```
In[1]:= << RISC`HolonomicFunctions`
```

HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
written by Christoph Koutschan
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Johannes Kepler University, Linz, Austria

```
--> Type  ?HolonomicFunctions  for help.
```

Univariate examples from the slides

$\mathrm{erf}\left(\sqrt{x+1}\right)^2 + \exp^2\left(\sqrt{x+1}\right)$

First, by executing the corresponding closure properties "by hand":

```
In[2]:= annErf = ToOrePolynomial[{Der[z]^2 + 2 z ** Der[z]}]
```

Out[2]= $\{D_z^2 + 2\, z\, D_z\}$

```
In[3]:= ApplyOreOperator[annErf, Erf[z]]
```

Out[3]= $\{0\}$

```
In[4]:= annErf1 = DFiniteSubstitute[annErf, {z → Sqrt[x + 1]}, Algebra → OreAlgebra[Der[x]]]
```

Out[4]= $\{(2 + 2\,x)\, D_x^2 + (3 + 2\,x)\, D_x\}$

```
In[5]:= annErf2 = DFiniteTimes[annErf1, annErf1]
```

Out[5]= $\{(2 + 2\,x)\, D_x^3 + (9 + 6\,x)\, D_x^2 + (8 + 4\,x)\, D_x\}$

```
In[6]:= ApplyOreOperator[annErf2, Erf[Sqrt[x + 1]]^2]
```

Out[6]=
$$\left\{\frac{2\,e^{-1-x}\,(8+4\,x)\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{\sqrt{\pi}\,\sqrt{1+x}} + (9+6\,x)\left(\frac{2\,e^{-2-2\,x}}{\pi\,(1+x)} - \frac{e^{-1-x}\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{\sqrt{\pi}\,(1+x)^{3/2}} - \frac{2\,e^{-1-x}\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{\sqrt{\pi}\,\sqrt{1+x}}\right) + (2+2\,x)\left(-\frac{3\,e^{-2-2\,x}}{\pi\,(1+x)^2} - \frac{6\,e^{-2-2\,x}}{\pi\,(1+x)} + \frac{3\,e^{-1-x}\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{2\,\sqrt{\pi}\,(1+x)^{5/2}} + \frac{2\,e^{-1-x}\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{\sqrt{\pi}\,(1+x)^{3/2}} + \frac{2\,e^{-1-x}\,\mathrm{Erf}\left[\sqrt{1+x}\right]}{\sqrt{\pi}\,\sqrt{1+x}}\right)\right\}$$

```
In[7]:= Together[%]
```

Out[7]= $\{0\}$

```
In[8]:= annExp = ToOrePolynomial[{Der[z] - 1}]
```

Out[8]= $\{D_z - 1\}$

```
In[9]:= annExp2 = DFiniteSubstitute[annExp, {z → 2 * Sqrt[x + 1]}, Algebra → OreAlgebra[Der[x]]]
```

Out[9]= $\{(2 + 2\,x)\, D_x^2 + D_x - 2\}$

In[10]:=
```
DFinitePlus[annErf2, annExp2]
```

Out[10]= $\{(20 + 56 x + 84 x^2 + 80 x^3 + 32 x^4) D_x^5 + (144 + 252 x + 348 x^2 + 336 x^3 + 96 x^4) D_x^4 + (249 + 154 x + 312 x^2 + 336 x^3 + 64 x^4) D_x^3 + (52 - 88 x - 32 x^2) D_x^2 + (-84 + 24 x - 64 x^2 - 64 x^3) D_x\}$

Second, by using the convenient Annihilator command:

In[11]:=
```
Annihilator[Erf[Sqrt[x + 1]]^2 + Exp[Sqrt[x + 1]]^2, Der[x]]
```

Out[11]= $\{(20 + 56 x + 84 x^2 + 80 x^3 + 32 x^4) D_x^5 + (144 + 252 x + 348 x^2 + 336 x^3 + 96 x^4) D_x^4 + (249 + 154 x + 312 x^2 + 336 x^3 + 64 x^4) D_x^3 + (52 - 88 x - 32 x^2) D_x^2 + (-84 + 24 x - 64 x^2 - 64 x^3) D_x\}$

$$\left(\sinh^2(x) + \frac{1}{\sin^2(x)}\right)\left(\cosh^2(x) + \frac{1}{\cos^2(x)}\right)$$

In[12]:=
```
Annihilator[(Sinh[x]^2 + Sin[x]^(-2)) * (Cosh[x]^2 + Cos[x]^(-2)), Der[x]]
```

Annihilator: The expression Sec[x] is not recognized to be ∂–finite. The result might not generate a zero–dimensional ideal.

