
 4, 8, 5, 5, 9, 0, 5, 10, 2, 2, 7, 0, 2, 4, 11, 11,
 13, 0, 11, 5, 1, 1, 12, 0, 1, 2, 14, 14, 15, 0, 14,
 11, 9, 9, 6, 0, 9, 1, 7, 7, 16, 0, 7, 14, 13, 13, 3,
 0, 13, 9, 12, 12, 8, 0, 12, 7, 15, 15, 10, 0, 15,
 13, 6, 6, 4, 0, 6, 12, 16, 16, 5, 0, 16, 15, 3, 3,
 2, 0, 3, 6, 8, 8, 11, 0, 8, 16, 10, 10, 1, 0, 10,
 3, 4, 4, 14, 0, 4, 8, 5, 5, 9, 0, 5, 10, 2, 2, 7}
```

f_n modulo powers of 17

In terms of generating functions, this is equivalent to

$$F(z) = 13 + \frac{5 + 12z}{1 + 15z + 7z^2} \quad \text{modulo } 17.$$

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More generally it seems:

Conjecture

Let α be a positive integer. The generating function $F(z) = 1 + \sum_{n \ge 1} f_n \, z^n$, when coefficients are reduced modulo 17^α, equals

$$\frac{P_\alpha(z)}{(1 + 15z + 7z^2)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

How to prove these conjectures?

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The generating function $F(z) = 1 + \sum_{n\ge1} f_n z^n$ *satisfies*

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where $P_\alpha(z)$ *is a polynomial in* z *over the integers.*

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where $P_\alpha(z)$ *is a polynomial in* z *over the integers.*

Maybe we make the Ansatz $F(z) = P_\alpha(z)/(1+2z)^\alpha$, substitute in the differential equation

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

and then it becomes clear that $P_\alpha(z)$ must be a polynomial, and which polynomial it is.

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If this does not work ..., we should maybe try a recursive approach: let us suppose that $F_\alpha(z) = P_\alpha(z)/(1+2z)^\alpha$ solves

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modulo 7^α. Then make the Ansatz $F_{\alpha+1}(z) = F_\alpha(z) + 7^\alpha G_{\alpha+1}(z)$, substitute in the differential equation, and solve for $G_{\alpha+1}(z)$.

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modulo 7^α. Then make the Ansatz $F_{\alpha+1}(z) = F_\alpha(z) + 7^\alpha G_{\alpha+1}(z)$, substitute in the differential equation, and solve for $G_{\alpha+1}(z)$.
This comes very close: in this manner one can prove that $F(z) = \bar{P}_\alpha(z)/(1+2z)^{e_\alpha}$ modulo 7^α, for some integer e_α.

How to prove these conjectures?

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New approach: Padé approximants!!

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Apparently, the generating function $F(z)$ is always rational when reduced modulo $7, 11, 13, 17, \dots$

On the other hand, over $\mathbb{Z}$, the solution to

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0$$

is certainly *not* rational. But we may approximate $F(z)$ by a rational function, say

$$F(z) = \frac{P_n(z)}{Q_n(z)} + O(z^{2n+1}),$$

where $P_n(z)$ and $Q_n(z)$ are polynomials of degree at most n.

How to prove these conjectures?

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substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

which, after clearing denominators, becomes

$$(1-4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - zP_n^2(z) - Q_n^2(z) = O(z^{2n+1}),$$

How to prove these conjectures?

So:

substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

which, after clearing denominators, becomes

$$(1-4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - zP_n^2(z) - Q_n^2(z) = \text{const}(n) \times z^{2n+1},$$

How to prove these conjectures?

So:
substitute $F(z) = P_n(z)/Q_n(z) + O(z^{2n+1})$ in

$$(1-4z)F(z) - 6z^2F'(z) - zF^2(z) - 1 = 0,$$

which, after clearing denominators, becomes

$$(1-4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - zP_n^2(z) - Q_n^2(z) = \text{const}(n) \times z^{2n+1},$$

and compute the polynomials $P_n(z)$ and $Q_n(z)$ for $n = 1, 2, 3, \dots$.
Then stare at these polynomials and try to come up with a guess for const(n), $P_n(z)$, $Q_n(z)$.

```
SolPQ[n_] := Module[{Erg},
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
     6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
     Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
 ]
SolPQ[1]
```

$$\left\{\frac{1-7z}{1-12z},\ -385\right\}$$

```
SolPQ[2]
```

```
SolPQ[n_] := Module[{Erg},
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
     6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
     Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
 ]
```

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$$\left\{\frac{1-7z}{1-12z}, -385\right\}$$

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SolPQ[2]
```

$$\left\{\frac{1-31z+91z^2}{\ldots}, \quad 85085\ldots\right\}$$

```
SolPQ[n_] := Module[{Erg},
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
      6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
      Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
 ]
```

```
SolPQ[1]
```

$$\left\{\frac{1-7z}{1-12z},\ -385\right\}$$

```
SolPQ[2]
```

$$\left\{\frac{1-31z+91z^2}{1-36z+211z^2},\ -85\,085\right\}$$

```
  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
     6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
     Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
 ]
```

```
SolPQ[1]
```

$$\left\{\frac{1-7z}{1-12z},\ -385\right\}$$

```
SolPQ[2]
```

$$\left\{\frac{1-31z+91z^2}{1-36z+211z^2},\ -85085\right\}$$

```
SolPQ[3]
```

```
  Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
  Erg = Expand[(1 - 4 z) P Q -
      6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
      Table[q[i], {i, 1, n}]]];
  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
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SolPQ[3]
```

```
      6 z^2 (D[P, z] Q - P D[Q, z]) - z P^2 - Q^2];
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
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  {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
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SolPQ[1]

$$\left\{\frac{1-7z}{1-12z}, -385\right\}$$

SolPQ[2]

$$\left\{\frac{1-31z+91z^2}{1-36z+211z^2}, -85\,085\right\}$$

SolPQ[3]

$$\left\{\frac{1-67z+986z^2-1729z^3}{\ldots}, -37\,182\,145\right\}$$