Out[12]= { }

$$\frac{\log\left(\sqrt{1-x^2}\right)}{\exp\left(\sqrt{1-x^2}\right)}$$

In[13]:=
```
Annihilator[Log[Sqrt[1 - x^2]] / Exp[Sqrt[1 - x^2]], Der[x]]
```

Out[13]= $\{(5 x^3 - 19 x^5 + 27 x^7 - 17 x^9 + 4 x^{11}) D_x^4 + (-30 x^2 + 72 x^4 - 46 x^6 - 4 x^8 + 8 x^{10}) D_x^3 + (75 x - 159 x^3 + 96 x^5 + 10 x^7 - 30 x^9 + 8 x^{11}) D_x^2 + (-75 + 159 x^2 - 96 x^4 + 14 x^6 - 12 x^8 + 8 x^{10}) D_x + (7 x^7 - 13 x^9 + 4 x^{11})\}$

$$\tan^{-1}(\exp(x))$$

In[14]:=
```
Annihilator[ArcTan[Exp[x]], Der[x]]
```

DFiniteSubstitute: The substitutions for continuous variables $\{e^x\}$ are supposed to be algebraic expressions. Not all of them are recognized to be algebraic. The result might not generate a ∂–finite ideal.

Annihilator: The expression (w.r.t. {Der[x]}) is not recognized to be ∂–finite. The result might not generate a zero–dimensional ideal.

Out[14]= { }

Finite Element Methods

In[15]:=
```
annphi = Annihilator[(1 - x) ^ i * JacobiP[j, 2 i + 1, 0, 2 x - 1] *
    LegendreP[i, 2 y / (1 - x) - 1], {S[i], S[j], Der[x], Der[y]}]
```

Out[15]=

$$\{(2+2i+3j+2ij+j^2)S_j + (3x+2ix+2jx-3x^2-2ix^2-2jx^2)D_x + (-3xy-2ixy-2jxy)D_y + (2+2i+3j+2ij+j^2-6x-7ix-2i^2x-7jx-4ijx-2j^2x), \cdots 1 \cdots, \cdots 4 \cdots, (\cdots 190 \cdots + 62j^4x^3+129ij^4x^3+85i^2j^4x^3+18i^3j^4x^3+4j^5x^3+6ij^5x^3+2i^2j^5x^3) S_i^2 + \cdots 3 \cdots + (\cdots 524 \cdots + \cdots 1 \cdots + 40ij^5xy^2+16i^2j^5xy^2)\}$$

large output | show less | show more | show all | set size limit...

In[16]:=
```
ByteCount[annphi]
```

Out[16]= 461 272

In[17]:=
```
Support[annphi]
```

Out[17]=

$$\{\{S_j, D_x, D_y, 1\}, \{D_y^2, D_y, 1\}, \{D_x D_y, S_i, D_x, D_y, 1\}, \{D_x^2, S_i, D_x, D_y, 1\}, \{S_i D_y, S_i, D_x, D_y, 1\}, \{S_i D_x, S_i, D_x, D_y, 1\}, \{S_i^2, S_i, D_x, D_y, 1\}\}$$

In[18]:=
```
FindRelation[annphi, Eliminate → {x, y}, Pattern → {_, _, 0 | 1, 0}]
```