```
  Var = Coefficient[Erg, z, 2 n + 1];
  Erg =
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
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SolPQ[3]
```

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  Erg =
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  Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
     Table[q[i], {i, 1, n}]]];
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```
FactorInteger[37 182 145]
```

```
Erg
   Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
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```
FactorInteger[37182145]
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{{5, 1}, {7, 1}, {11, 1}

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```
FactorInteger[37 182 145]
```

```
{{5, 1}, {7, 1}, {11, 1},
  {13, 1}, {17, 1}, {19, 1}, {23, 1}}
```

```
    Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
        Table[q[i], {i, 1, n}]]];
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FactorInteger[37182145]
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{{5, 1}, {7, 1}, {11, 1}, {13, 1}, {17, 1}, {19, 1}, {23, 1}}

So, apparently, there exist polynomials $P_n(z)$ and $Q_n(z)$ of degree n such that

$$(1-4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - zP_n^2(z) - Q_n^2(z) = -5z^{2n+1}\prod_{\ell=1}^{n}(6\ell+1)(6\ell+5),$$

or, equivalently, $R_n(z) = P_n(z)/Q_n(z)$ satisfies

$$(1-4z)R_n(z) - 6z^2R_n'(z) - zR_n^2(z) - 1 \\ = -\frac{5z^{2n+1}}{Q_n^2(z)}\prod_{\ell=1}^{n}(6\ell+1)(6\ell+5).$$

Let us pause here for a moment!

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This would prove immediately that, modulo any prime power p^α with $p \geq 5$, our generating function $F(z)$ is rational. In particular, the sequence $(f_n)_{n\geq 1}$ is eventually periodic, if considered modulo a prime power p^α with $p \geq 5$!

So, we know what we have to do:

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After having guessed const(n), we must now guess polynomials $P_n(z)$ and $Q_n(z)$ of degree n such that

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Well: „*Knapp daneben ist auch vorbei!*“

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$$(1-4z)P_n(z)Q_n(z) - 6z^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - zP_n^2(z) - Q_n^2(z) = -5z^{2n+1}\prod_{\ell=1}^{n}(6\ell+1)(6\ell+5).$$

Well: *„Knapp daneben ist auch vorbei!“*
(*“A miss is as good as a mile!”*)

Generalise!

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We had:

$$(1-4z)F(z)-6z^2F'(z)-zF^2(z)-1=0.$$

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Let us consider

$$(1-Az)F(z)-Bz^2F'(z)-CzF^2(z)-1-Dz=0.$$

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We had:

$$(1-4z)F(z)-6z^2F'(z)-zF^2(z)-1=0.$$

Let us consider

$$(1-Az)F(z)-Bz^2F'(z)-CzF^2(z)-1-Dz=0.$$

For the Padé approximant $F(z)=P_n(z)/Q_n(z)+O(z^{2n+1})$, we then have

$$\begin{aligned}(1-Az)P_n(z)Q_n(z)-Bz^2(P_n'(z)Q_n(z)-P_n(z)Q_n'(z))\\ -CzP_n^2(z)-(1+Dz)Q_n^2(z)=\operatorname{const}(n)\times z^{2n+1}.\end{aligned}$$

```
SolABCD[n_] := SolABCD[n] = Module[{Erg},
   P = (Sum[p[i] z^i, {i, 1, n}] + 1);
   Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
   Erg = Expand[(1 - A z) P Q -
      B z^2 (D[P, z] Q - P D[Q, z]) -
      CC z P^2 - Q^2 - DD z Q^2];
   Var = Coefficient[Erg, z, 2 n + 1];
   Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
   Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
      Table[q[i], {i, 1, n}]]];
   {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
  ]

SolABCD[1]
```

$$\left\{\frac{1+(-B-CC+DD)\,z}{1+(-A-B-2\,CC)\,z},\right.$$

```
SolABCD[n_] := SolABCD[n] = Module[{Erg},
   P = (Sum[p[i] z^i, {i, 1, n}] + 1);
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    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
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   {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
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SolABCD[1]
```

$$\left\{\frac{1 + (-B - CC + DD)\, z}{1 + (-A - B - 2\, CC)\, z},\right.$$

$$\left. -(A + CC + DD)\,\left(A\, B + B^2 + A\, CC + 2\, B\, CC + CC^2 + CC\, DD\right)\right\}$$

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SolABCD[n_] := SolABCD[n] = Module[{Erg},
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      B z^2 (D[P, z] Q - P D[Q, z]) -
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   Var = Coefficient[Erg, z, 2 n + 1];
   Erg =
    Table[Coefficient[Erg, z, ii] == 0, {ii, 0, 2 n}];
   Erg = Solve[Erg, Join[Table[p[i], {i, 1, n}],
      Table[q[i], {i, 1, n}]]];
   {(P / Q) /. Erg[[1]], Factor[Var /. Erg[[1]]]}
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SolABCD[1]
```

$$\left\{\frac{1+(-B-CC+DD)\,z}{1+(-A-B-2\,CC)\,z},\right.$$
$$\left.-(A+CC+DD)\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)\right\}$$

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  P = (Sum[p[i] z^i, {i, 1, n}] + 1);
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      B z^2 (D[P, z] Q - P D[Q, z]) -
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SolABCD[1]
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Q = (Sum[q[i] z^i, {i, 1, n}] + 1);
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 ]
```

```
SolABCD[1]
```

$$\left\{\frac{1+(-B-CC+DD)\,z}{1+(-A-B-2\,CC)\,z},\; -(A+CC+DD)\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)\right\}$$

SolABCD[2]

$$\{(1+(-A-4B-3CC+DD)z+(2B^2+3B\,CC+CC^2-A\,DD-3B\,DD-3CC\,DD)z^2)/(1-2(A+2B+2CC)z+(A^2+3AB+2B^2+3A\,CC+6B\,CC+3CC^2-CC\,DD)z^2),\ -(A+CC+DD)(AB+B^2+A\,CC+2B\,CC+CC^2+CC\,DD)(2AB+4B^2+A\,CC+4B\,CC+CC^2+CC\,DD)\}$$

SolABCD[3]

$$\{(1+(-2A-9B-5CC+DD)z+(A^2+7AB+18B^2+4A\,CC+22B\,CC+6CC^2-2A\,DD-8B\,DD-6CC\,DD)z^2+(-6B^3-11B^2CC-6B\,CC^2-CC^3+A^2DD+6AB\,DD+11B^2DD+4A\,CC\,DD+18B\,CC\,DD+6CC^2DD-CC\,DD^2)z^3)/(1-3(A+3B+2CC)z+(3A^2+15AB+18B^2+$$

```
SolABCD[2]
```

$$\{(1+(-A-4B-3CC+DD)z+(2B^2+3B\,CC+CC^2-A\,DD-3B\,DD-3CC\,DD)z^2)/(1-2(A+2B+2CC)z+(A^2+3AB+2B^2+3A\,CC+6B\,CC+3CC^2-CC\,DD)z^2),\ -(A+CC+DD)(AB+B^2+A\,CC+2B\,CC+CC^2+CC\,DD)(2AB+4B^2+A\,CC+4B\,CC+CC^2+CC\,DD)\}$$