Out[18]=

$$\{(-25-20i-4i^2-15j-6ij-2j^2)S_i S_j^2 D_x + (-15-6i-11j-2ij-2j^2)S_j^3 D_x + (-18-18i-4i^2-6j-4ij)S_i S_j D_x + (6+14i+4i^2+2j+4ij)S_j^2 D_x + (210+214i+72i^2+8i^3+214j+144ij+24i^2j+72j^2+24ij^2+8j^3)S_i S_j + (7+2i+9j+2ij+2j^2)S_i D_x + (210+214i+72i^2+8i^3+214j+144ij+24i^2j+72j^2+24ij^2+8j^3)S_j^2 + (21+20i+4i^2+13j+6ij+2j^2)S_j D_x\}$$

In[19]:=
```
ApplyOreOperator[Factor[First[%]], ϕi,j[x]]
```

Out[19]=

$$2(3+i+j)(5+2i+2j)(7+2i+2j)\phi_{i,2+j}[x] + 2(3+i+j)(5+2i+2j)(7+2i+2j)\phi_{1+i,1+j}[x] + (3+2i+j)(7+2i+2j)\phi_{i,1+j}'[x] + 2(1+2i)(3+i+j)\phi_{i,2+j}'[x] - (3+j)(5+2i+2j)\phi_{i,3+j}'[x] + (1+j)(7+2i+2j)\phi_{1+i,j}'[x] - 2(3+2i)(3+i+j)\phi_{1+i,1+j}'[x] - (5+2i+j)(5+2i+2j)\phi_{1+i,2+j}'[x]$$

Example from Gradshteyn & Ryzhik

In[20]:=
```
CreativeTelescoping[(1 - x^2)^(nu - 1/2) * Exp[I * a * x] * GegenbauerC[n, nu, x],
  Der[x], {S[n], Der[a]}] // Timing
```

Out[20]=

$$\{0.342969, \{\{(a + a\,n)\,S_n + (i\,a\,n + 2\,i\,a\,nu)\,D_a + (-i\,n^2 - 2\,i\,n\,nu),\ a^2\,D_a^2 + (a + 2\,a\,nu)\,D_a + (a^2 - n^2 - 2\,n\,nu)\},\ \{i\,(1 + n)\,S_n - i\,(n\,x + 2\,nu\,x),\ (1 + n)\,S_n - i\,(-a - i\,n\,x - 2\,i\,nu\,x + a\,x^2)\}\}\}$$

In[21]:=
```
Printlevel = 5;
CreativeTelescoping[
 (1 - x^2)^(nu - 1/2) * Exp[I * a * x] * GegenbauerC[n, nu, x], Der[x], {S[n], Der[a]}]
```

```
    Annihilator called with E^(I*a*x)*(1 - x^2)^(-1/2 + nu)*GegenbauerC[n, nu, x].
    Annihilator: The factors that contain not-to-be-evaluated elements are {}
    Annihilator: The remaining factors
 are {E^(I*a*x), (1 - x^2)^(-1/2 + nu), GegenbauerC[n, nu, x]}
    Annihilator: Factors that are not
 hypergeometric and hyperexponential: {GegenbauerC[n, nu, x]}
        Monomials: {1}
        Monomials: {Der[a], S[n], Der[x]}
        Monomials: {S[n], Der[x]}
        Monomials: {Der[x], Der[a]*S[n], S[n]^2, Der[x]*S[n]}
        Monomials: {Der[a]*S[n], S[n]^2, Der[x]*S[n]}
        Monomials: {S[n]^2, Der[x]*S[n]}
        Monomials: {Der[x]*S[n]}
 CreativeTelescoping: using Method -> "Chyzak".
 CreativeTelescoping: Trying d = 0,
 ansatz = Der[x]**(phi[1][x]**1 + phi[2][x]**S[n]) + eta[0]**1
 LocalOreReduce: Reducing {0, 0, 0}
 LocalOreReduce: Reducing {0, 1, 0}
 LocalOreReduce: Reducing {1, 0, 0}
 LocalOreReduce: Reducing {1, 1, 0}
   Start uncoupling.
       OreGroebnerBasis: Number of pairs: 1
       OreGroebnerBasis: Taking {3, {2, 1}, 1, 2}
         OreReduce: LPP = {2, 0}
         OreReduce: reduced with nr. 1
         OreReduce: LPP = {1, 2}
         OreReduce: Leading term cannot be reduced. Stop the reduction process.
```

```
    OreGroebnerBasis: Does not reduce to zero -> number 4 in the basis.
                      The lpp is {1, 2}. The ByteCount is 3392.
    OreReduce: LPP = {0, 1}
    OreReduce: reduced with nr. 1
    OreReduce: LPP = {1, 2}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {1, 1}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {1, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {0, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {2, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {1, 1}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {1, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {2, 1}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {2, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {1, 0}
    OreReduce: Put leading term into remainder.
    OreReduce: LPP = {0, 0}
    OreReduce: Put leading term into remainder.
Finished uncoupling.
Start to solve scalar equation...
  DSolveRational: got a differential equation of order 2
  DSolveRational: Denominator v = 1
    DSolvePolynomial: bound = 0
Solved scalar equation.
Start to solve scalar equation...
  RSolveRational: got a recurrence of order 0
Solved scalar equation.
```

```
CreativeTelescoping: Trying d = 1, ansatz =
Der[x]**(phi[1][x]**1 + phi[2][x]**S[n]) + eta[0]**1 + eta[1]**Der[a]
LocalOreReduce: Reducing {0, 0, 1}
  Start uncoupling.
      OreGroebnerBasis: Number of pairs: 1
      OreGroebnerBasis: Taking {3, {2, 1}, 1, 2}
        OreReduce: LPP = {2, 0}
        OreReduce: reduced with nr. 1
        OreReduce: LPP = {1, 2}
        OreReduce: Leading term cannot be reduced. Stop the reduction process.
      OreGroebnerBasis: Does not reduce to zero -> number 4 in the basis.
                        The lpp is {1, 2}. The ByteCount is 3784.
      OreReduce: LPP = {0, 1}
      OreReduce: reduced with nr. 1
      OreReduce: LPP = {1, 2}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {0, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {0, 0}
      OreReduce: Put leading term into remainder.
```

```
  Finished uncoupling.
  Start to solve scalar equation...
    DSolveRational: got a differential equation of order 2
    DSolveRational: Denominator v = 1
      DSolvePolynomial: bound = 0
  Solved scalar equation.
  Start to solve scalar equation...
    RSolveRational: got a recurrence of order 0
  Solved scalar equation.
CreativeTelescoping: Trying d = 2, ansatz = Der[x]**(phi[1][x]**1
+ phi[2][x]**S[n]) + eta[0]**1 + eta[1]**Der[a] + eta[2]**S[n]
  Start uncoupling.
      OreGroebnerBasis: Number of pairs: 1
      OreGroebnerBasis: Taking {3, {2, 1}, 1, 2}
        OreReduce: LPP = {2, 0}
        OreReduce: reduced with nr. 1
        OreReduce: LPP = {1, 2}
        OreReduce: Leading term cannot be reduced. Stop the reduction process.
      OreGroebnerBasis: Does not reduce to zero -> number 4 in the basis.
                        The lpp is {1, 2}. The ByteCount is 5104.
      OreReduce: LPP = {0, 1}
      OreReduce: reduced with nr. 1
      OreReduce: LPP = {1, 2}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {0, 1}
      OreReduce: reduced with nr. 1
      OreReduce: LPP = {0, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
```

```
        OreReduce: LPP = {1, 0}
        OreReduce: Put leading term into remainder.
        OreReduce: LPP = {0, 0}
        OreReduce: Put leading term into remainder.
        OreReduce: LPP = {2, 1}
        OreReduce: Put leading term into remainder.
        OreReduce: LPP = {2, 0}
        OreReduce: Put leading term into remainder.
        OreReduce: LPP = {1, 0}
        OreReduce: Put leading term into remainder.
        OreReduce: LPP = {0, 0}
        OreReduce: Put leading term into remainder.
    Finished uncoupling.
    Start to solve scalar equation...
      DSolveRational: got a differential equation of order 2
      DSolveRational: Denominator v = 1
        DSolvePolynomial: bound = 0
    Solved scalar equation.
    Start to solve scalar equation...
      RSolveRational: got a recurrence of order 0
    Solved scalar equation.
  CreativeTelescoping: Trying d = 2, ansatz = Der[x]**(phi[1][x]**1
  + phi[2][x]**S[n]) + eta[0]**1 + eta[1]**Der[a] + eta[2]**Der[a]^2
  LocalOreReduce: Reducing {0, 0, 2}
    Start uncoupling.
        OreGroebnerBasis: Number of pairs: 1
        OreGroebnerBasis: Taking {3, {2, 1}, 1, 2}
          OreReduce: LPP = {2, 0}
          OreReduce: reduced with nr. 1
          OreReduce: LPP = {1, 2}
          OreReduce: Leading term cannot be reduced. Stop the reduction process.
        OreGroebnerBasis: Does not reduce to zero -> number 4 in the basis.
                          The lpp is {1, 2}. The ByteCount is 4192.
        OreReduce: LPP = {0, 1}
        OreReduce: reduced with nr. 1
        OreReduce: LPP = {1, 2}
```

```
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {0, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 1}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {2, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {1, 0}
      OreReduce: Put leading term into remainder.
      OreReduce: LPP = {0, 0}
      OreReduce: Put leading term into remainder.
  Finished uncoupling.
  Start to solve scalar equation...
    DSolveRational: got a differential equation of order 2
    DSolveRational: Denominator v = 1
      DSolvePolynomial: bound = 0
  Solved scalar equation.
  Start to solve scalar equation...
    RSolveRational: got a recurrence of order 0
  Solved scalar equation.
```