```
SolABCD[3]
```

$$\{(1+(-2A-9B-5CC+DD)z+(A^2+7AB+18B^2+4A\,CC+22B\,CC+6CC^2-2A\,DD-8B\,DD-6CC\,DD)z^2+(-6B^3-11B^2\,CC-6B\,CC^2-CC^3+A^2\,DD+6AB\,DD+11B^2\,DD+4A\,CC\,DD+18B\,CC\,DD+6CC^2\,DD-CC\,DD^2)z^3)/(1-3(A+3B+2CC)z+(3A^2+15AB+18B^2+10A\,CC+30B\,CC+10CC^2-2CC\,DD)z^2+$$

SolABCD[2]

$$\Big\{\Big(1+(-A-4B-3CC+DD)\,z+\big(2B^2+3B\,CC+CC^2-A\,DD-3B\,DD-3CC\,DD\big)\,z^2\Big)\Big/\Big(1-2(A+2B+2CC)\,z+\big(A^2+3AB+2B^2+3A\,CC+6B\,CC+3CC^2-CC\,DD\big)\,z^2\Big),$$

$$-(A+CC+DD)\big(AB+B^2+A\,CC+2B\,CC+CC^2+CC\,DD\big)\big(2AB+4B^2+A\,CC+4B\,CC+CC^2+CC\,DD\big)\Big\}$$

SolABCD[3]

$$\Big\{\Big(1+(-2A-9B-5CC+DD)\,z+\big(A^2+7AB+18B^2+4A\,CC+22B\,CC+6CC^2-2A\,DD-8B\,DD-6CC\,DD\big)\,z^2+\big(-6B^3-11B^2\,CC-6B\,CC^2-CC^3+A^2\,DD+6AB\,DD+11B^2\,DD+4A\,CC\,DD+18B\,CC\,DD+6CC^2\,DD-CC\,DD^2\big)\,z^3\Big)\Big/$$

$$\Big(1-3(A+3B+2CC)\,z+\big(3A^2+15AB+18B^2+10A\,CC+30B\,CC+10CC^2-2CC\,DD\big)\,z^2+$$

$$\big(-A^3-6A^2B-11AB^2-6B^3-4A^2\,CC-18AB\,CC-$$

$$\{(1 + (-A - 4\,B - 3\,CC + DD)\,z +$$

$$(2\,B^2 + 3\,B\,CC + CC^2 - A\,DD - 3\,B\,DD - 3\,CC\,DD)\,z^2)\,/$$

$$(1 - 2\,(A + 2\,B + 2\,CC)\,z +$$

$$(A^2 + 3\,A\,B + 2\,B^2 + 3\,A\,CC + 6\,B\,CC + 3\,CC^2 - CC\,DD)\,z^2),$$

$$-(A + CC + DD)\,(A\,B + B^2 + A\,CC + 2\,B\,CC + CC^2 + CC\,DD)$$

$$(2\,A\,B + 4\,B^2 + A\,CC + 4\,B\,CC + CC^2 + CC\,DD)\}$$

SolABCD[3]

$$\{(1 + (-2\,A - 9\,B - 5\,CC + DD)\,z + (A^2 + 7\,A\,B + 18\,B^2 + 4\,A\,CC +$$

$$22\,B\,CC + 6\,CC^2 - 2\,A\,DD - 8\,B\,DD - 6\,CC\,DD)\,z^2 +$$

$$(-6\,B^3 - 11\,B^2\,CC - 6\,B\,CC^2 - CC^3 + A^2\,DD + 6\,A\,B\,DD + 11\,B^2$$

$$DD + 4\,A\,CC\,DD + 18\,B\,CC\,DD + 6\,CC^2\,DD - CC\,DD^2)\,z^3)\,/$$

$$(1 - 3\,(A + 3\,B + 2\,CC)\,z + (3\,A^2 + 15\,A\,B + 18\,B^2 +$$

$$10\,A\,CC + 30\,B\,CC + 10\,CC^2 - 2\,CC\,DD)\,z^2 +$$

$$(-A^3 - 6\,A^2\,B - 11\,A\,B^2 - 6\,B^3 - 4\,A^2\,CC - 18\,A\,B\,CC -$$

$$22\,B^2\,CC - 6\,A\,CC^2 - 18\,B\,CC^2 - 4\,CC^3$$

$\left(2\,B^2 + 3\,B\,CC + CC^2 - A\,DD - 3\,B\,DD - 3\,CC\,DD\right)\,z^2\big)\;/$

$\big(1 - 2\,(A + 2\,B + 2\,CC)\,z +$

$\left(A^2 + 3\,A\,B + 2\,B^2 + 3\,A\,CC + 6\,B\,CC + 3\,CC^2 - CC\,DD\right)\,z^2\big),$

$-(A + CC + DD)\,\left(A\,B + B^2 + A\,CC + 2\,B\,CC + CC^2 + CC\,DD\right)$

$\left(2\,A\,B + 4\,B^2 + A\,CC + 4\,B\,CC + CC^2 + CC\,DD\right)\}$

SolABCD[3]

$\{\big(1 + (-2\,A - 9\,B - 5\,CC + DD)\,z + \big(A^2 + 7\,A\,B + 18\,B^2 + 4\,A\,CC +$

$22\,B\,CC + 6\,CC^2 - 2\,A\,DD - 8\,B\,DD - 6\,CC\,DD\big)\,z^2 +$

$\big(-6\,B^3 - 11\,B^2\,CC - 6\,B\,CC^2 - CC^3 + A^2\,DD + 6\,A\,B\,DD + 11\,B^2$

$DD + 4\,A\,CC\,DD + 18\,B\,CC\,DD + 6\,CC^2\,DD - CC\,DD^2\big)\,z^3\big)\;/$

$\big(1 - 3\,(A + 3\,B + 2\,CC)\,z + \big(3\,A^2 + 15\,A\,B + 18\,B^2 +$

$10\,A\,CC + 30\,B\,CC + 10\,CC^2 - 2\,CC\,DD\big)\,z^2 +$

$\big(-A^3 - 6\,A^2\,B - 11\,A\,B^2 - 6\,B^3 - 4\,A^2\,CC - 18\,A\,B\,CC -$

$22\,B^2\,CC - 6\,A\,CC^2 - 18\,B\,CC^2 - 4\,CC^3 +$

$$\left(1-2\,(A+2\,B+2\,CC)\,z+\left(A^2+3\,A\,B+2\,B^2+3\,A\,CC+6\,B\,CC+3\,CC^2-CC\,DD\right)z^2\right),$$

$$-(A+CC+DD)\,\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)\left(2\,A\,B+4\,B^2+A\,CC+4\,B\,CC+CC^2+CC\,DD\right)\}$$

```
SolABCD[3]
```

$$\{\left(1+(-2\,A-9\,B-5\,CC+DD)\,z+\left(A^2+7\,A\,B+18\,B^2+4\,A\,CC+22\,B\,CC+6\,CC^2-2\,A\,DD-8\,B\,DD-6\,CC\,DD\right)z^2+\left(-6\,B^3-11\,B^2\,CC-6\,B\,CC^2-CC^3+A^2\,DD+6\,A\,B\,DD+11\,B^2\,DD+4\,A\,CC\,DD+18\,B\,CC\,DD+6\,CC^2\,DD-CC\,DD^2\right)z^3\right)\Big/$$

$$\left(1-3\,(A+3\,B+2\,CC)\,z+\left(3\,A^2+15\,A\,B+18\,B^2+10\,A\,CC+30\,B\,CC+10\,CC^2-2\,CC\,DD\right)z^2+\left(-A^3-6\,A^2\,B-11\,A\,B^2-6\,B^3-4\,A^2\,CC-18\,A\,B\,CC-22\,B^2\,CC-6\,A\,CC^2-18\,B\,CC^2-4\,CC^3+2\,A\,CC\,DD+6\,B\,CC\,DD+4\,CC^2\,DD\right)z^3\right),$$

$$\left(A^2+3\,A\,B+2\,B^2+3\,A\,CC+6\,B\,CC+3\,CC^2-CC\,DD\right)\,z^2\Big),$$

$$-(A+CC+DD)\,\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)$$
$$\left(2\,A\,B+4\,B^2+A\,CC+4\,B\,CC+CC^2+CC\,DD\right)\Big\}$$