Out[22]= $\{\{(a + a\,n)\,S_n + (i\,a\,n + 2\,i\,a\,nu)\,D_a + (-i\,n^2 - 2\,i\,n\,nu),\ a^2\,D_a^2 + (a + 2\,a\,nu)\,D_a + (a^2 - n^2 - 2\,n\,nu)\},$
$\{i\,(1 + n)\,S_n - i\,(n\,x + 2\,nu\,x),\ (1 + n)\,S_n - i\,(-a - i\,n\,x - 2\,i\,nu\,x + a\,x^2)\}\}$

In[23]:= **Printlevel = 0;**

In[24]:=
```
Annihilator[Pi * 2 ^ (1 - nu) * I ^ n *
   Gamma[2 nu + n] / n ! / Gamma[nu] * a ^ (-nu) * BesselJ[nu + n, a], {S[n], Der[a]}]
```

Out[24]= $\{(a + a\,n)\,S_n + (i\,a\,n + 2\,i\,a\,nu)\,D_a + (-i\,n^2 - 2\,i\,n\,nu),\ a^2\,D_a^2 + (a + 2\,a\,nu)\,D_a + (a^2 - n^2 - 2\,n\,nu)\}$

Holonomic Special Function Identities

(1) $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{k+n}{k}\sum_{j=0}^{k}\binom{k}{j}^3$

In[25]:=
```
CreativeTelescoping[Binomial[k, j]^3, S[j] - 1, {S[k], S[n]}][[1]]
```

Out[25]= $\{S_n - 1,\ (4 + 4\,k + k^2)\,S_k^2 + (-16 - 21\,k - 7\,k^2)\,S_k + (-8 - 16\,k - 8\,k^2)\}$

In[26]:=
```
DFiniteTimes[Annihilator[Binomial[n, k] * Binomial[k + n, k], {S[k], S[n]}], %]
```

Out[26]= $\{(-1 + k - n)\,S_n + (1 + k + n),\ (16 + 32\,k + 24\,k^2 + 8\,k^3 + k^4)\,S_k^2 + (32 + 90\,k + 93\,k^2 + 42\,k^3 + 7\,k^4 - 16\,n - 21\,k\,n - 7\,k^2\,n - 16\,n^2 - 21\,k\,n^2 - 7\,k^2\,n^2)\,S_k + (-16\,k - 40\,k^2 - 32\,k^3 - 8\,k^4 + 16\,n + 32\,k\,n + 16\,k^2\,n + 8\,n^2 + 32\,k\,n^2 + 16\,k^2\,n^2 - 16\,n^3 - 8\,n^4)\}$