```
SolABCD[3]
```

$$\Big\{\Big(1+(-2\,A-9\,B-5\,CC+DD)\,z+\left(A^2+7\,A\,B+18\,B^2+4\,A\,CC+\right.$$
$$\left.22\,B\,CC+6\,CC^2-2\,A\,DD-8\,B\,DD-6\,CC\,DD\right)\,z^2+$$
$$\left(-6\,B^3-11\,B^2\,CC-6\,B\,CC^2-CC^3+A^2\,DD+6\,A\,B\,DD+11\,B^2\right.$$
$$\left.DD+4\,A\,CC\,DD+18\,B\,CC\,DD+6\,CC^2\,DD-CC\,DD^2\right)\,z^3\Big)\Big/$$
$$\Big(1-3\,(A+3\,B+2\,CC)\,z+\left(3\,A^2+15\,A\,B+18\,B^2+\right.$$
$$\left.10\,A\,CC+30\,B\,CC+10\,CC^2-2\,CC\,DD\right)\,z^2+$$
$$\left(-A^3-6\,A^2\,B-11\,A\,B^2-6\,B^3-4\,A^2\,CC-18\,A\,B\,CC-\right.$$
$$22\,B^2\,CC-6\,A\,CC^2-18\,B\,CC^2-4\,CC^3+$$
$$\left.2\,A\,CC\,DD+6\,B\,CC\,DD+4\,CC^2\,DD\right)\,z^3\Big),$$

$$-(A+CC+DD)\,\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)$$

$$\left(A^{2}+3\,A\,B+2\,B^{2}+3\,A\,CC+6\,B\,CC+3\,CC^{2}-CC\,DD\right)z^{2}\Big),$$

$$-(A+CC+DD)\left(A\,B+B^{2}+A\,CC+2\,B\,CC+CC^{2}+CC\,DD\right)\left(2\,A\,B+4\,B^{2}+A\,CC+4\,B\,CC+CC^{2}+CC\,DD\right)\Big\}$$

SolABCD[3]

$$\Big\{\Big(1+(-2\,A-9\,B-5\,CC+DD)\,z+\left(A^{2}+7\,A\,B+18\,B^{2}+4\,A\,CC+22\,B\,CC+6\,CC^{2}-2\,A\,DD-8\,B\,DD-6\,CC\,DD\right)z^{2}+\left(-6\,B^{3}-11\,B^{2}\,CC-6\,B\,CC^{2}-CC^{3}+A^{2}\,DD+6\,A\,B\,DD+11\,B^{2}\,DD+4\,A\,CC\,DD+18\,B\,CC\,DD+6\,CC^{2}\,DD-CC\,DD^{2}\right)z^{3}\Big)\Big/$$

$$\Big(1-3\,(A+3\,B+2\,CC)\,z+\left(3\,A^{2}+15\,A\,B+18\,B^{2}+10\,A\,CC+30\,B\,CC+10\,CC^{2}-2\,CC\,DD\right)z^{2}+\left(-A^{3}-6\,A^{2}\,B-11\,A\,B^{2}-6\,B^{3}-4\,A^{2}\,CC-18\,A\,B\,CC-22\,B^{2}\,CC-6\,A\,CC^{2}-18\,B\,CC^{2}-4\,CC^{3}+2\,A\,CC\,DD+6\,B\,CC\,DD+4\,CC^{2}\,DD\right)z^{3}\Big),$$

$$-(A+CC+DD)\left(A\,B+B^{2}+A\,CC+2\,B\,CC+CC^{2}+CC\,DD\right)$$

$$\left(2\,A\,B+4\,B^{2}+A\,CC+4\,B\,CC+CC^{2}+CC\,DD\right)$$

$$-(A+CC+DD)\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)$$
$$\left(2\,A\,B+4\,B^2+A\,CC+4\,B\,CC+CC^2+CC\,DD\right)\}$$

SolABCD[3]

$$\{\Big(1+(-2\,A-9\,B-5\,CC+DD)\,z+\big(A^2+7\,A\,B+18\,B^2+4\,A\,CC+22\,B\,CC+6\,CC^2-2\,A\,DD-8\,B\,DD-6\,CC\,DD\big)\,z^2+\big(-6\,B^3-11\,B^2\,CC-6\,B\,CC^2-CC^3+A^2\,DD+6\,A\,B\,DD+11\,B^2\,DD+4\,A\,CC\,DD+18\,B\,CC\,DD+6\,CC^2\,DD-CC\,DD^2\big)\,z^3\Big)\Big/$$
$$\Big(1-3\,(A+3\,B+2\,CC)\,z+\big(3\,A^2+15\,A\,B+18\,B^2+10\,A\,CC+30\,B\,CC+10\,CC^2-2\,CC\,DD\big)\,z^2+\big(-A^3-6\,A^2\,B-11\,A\,B^2-6\,B^3-4\,A^2\,CC-18\,A\,B\,CC-22\,B^2\,CC-6\,A\,CC^2-18\,B\,CC^2-4\,CC^3+2\,A\,CC\,DD+6\,B\,CC\,DD+4\,CC^2\,DD\big)\,z^3\Big),$$

$$-(A+CC+DD)\left(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\right)$$
$$\left(2\,A\,B+4\,B^2+A\,CC+4\,B\,CC+CC^2+CC\,DD\right)$$

$\left(2\,A\,B + 4\,B^2 + A\,CC + 4\,B\,CC + CC^2 + CC\,DD\right)\}$

SolABCD[3]

$$\{\left(1 + (-2\,A - 9\,B - 5\,CC + DD)\,z + \left(A^2 + 7\,A\,B + 18\,B^2 + 4\,A\,CC + 22\,B\,CC + 6\,CC^2 - 2\,A\,DD - 8\,B\,DD - 6\,CC\,DD\right)\,z^2 + \left(-6\,B^3 - 11\,B^2\,CC - 6\,B\,CC^2 - CC^3 + A^2\,DD + 6\,A\,B\,DD + 11\,B^2\,DD + 4\,A\,CC\,DD + 18\,B\,CC\,DD + 6\,CC^2\,DD - CC\,DD^2\right)\,z^3\right) \Big/$$

$$\left(1 - 3\,(A + 3\,B + 2\,CC)\,z + \left(3\,A^2 + 15\,A\,B + 18\,B^2 + 10\,A\,CC + 30\,B\,CC + 10\,CC^2 - 2\,CC\,DD\right)\,z^2 + \left(-A^3 - 6\,A^2\,B - 11\,A\,B^2 - 6\,B^3 - 4\,A^2\,CC - 18\,A\,B\,CC - 22\,B^2\,CC - 6\,A\,CC^2 - 18\,B\,CC^2 - 4\,CC^3 + 2\,A\,CC\,DD + 6\,B\,CC\,DD + 4\,CC^2\,DD\right)\,z^3\right),$$

$$-(A + CC + DD)\,\left(A\,B + B^2 + A\,CC + 2\,B\,CC + CC^2 + CC\,DD\right)\left(2\,A\,B + 4\,B^2 + A\,CC + 4\,B\,CC + CC^2 + CC\,DD\right)\left(3\,A\,B + 9\,B^2 + A\,CC + 6\,B\,CC + CC^2 + CC\,DD\right)\}$$

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SolABCD[3]
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$$\Big\{\Big(1+(-2\,A-9\,B-5\,CC+DD)\,z+\big(A^2+7\,A\,B+18\,B^2+4\,A\,CC+22\,B\,CC+6\,CC^2-2\,A\,DD-8\,B\,DD-6\,CC\,DD\big)\,z^2+\big(-6\,B^3-11\,B^2\,CC-6\,B\,CC^2-CC^3+A^2\,DD+6\,A\,B\,DD+11\,B^2\,DD+4\,A\,CC\,DD+18\,B\,CC\,DD+6\,CC^2\,DD-CC\,DD^2\big)\,z^3\Big)\Big/$$

$$\Big(1-3\,(A+3\,B+2\,CC)\,z+\big(3\,A^2+15\,A\,B+18\,B^2+10\,A\,CC+30\,B\,CC+10\,CC^2-2\,CC\,DD\big)\,z^2+\big(-A^3-6\,A^2\,B-11\,A\,B^2-6\,B^3-4\,A^2\,CC-18\,A\,B\,CC-22\,B^2\,CC-6\,A\,CC^2-18\,B\,CC^2-4\,CC^3+2\,A\,CC\,DD+6\,B\,CC\,DD+4\,CC^2\,DD\big)\,z^3\Big),$$

$$-(A+CC+DD)\,\big(A\,B+B^2+A\,CC+2\,B\,CC+CC^2+CC\,DD\big)\,\big(2\,A\,B+4\,B^2+A\,CC+4\,B\,CC+CC^2+CC\,DD\big)\,\big(3\,A\,B+9\,B^2+A\,CC+6\,B\,CC+CC^2+CC\,DD\big)\Big\}$$

So, apparently,

So, apparently, for every positive integer n, there exist polynomials $P_n(z) = 1 + \sum_{k=1}^{n} p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^{n} q_{n,k} z^k$, such that

$$(1 - Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - CzP_n^2(z) - (1 + Dz)Q_n^2(z) \\ = -z^{2n+1}(A + C + D) \prod_{\ell=1}^{n} (\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2).$$

Let $P_n(z) = 1 + \sum_{k=1}^{n} p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^{n} q_{n,k} z^k$.

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By staring at the first few polynomials, one is led to conjecture:

Let $P_n(z)=1+\sum_{k=1}^n p_{n,k}z^k$ and $Q_n(z)=1+\sum_{k=1}^n q_{n,k}z^k$.

By staring at the first few polynomials, one is led to conjecture:

$$
\begin{aligned}
p_{n,1}&=-(n-1)A-n^2B-(2n-1)C+D,\\
q_{n,1}&=-nA-n^2B-2nC,\\
p_{n,2}&=\tfrac{1}{2}(n-2)(n-1)A^2+\tfrac{1}{2}(n-2)(n-1)(2n+1)AB\\
&\quad+2(n-2)(n-1)AC-(n-1)AD+\tfrac{1}{2}(n-1)^2n^2B^2\\
&\quad+(n-1)\left(2n^2-2n-1\right)BC\\
&\quad-(n-1)(n+1)BD+(n-1)(2n-3)C^2-3(n-1)CD,\\
q_{n,2}&=\tfrac{1}{2}(n-1)nA^2+\tfrac{1}{2}(n-1)n(2n-1)AB+(n-1)(2n-1)AC\\
&\quad+\tfrac{1}{2}(n-1)^2n^2B^2+(n-1)n(2n-1)BC\\
&\quad+(n-1)(2n-1)C^2-(n-1)CD,
\end{aligned}
$$

$$
\begin{aligned}
p_{n,3} = &-\tfrac{1}{6}(n-3)(n-2)(n-1)A^3 - \tfrac{1}{2}(n-3)(n-2)\left(n^2-n-1\right)A^2B\\
&- \tfrac{1}{2}(n-3)(n-2)(2n-3)A^2C + \tfrac{1}{2}(n-2)(n-1)A^2D\\
&- \tfrac{1}{6}(n-3)(n-2)\left(3n^3-3n^2-n-2\right)AB^2\\
&- \tfrac{1}{2}(n-3)(n-2)(2n-3)(2n+1)ABC\\
&+ \tfrac{1}{2}(n-2)(n+1)(2n-3)ABD\\
&- (n-3)(n-2)(2n-3)AC^2 + (n-2)(3n-5)ACD\\
&- \tfrac{1}{6}(n-2)^2(n-1)^2n^2B^3\\
&- \tfrac{1}{6}(n-2)(2n-3)\left(3n^3-6n^2-n-2\right)B^2C\\
&+ \tfrac{1}{2}(n-2)\left(n^3-n-2\right)B^2D\\
&- (n-2)(2n-3)\left(n^2-2n-1\right)BC^2\\
&+ (n-2)\left(3n^2-2n-3\right)BCD - \tfrac{1}{3}(n-2)(2n-5)(2n-3)C^3\\
&+ 2(n-2)(2n-3)DC^2 - (n-2)CD^2,
\end{aligned}
$$

$$
\begin{aligned}
q_{n,3} = &-\tfrac{1}{6}(n-2)(n-1)nA^3 - \tfrac{1}{2}(n-2)(n-1)^2 nA^2 B \\
&- (n-2)(n-1)^2 A^2 C \\
&- \tfrac{1}{6}(n-2)(n-1)n\left(3n^2 - 6n + 2\right) AB^2 \\
&- (n-2)(n-1)n(2n-3)ABC \\
&- (n-2)(n-1)(2n-3)AC^2 + (n-2)(n-1)ACD \\
&- \tfrac{1}{6}(n-2)^2(n-1)^2 n^2 B^3 \\
&- \tfrac{1}{3}(n-2)(n-1)n\left(3n^2 - 6n + 2\right) B^2 C \\
&- (n-2)(n-1)n(2n-3)BC^2 + (n-2)(n-1)nBCD \\
&- \tfrac{2}{3}(n-2)(n-1)(2n-3)C^3 + 2(n-2)(n-1)C^2 D.
\end{aligned}
$$

??

An interim thought

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If we should be able to come up with a guess for $P_n(z)$ and $Q_n(z)$, how would we ever be able to prove that this guess is correct?

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If we should be able to come up with a guess for $P_n(z)$ and $Q_n(z)$, how would we ever be able to prove that this guess is correct?

If we should be able to prove anything, then this proof must come from hypergeometrics!

An interim thought

Hypergeometrics?

An interim thought

Hypergeometrics?

We are looking at the differential equation:

$$(1-Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - CzP_n^2(z) - (1+Dz)Q_n^2(z) \\ = -z^{2n+1}(A+C+D)\prod_{\ell=1}^{n}(\ell AB + AC + CD + \ell^2B^2 + 2\ell BC + C^2).$$

An interim thought

Hypergeometrics?

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If this wants to be part of the realm of hypergeometrics, then we should better factor the product on the right-hand side into linear factors.

An interim thought

Here is this product:

$$(A+C+D)\prod_{\ell=1}^{n}(\ell AB+AC+CD+\ell^2B^2+2\ell BC+C^2).$$

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$$(A+C+D)\prod_{\ell=1}^{n}(\ell AB+AC+CD+\ell^2B^2+2\ell BC+C^2).$$

If we make the substitution $E^2=A^2-4CD$, then this becomes

$$(A+C+D)\prod_{\ell=1}^{n}\left(\ell B+\tfrac{A+2C+E}{2}\right)\left(\ell B+\tfrac{A+2C-E}{2}\right)$$
$$=\frac{1}{C}\prod_{\ell=0}^{n}\left(\ell B+\tfrac{A+2C+E}{2}\right)\left(\ell B+\tfrac{A+2C-E}{2}\right).$$

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$$=\frac{1}{C}\prod_{\ell=0}^{n}\left(\ell B+\tfrac{A+2C+E}{2}\right)\left(\ell B+\tfrac{A+2C-E}{2}\right).$$

Let us write

$$\Pi_+=\prod_{\ell=0}^{n}\left(\ell B+\tfrac{A+2C+E}{2}\right)\quad\text{and}\quad\Pi_-=\prod_{\ell=0}^{n}\left(\ell B+\tfrac{A+2C-E}{2}\right).$$

Our "new" problem: find polynomials $P_n(z)$ and $Q_n(z)$ of degree n with

$$(1 - Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z)) \\ - CzP_n^2(z) - (1 + Dz)Q_n^2(z) = -\tfrac{1}{C}\Pi_+\Pi_-\, z^{2n+1}.$$

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Maybe we should "work from the other end"?!

In other words: let us try to come up with guesses for $p_{n,n}$, $q_{n,n}$, $p_{n,n-1}$, $q_{n,n-1}$, etc.

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Maybe we should "work from the other end"?!

In other words: let us try to come up with guesses for $p_{n,n}$, $q_{n,n}$, $p_{n,n-1}$, $q_{n,n-1}$, etc.

Comparing coefficients of z^{2n+1} in our differential equation above, we obtain

$$-Ap_{n,n}q_{n,n} - Cp_{n,n}^2 - Dq_{n,n}^2 = -\frac{1}{C}\Pi_+\Pi_-.$$

The equation again:

$$-Ap_{n,n}q_{n,n} - Cp_{n,n}^2 - Dq_{n,n}^2 = -\frac{1}{C}\Pi_+\Pi_-.$$

Performing the substitution $E^2 = A^2 - 4CD$, the left-hand side factors:

$$\left(Cp_{n,n} + \tfrac{1}{2}(A-E)q_{n,n}\right)\left(Cp_{n,n} + \tfrac{1}{2}(A+E)q_{n,n}\right) = \Pi_+\Pi_-.$$

The equation again:

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Performing the substitution $E^2 = A^2 - 4CD$, the left-hand side factors:

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Indeed, the computer says:

$$Cp_{n,n} + \tfrac{1}{2}(A - E)q_{n,n} = \Pi_-,$$
$$Cp_{n,n} + \tfrac{1}{2}(A + E)q_{n,n} = \Pi_+.$$

Indeed, the computer says:

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\begin{aligned}
Cp_{n,n} + \tfrac{1}{2}(A - E)q_{n,n} &= \Pi_-,\\
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\end{aligned}
$$