In[27]:=
```
CreativeTelescoping[%, S[k] - 1][[1]]
```

Out[27]= $\{(8 + 12\,n + 6\,n^2 + n^3)\,S_n^2 + (-117 - 231\,n - 153\,n^2 - 34\,n^3)\,S_n + (1 + 3\,n + 3\,n^2 + n^3)\}$

In[28]:=
```
CreativeTelescoping[Binomial[n, k]^2 * Binomial[n + k, k]^2, S[k] - 1, S[n]]
```

Out[28]= $\{\{(8 + 12\,n + 6\,n^2 + n^3)\,S_n^2 + (-117 - 231\,n - 153\,n^2 - 34\,n^3)\,S_n + (1 + 3\,n + 3\,n^2 + n^3)\},\ \{(4\,(3 + 2\,n)\,(8\,k^4 + 3\,k^5 - 2\,k^6 + 12\,k^4\,n + 4\,k^4\,n^2)) / (2 - 3\,k + k^2 + 3\,n - 2\,k\,n + n^2)^2\}\}$

(4) $\int_{-\infty}^{\infty}\left(\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\frac{H_m(x)\,H_n(x)\,r^m\,s^n\exp(-x^2)}{m!\,n!}\right)\right)dx = \sqrt{\pi}\,\exp(2\,r\,s)$

In[29]:=
```
CreativeTelescoping[HermiteH[m, x] * HermiteH[n, x] * r^m * s^n * Exp[-x^2] / m! / n!,
   S[n] - 1, {S[m], Der[x], Der[r], Der[s]}][[1]]
```

Out[29]= $\{D_s + (2\,s - 2\,x),\ r\,D_r - m,\ (1 + m)\,S_m + r\,D_x - 2\,r\,s,\ D_x^2 + (-4\,s + 2\,x)\,D_x + (2 + 2\,m + 4\,s^2 - 4\,s\,x)\}$

In[30]:=
```
CreativeTelescoping[%, S[m] - 1][[1]]
```

Out[30]= $\{D_s + (2\,s - 2\,x),\ D_r + (2\,r - 2\,x),\ D_x + (-2\,r - 2\,s + 2\,x)\}$

In[31]:=
```
CreativeTelescoping[%, Der[x]][[1]]
```

Out[31]= $\{D_s - 2\,r,\ D_r - 2\,s\}$

Proof of Di Francesco's Conjecture

Doron Zeilberger wrote (23.06.2021):

Philippe Di Francesco just gave a great talk at the Lattice path conference mentioning, inter alia, a certain conjectured determinant.

It is Conj. 8.1 (combined with Th. 8.2) in https://arxiv.org/pdf/2102.02920.pdf
I am curious if you can prove it by the Koutschan-Zeilberger-Aek holonomic ansatz method.
If you can do it before Friday, June 25, 2021, 17:00 Paris time, I will mention it in my talk in that conference.

In[32]:=
```
(* The matrix entries *)
mya[i_, j_] :=
  FunctionExpand[2^i * Binomial[i + 2 j + 1, 2 j + 1] - Binomial[i - 1, 2 j + 1]];
TableForm[Table[mya[i, j], {i, 0, 5}, {j, 0, 5}]]
```

Out[33]//TableForm=

2	2	2	2	2	2
4	8	12	16	20	24
11	40	84	144	220	312
30	160	448	960	1760	2912
77	559	2016	5280	11 440	21 840
188	1788	8064	25 344	64 064	139 776

In[34]:=
```
(* Annihilator of the matrix entries a_{i,j} *)
anna = Annihilator[mya[i, j], {S[i], S[j]}];
Factor[anna]
```