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$$Cp_{n,n} + \tfrac{1}{2}(A+E)q_{n,n} = \Pi_+.$$

Solve for $p_{n,n}$ and $q_{n,n}$:

$$p_{n,n} = \frac{(-1)^{n+1}}{2CE}\big((A-E)\Pi_+ - (A+E)\Pi_-\big),$$
$$q_{n,n} = \frac{(-1)^{n}}{E}\big(\Pi_+ - \Pi_-\big),$$

where $\Pi_+ = B^{n+1}\left(\frac{A+2C+E}{2B}\right)_{n+1}$ and $\Pi_- = B^{n+1}\left(\frac{A+2C-E}{2B}\right)_{n+1}$.

We continue:

Comparing coefficients of z^{2n} in our differential equation, we obtain

$$p_{n,n}q_{n,n} - Ap_{n,n}q_{n,n-1} - Ap_{n,n-1}q_{n,n} - Bp_{n,n}q_{n,n-1} + Bp_{n,n-1}q_{n,n} \\ - 2Cp_{n,n-1}p_{n,n} - q_{n,n}^2 - 2Dq_{n,n-1}q_{n,n} = 0.$$

We consider this equation modulo $\Pi_+ - \Pi_-$, and modulo $\Pi_+ + \Pi_-$. This yields two linear congruences:

$$C_1 p_{n,n-1} + C_2 q_{n,n-1} + C_3 \equiv 0 \pmod{\Pi_+ - \Pi_-},$$
$$C_4 p_{n,n-1} + C_5 q_{n,n-1} + C_6 \equiv 0 \pmod{\Pi_+ + \Pi_-},$$

with explicitly known $C_1, C_2, C_3, C_4, C_5, C_6$. In other words,

$$C_1 p_{n,n-1} + C_2 q_{n,n-1} + C_3 = D_1(\Pi_+ - \Pi_-),$$
$$C_4 p_{n,n-1} + C_5 q_{n,n-1} + C_6 = D_2(\Pi_+ + \Pi_-),$$

for some (unknown) D_1 and D_2.

$$p_{n,n} = \frac{(-1)^{n+1}}{2CE}\big((A-E)\Pi_+ - (A+E)\Pi_-\big),$$

$$q_{n,n} = \frac{(-1)^{n}}{E}\big(\Pi_+ - \Pi_-\big),$$

$$\begin{aligned} p_{n,n-1} = {} & \frac{(-1)^{n+1}}{2C\,(E-B)E(E+B)} \\ & \times \Big(\Pi_+\big(n\,(-E+B)\,(A-E) + (A+2C-E)\,(A+B)\big) \\ & \qquad - \Pi_-\big(\,(E+B)\,(A+E) + (A+2C+E)\,(A+B)\big)\Big), \end{aligned}$$

$$\begin{aligned} q_{n,n-1} = {} & \frac{(-1)^{n}}{(E-B)E(E+B)} \\ & \times \Big(\Pi_+\big(n\,(-E+B) + (A+2C-E)\,\big) \\ & \qquad - \Pi_-\big(n\,(E+B) + (A+2C+E)\,\big)\Big), \end{aligned}$$

$$
\begin{aligned}
p_{n,n-2} &= \frac{(-1)^{n+1}}{2C\,(E-2B)(E-B)E(E+B)(E+2B)} \\
&\times \Bigg(\Pi_+ \Big(\tfrac{1}{2} n(n-1)\,(-E+B)\,(-E+2B)\,(A-E) \\
&\qquad + \tfrac{3}{2}(n-1)\,(-E+2B)\,(A+2C-E)\,\left(A+\tfrac{4}{3}B-\tfrac{1}{3}E\right) \\
&\qquad + \tfrac{3}{2}\,(A+2C-E)\,(A+2C-E+2B)\,(A+2B) \Big) \Bigg) \\
&\quad - \Pi_- \Big(\tfrac{1}{2} n(n-1)\,(E+B)\,(E+2B)\,(A+E) \\
&\qquad + \tfrac{3}{2}(n-1)\,(E+2B)\,(A+2C+E)\,\left(A+\tfrac{4}{3}B+\tfrac{1}{3}E\right) \\
&\qquad + \tfrac{3}{2}\,(A+2C+E)\,(A+2C+E+2B)\,(A+2B) \Big) \Bigg),
\end{aligned}
$$

$$
\begin{aligned}
q_{n,n-2} = {} & \frac{(-1)^n}{(E-2B)(E-B)E(E+B)(E+2B)} \\
& \times \Bigg(\Pi_+ \bigg(\tfrac{1}{2} n(n-1)\,(-E+B)\,(-E+2B) \\
& \qquad + \tfrac{3}{2}(n-1)\,(-E+2B)\,(A+2C-E) \\
& \qquad + \tfrac{3}{2}\,(A+2C-E)\,(A+2C-E+2B) \bigg) \\
& \quad - \Pi_- \bigg(\tfrac{1}{2} n(n-1)\,(E+B)\,(E+2B) \\
& \qquad + \tfrac{3}{2}(n-1)\,(E+2B)\,(A+2C+E) \\
& \qquad + \tfrac{3}{2}\,(A+2C+E)\,(A+2C+E+2B) \bigg) \Bigg)
\end{aligned}
$$

$$q_{n,n-k} = \frac{(-1)^n B^{n-k}}{(\frac{E}{B} - k)_{2k+1}}$$

$$\times \Bigg(\left(\tfrac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^{k} q_{n,k,j} (n-k+1)_{k-j} \left(-\tfrac{E}{B} + j + 1\right)_{k-j} \left(\tfrac{A+2C-E}{2B}\right)_j$$

$$- \left(\tfrac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^{k} q_{n,k,j} (n-k+1)_{k-j} \left(\tfrac{E}{B} + j + 1\right)_{k-j} \left(\tfrac{A+2C+E}{2B}\right)_j \Bigg).$$

$$
\begin{aligned}
q_{n,n-k} &= \frac{(-1)^n B^{n-k}}{(\frac{E}{B}-k)_{2k+1}} \\
&\times \Bigg(\left(\tfrac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^{k} q_{n,k,j} (n-k+1)_{k-j} \left(-\tfrac{E}{B}+j+1\right)_{k-j} \left(\tfrac{A+2C-E}{2B}\right)_j \\
&\quad - \left(\tfrac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^{k} q_{n,k,j} (n-k+1)_{k-j} \left(\tfrac{E}{B}+j+1\right)_{k-j} \left(\tfrac{A+2C+E}{2B}\right)_j \Bigg).
\end{aligned}
$$

How can we guess the coefficients $q_{n,k,j}$?

How can we guess the coefficients $q_{n,k,j}$?

Let us assume the form of $q_{n,k}$ from the previous page, and the earlier observed polynomiality of $q_{n,k}$ in A, B, C, D, and, hence, also in A, B, C, E.

$(\frac{E}{B} - k)_{2k+1}$ must divide the expression between parentheses, that is, this expression must vanish for $E = Bs$, $s = -k, -k+1, \dots, k$. Let us do the substitution $E = Bs$ in this expression, and let us suppose that $s > 0$.

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Then we have

$$\left(\frac{A+2C+E}{2B}\right)_{n+1} = \left(\frac{A+2C+Bs}{2B}\right)_{n+1} = \left(\frac{A+2C+Bs}{2B}\right)_{n+1-s}\left(\frac{A+2C-Bs}{2B}+n+1\right)_s$$

Comparing with

$$\left(\frac{A+2C-E}{2B}\right)_{n+1} = \left(\frac{A+2C-Bs}{2B}\right)_{n+1},$$

we infer that $\left(\frac{A+2C-Bs}{2B}+n+1\right)_s$ must divide the second sum over j *as a polynomial in* n! This provides many non-trivial vanishing conditions for this sum over j, viewed as polynomial in n, and this suffices to compute the coefficients $q_{n,k,j}$ for a large range of k's and corresponding j's.

Voilà! Here is our guess for $q_{n,n-k}$:

$$q_{n,n-k} = \frac{(-1)^n B^{n-k}}{(\frac{E}{B}-k)_{2k+1}}$$
$$\times \Bigg(\left(\tfrac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^{k} \binom{k+j}{k} \binom{n-j}{k-j} \left(-\tfrac{E}{B}+j+1\right)_{k-j} \left(\tfrac{A+2C-E}{2B}\right)_j$$
$$- \left(\tfrac{A+2C-E}{2B}\right)_{n+1} \sum_{j=0}^{k} \binom{k+j}{k} \binom{n-j}{k-j} \left(\tfrac{E}{B}+j+1\right)_{k-j} \left(\tfrac{A+2C+E}{2B}\right)_j \Bigg).$$

Similarly, here is our guess for $p_{n,n-k}$:

$$p_{n,n-k} = \frac{(-1)^{n+1} B^{n-k}}{2C\,(\frac{E}{B}-k)_{2k+1}}$$
$$\times \Bigg(\left(\tfrac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^{k} \binom{k+j}{k} \binom{n-j}{k-j} \left(-\tfrac{E}{B}+j+1\right)_{k-j}$$
$$\cdot \left(\tfrac{A+2C-E}{2B}\right)_j \left(A + \tfrac{2kj}{k+j} B - \tfrac{k-j}{k+j} E\right)$$
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$$\cdot \left(\tfrac{A+2C+E}{2B}\right)_j \left(A + \tfrac{2kj}{k+j} B + \tfrac{k-j}{k+j} E\right) \Bigg),$$

We have a full-fledged guess:

Conjecture

For every positive integer n, there exist uniquely determined polynomials $P_n(z) = 1 + \sum_{k=1}^n p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^n q_{n,k} z^k$, where $p_{n,k}$ and $q_{n,k}$ are homogeneous polynomials in A, B, C, D of degree k over the integers such that the rational function $R_n(z) = P_n(z)/Q_n(z)$ satisfies

$$(1 - Az)R_n(z) - Bz^2 R_n'(z) - CzR_n^2(z) - 1 - Dz$$
$$= -\frac{z^{2n+1}}{Q_n^2(z)}(A+C+D)\prod_{\ell=1}^{n}(\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2).$$

Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

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Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

So far, we did not prove anything!!

Here is what we would like to prove:

Theorem?

For every positive integer n, there exist uniquely determined polynomials $P_n(z) = 1 + \sum_{k=1}^{n} p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^{n} q_{n,k} z^k$, where $p_{n,k}$ and $q_{n,k}$ are homogeneous polynomials in A, B, C, D of degree k over the integers such that the rational function $R_n(z) = P_n(z)/Q_n(z)$ satisfies

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Moreover, the coefficients $p_{n,k}$ and $q_{n,k}$ are given by the previous formulae.

So far, we did not prove anything!!

However, in principle, as soon as the guess for the polynomials $P_n(z) = 1 + \sum_{k=1}^{n} p_{n,k} z^k$ and $Q_n(z) = 1 + \sum_{k=1}^{n} q_{n,k} z^k$ is written down, it is already proved!

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Sketch of proof that $P_n(z)/Q_n(z)$ satisfies the differential equation

The differential equation:

$$\begin{aligned}(1-Az)P_n(z)Q_n(z) - Bz^2(P_n'(z)Q_n(z) - P_n(z)Q_n'(z))\\ - CzP_n^2(z) - (1+Dz)Q_n^2(z)\\ = -z^{2n+1}(A+C+D)\prod_{\ell=1}^{n}(\ell AB+AC+CD+\ell^2B^2+2\ell BC+C^2).\end{aligned}$$

We actually already verified that the coefficients of z^{2n+1} match. (This is how we found the formulae for $p_{n,n}$ and $q_{n,n}$.)

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$$= -z^{2n+1}(A+C+D)\prod_{\ell=1}^{n}(\ell AB + AC + CD + \ell^2 B^2 + 2\ell BC + C^2).$$

Now we divide both sides by z^{2n+1}, then differentiate them with respect to z, and finally multiply both sides of the resulting equation by z^{2n+2}. In this way, we obtain

$$P_n(z)\big(2CnzP_n(z) - 2Cz^2P_n'(z) - (2n+1)Q_n(z) + zQ_n'(z) \\ - z^2(A+2Bn-B)Q_n'(z) + 2AnzQ_n(z) + Bz^3Q_n''(z)\big) \\ +Q_n(z)\big(zP_n'(z) - z^2(A-2Bn+B)P_n'(z) - Bz^3P_n''(z) \\ + (2n+1)Q_n(z) + 2DnzQ_n(z) - 2Dz^2Q_n'(z) - 2zQ_n'(z)\big) = 0.$$

This is equivalent to the original equation!

We claim that, in fact,

$$2CnzP_n(z)-2Cz^2P_n'(z)-(2n+1)Q_n(z)+zQ_n'(z)-z^2(A+2Bn-B)Q_n'(z)$$
$$+2AnzQ_n(z)+Bz^3Q_n''(z)=-(2n+1-nAz+n^2Bz)Q_n(z)$$

and

$$zP_n'(z)-z^2(A-2Bn+B)P_n'(z)-Bz^3P_n''(z)+(2n+1)Q_n(z)+2DnzQ_n(z)$$
$$-2Dz^2Q_n'(z)-2zQ_n'(z)=(2n+1-nAz+n^2Bz)P_n(z).$$

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and

$$zP_n'(z)-z^2(A-2Bn+B)P_n'(z)-Bz^3P_n''(z)+(2n+1)Q_n(z)+2DnzQ_n(z) - 2Dz^2Q_n'(z) - 2zQ_n'(z) = (2n+1-nAz+n^2Bz)P_n(z).$$

This is not too difficult to prove by comparing coefficients of powers of z. The Gosper algorithm is used to prove one of the arising identities.

Sketch of proof of polynomiality of $p_{n,k}$ and $q_{n,k}$

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Recall:

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$$\times \Bigg(\left(\tfrac{A+2C+E}{2B}\right)_{n+1} \sum_{j=0}^{k} \binom{k+j}{k} \binom{n-j}{k-j} \left(-\tfrac{E}{B} + j + 1\right)_{k-j} \left(\tfrac{A+2C-E}{2B}\right)_j$$

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$(\frac{E}{B}-k)_{2k+1}$ must divide the term between parentheses. That is, if we put $E = Bs$ for $s \in \{-k, -k+1, \dots, k\}$, then the term between parentheses must vanish.

So, what we have to establish is the identity

$$\left(\tfrac{A+2C+Bs}{2B}\right)_{n+1}\sum_{j=0}^{k}\binom{k+j}{k}\binom{n-j}{k-j}(-s+j+1)_{k-j}\left(\tfrac{A+2C-Bs}{2B}\right)_{j}$$

$$-\left(\tfrac{A+2C-Bs}{2B}\right)_{n+1}\sum_{j=0}^{k}\binom{k+j}{k}\binom{n-j}{k-j}(s+j+1)_{k-j}\left(\tfrac{A+2C+Bs}{2B}\right)_{j}=0,$$

for $s\in\{-k,-k+1,\dots,k\}$.

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for $s\in\{-k,-k+1,\dots,k\}$.
The old hypergeometric transformation (Thomae (1879))

$${}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right]=\frac{\Gamma(e)\,\Gamma(d+e-a-b-c)}{\Gamma(e-a)\,\Gamma(d+e-b-c)}\,{}_3F_2\left[\begin{matrix}a,d-b,d-c\\ d,d+e-b-c\end{matrix};1\right]$$

does the job.

Back to $p = 7$:

Conjecture

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 7^α, is eventually periodic, with period length $6 \cdot 7^{\alpha-1}$.

Moreover, the generating function $F(z) = 1 + \sum_{n\ge1} f_n\, z^n$, when coefficients are reduced modulo 7^α, equals

$$\frac{P_\alpha(z)}{(1+2z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

For the rest of the talk, we shall always talk about the special case $A = 4,\ B = 6,\ C = 1,\ D = 0,\ E = 4$, corresponding to the differential equation for the free subgroup numbers for $PSL_2(\mathbb{Z})$.

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Lemma

Let p be a prime with $p \equiv 1 \pmod 6$. For $n \equiv \frac{p-1}{6} \pmod p$, we have

$$Q_n(z) = Q_{(p-1)/6}(z) \quad \text{modulo } p.$$

Furthermore, the polynomial $Q_{(p-1)/6}(z)$ has degree $(p-1)/6$ in z.

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Furthermore, the polynomial $Q_{(p-1)/6}(z)$ has degree $(p-1)/6$ in z.

Sketch of proof. One applies the same ${}_3F_2$-transformation again to the sums over j in the definition of $q_{n,n-k}$, to obtain

$$q_{n,n-k} = (-1)^n 6^{n-k} \left(\sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \frac{\left(\frac{5}{6} - j\right)_{n+k+1}}{\left(\frac{2}{3} - j\right)_{k+1}} + \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \frac{\left(\frac{1}{6} - j\right)_{n+k+1}}{\left(-\frac{2}{3} - j\right)_{k+1}} \right).$$

Now elementary (though tedious) p-adic analysis shows that each summand is divisible by p for $n \geq k + \frac{p-1}{6} + 1$. □

Theorem

Let α be a positive integer. If p is a prime with $p \equiv 1 \pmod 6$, then the generating function $F(z) = 1 + \sum_{n\ge1} f_n\, z^n$, when coefficients are reduced modulo p^α, equals $A_\alpha(z)/Q^\alpha_{(p-1)/6}(z)$, where $A_\alpha(z)$ is a polynomial in z over the integers.

There is a similar statement for $p \equiv 5 \pmod 6$.

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Induction step: Choose $n \equiv \frac{p-1}{6} \pmod p$ large enough such that the product over ℓ on the right-hand side of the "Padé-approximated" differential equation vanishes modulo $p^{\alpha+1}$, and thus $P_n(z)/Q_n(z)$ solves the differential equation modulo $p^{\alpha+1}$. Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} \quad \text{modulo } p^\alpha.$$

PROOF (CONTINUED).

Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} \quad \text{modulo } p^\alpha.$$

PROOF (CONTINUED).

Then

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} \quad \text{modulo } p^\alpha.$$

Consequently, in the difference

$$\frac{P_n(z)}{Q_n(z)} - \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} = \frac{P_n(z)Q^\alpha_{(p-1)/6}(z) - A_\alpha(z)Q_n(z)}{Q^\alpha_{(p-1)/6}(z)Q_n(z)},$$

all coefficients of the integer polynomial in the numerator of the last fraction must be divisible by p^α. In other words, there is an integer polynomial $B_\alpha(z)$ such that

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} + p^\alpha \frac{B_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)Q_n(z)}.$$

By the previous lemma, this leads to

$$\frac{P_n(z)}{Q_n(z)} = \frac{A_\alpha(z)}{Q^\alpha_{(p-1)/6}(z)} + p^\alpha \frac{B_\alpha(z)}{Q^{\alpha+1}_{(p-1)/6}(z)} \quad \text{modulo } p^{\alpha+1}. \qquad \square$$

Our differential equation for the generating function for the numbers of free subgroups in $PSL_2(\mathbb{Z})$:

$$(1-4z)F(z)-6z^2F'(z)-zF^2(z)-1=0.$$

Theorem

Let α be a positive integer. If p is a prime with $p \equiv 1 \pmod 6$, then the generating function $F(z)=1+\sum_{n\ge1} f_n\, z^n$, when coefficients are reduced modulo p^α, equals $A_\alpha(z)/Q^\alpha_{(p-1)/6}(z)$, where $A_\alpha(z)$ is a polynomial in z over the integers.

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A *Mathematica* implementation of the algorithm just described that proves the above theorem is available.

f_n modulo powers of 7

Corollary

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 7^α, is eventually periodic, with period length $6\cdot 7^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 7^α, equals

$$\frac{P_\alpha(z)}{(1+2z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 11

Corollary

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 11^α, is eventually periodic, with period length $11^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 11^α, equals

$$\frac{P_\alpha(z)}{(1-z)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 13

Corollary

Let α be a positive integer. The sequence $(f_n)_{n\ge1}$, considered modulo 13^α, is eventually periodic, with period length $12\cdot 13^{\alpha-1}$.

Moreover, the generating function $F(z)=1+\sum_{n\ge1} f_n\,z^n$, when coefficients are reduced modulo 13^α, equals

$$\frac{P_\alpha(z)}{\big((1-2z)(1+5z)\big)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.

f_n modulo powers of 17

Corollary

Let α be a positive integer. The generating function $F(z) = 1 + \sum_{n\ge1} f_n\, z^n$, when coefficients are reduced modulo 17^α, equals

$$\frac{P_\alpha(z)}{(1+15z+7z^2)^\alpha},$$

where $P_\alpha(z)$ is a polynomial in z over the integers.