Out[35]= $\{2 i (1 + i) (-1 + i - 2 j) (5 + 4 j) S_i - 2 (1 + j) (3 + 2 j) (-4 + i + i^2 - 12 j - 8 j^2) S_j + (2 + i + 2 j) (-12 + 19 i - 16 i^2 + i^3 - 44 j + 46 i j - 14 i^2 j - 48 j^2 + 24 i j^2 - 16 j^3),$
$4 (1 + j) (2 + j) (3 + 2 j) (5 + 2 j) (5 + 4 j) S_j^2 - 4 (1 + j) (3 + 2 j) (7 + 4 j) (11 + i^2 + 14 j + 4 j^2) S_j + (-3 + i - 2 j) (-2 + i - 2 j) (2 + i + 2 j) (3 + i + 2 j) (9 + 4 j)\}$

In[36]:=
```
(* Test conjectured identity. *)
Table[Det[Table[mya[i, j], {i, 0, n - 1}, {j, 0, n - 1}]] /
  (2 * Product[2^(i - 1) * (4 i - 2) ! / (n + 2 i - 1) !, {i, n}]), {n, 10}]
```

Out[36]= {1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

In[37]:=
```
data = Table[
   ns = NullSpace[Table[mya[i, j], {i, 0, n - 2}, {j, 0, n - 1}]][[1]];
   Together[ns / Last[ns]]
   , {n, 2, 30}];
```

In[38]:=
```
Take[data, 10]
```

Out[38]=

$$\left\{\{-1, 1\}, \{1, -2, 1\}, \left\{-\frac{16}{15}, \frac{47}{15}, -\frac{46}{15}, 1\right\}, \left\{\frac{16}{13}, -\frac{60}{13}, \frac{85}{13}, -\frac{54}{13}, 1\right\}, \left\{-\frac{20}{13}, \frac{88}{13}, -\frac{633}{52}, \frac{291}{26}, -\frac{21}{4}, 1\right\}, \left\{\frac{2008}{969}, -\frac{9808}{969}, \frac{2441}{114}, -\frac{8107}{323}, \frac{33115}{1938}, -\frac{362}{57}, 1\right\}, \left\{-\frac{10592}{3553}, \frac{55360}{3553}, -\frac{7712}{209}, \frac{16567}{323}, -\frac{159022}{3553}, \frac{5062}{209}, -\frac{82}{11}, 1\right\}, \left\{\frac{2608}{575}, -\frac{2848}{115}, \frac{36496}{575}, -\frac{57388}{575}, \frac{59828}{575}, -\frac{41696}{575}, \frac{18739}{575}, -\frac{214}{25}, 1\right\}, \left\{-\frac{32432}{4485}, \frac{182176}{4485}, -\frac{12656}{115}, \frac{849728}{4485}, -\frac{1011076}{4485}, \frac{56467}{299}, -\frac{492191}{4485}, \frac{8228}{195}, -\frac{29}{3}, 1\right\}, \left\{\frac{161632}{13485}, -\frac{924992}{13485}, \frac{2606624}{13485}, -\frac{4799104}{13485}, \frac{1262497}{2697}, -\frac{6078586}{13485}, \frac{4266601}{13485}, -\frac{425608}{2697}, \frac{47679}{899}, -\frac{334}{31}, 1\right\}\right\}$$

In[39]:=
```
<< RISC`Guess`
```

Package GeneratingFunctions version 0.8 written by Christian Mallinger
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

Guess Package version 0.52
written by Manuel Kauers
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[40]:=
```
guess = GuessMultRE[PadRight[data], {c[n, j], c[n + 1, j], c[n, j + 1], c[n + 2, j],
    c[n + 1, j + 1], c[n, j + 2]}, {n, j}, 8, StartPoint → {2, 0}, Infolevel → 5];
```

```
270 terms
collecting nonzero points...
modular system: 370 eqns, 270 vars
6 solutions predicted.
refined system: 324 eqns, 224 vars
Q.
9 223 372 036 854 775 783
9 223 372 036 854 775 643
{0.137638, 0, 0.065053, 0.148379}
6 solutions.
```

In[41]:=
```
annc = OreGroebnerBasis[NormalizeCoefficients /@ ToOrePolynomial[guess, c[n, j]]];
#[annc] & /@ {ByteCount, Support}
```

Out[42]= $\{164\,120, \{\{S_j^2, S_n, S_j, 1\}, \{S_n S_j, S_n, S_j, 1\}, \{S_n^2, S_n, S_j, 1\}\}\}$

In[43]:=
```
annc // Factor
```

Out[43]=

$$
\begin{aligned}
&\{-(1+j)\,(6+2j-n)\,(7+2j-n)\,(3+j+n)\,(1+3n)\\
&\quad(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-108jn-168j^2n-\\
&\quad\quad 132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2)\,S_j^2+\\
&\quad 3\,(2+j)\,(21+56j+46j^2+16j^3+2j^4)\,(j-n)\,n^2\,(1+2n)\,(-1+3n)\,(1+3n)\,S_n+\\
&\quad 2\,(1+j)\,(1+3n)\,(1872j+9168j^2+19\,060j^3+21\,952j^4+15\,320j^5+6640j^6+1748j^7+\\
&\quad\quad 256j^8+16j^9-2106n-11\,580jn-27\,220j^2n-35\,864j^3n-28\,700j^4n-\\
&\quad\quad 14\,184j^5n-4220j^6n-692j^7n-48j^8n+1986n^2+9197jn^2+17\,672j^2n^2+\\
&\quad\quad 18\,604j^3n^2+11\,396j^4n^2+4032j^5n^2+764j^6n^2+60j^7n^2+111n^3+469jn^3+\\
&\quad\quad 820j^2n^3+768j^3n^3+408j^4n^3+112j^5n^3+12j^6n^3-123n^4-458jn^4-672j^2n^4-\\
&\quad\quad 496j^3n^4-180j^4n^4-24j^5n^4+78n^5+248jn^5+280j^2n^5+144j^3n^5+24j^4n^5)\,S_j-\\
&\quad (1+2j+n)\,(720j+3648j^2+7844j^3+9340j^4+6736j^5+3016j^6+820j^7+124j^8+\\
&\quad\quad 8j^9-810n-3414jn-4346j^2n+1316j^3n+9104j^4n+10\,452j^5n+6048j^6n+\\
&\quad\quad 1954j^7n+336j^8n+24j^9n-645n^2-7592jn^2-26\,703j^2n^2-45\,406j^3n^2-\\
&\quad\quad 43\,318j^4n^2-24\,394j^5n^2-8054j^6n^2-1442j^7n^2-108j^8n^2+4146n^3+22\,215jn^3+\\
&\quad\quad 48\,959j^2n^3+56\,710j^3n^3+37\,202j^4n^3+13\,922j^5n^3+2774j^6n^3+228j^7n^3-\\
&\quad\quad 2721n^4-12\,651jn^4-22\,382j^2n^4-19\,296j^3n^4-8800j^4n^4-2052j^5n^4-\\
&\quad\quad 192j^6n^4+1542n^5+4898jn^5+5588j^2n^5+3072j^3n^5+832j^4n^5+88j^5n^5),\\
&-3\,(1+j)\,(3+j+n)\,(-1+3n)\,(1+3n)\,(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-\\
&\quad\quad 108jn-168j^2n-132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2)\\
&\quad S_n S_j+3\,(2+j)\,(j-n)\,(-1+3n)\,(1+3n)\\
&\quad(72+288j+466j^2+390j^3+178j^4+42j^5+4j^6+81n+288jn+372j^2n+\\
&\quad\quad 228j^3n+68j^4n+8j^5n+54n^2+156jn^2+144j^2n^2+56j^3n^2+8j^4n^2)\,S_n+\\
&\quad 2\,(1+j)\,(5+2j-n)\,(-1+4n)\,(1+4n)\,(36+186j+380j^2+404j^3+232j^4+\\
&\quad\quad 68j^5+8j^6+45n+168jn+240j^2n+172j^3n+60j^4n+8j^5n+9n^2+30jn^2+\\
&\quad\quad 36j^2n^2+20j^3n^2+4j^4n^2)\,S_j-2\,(2+j)\,(1+2j+n)\,(-1+4n)\,(1+4n)\\
&\quad(90+366j+632j^2+580j^3+292j^4+76j^5+8j^6-27n-132jn-228j^2n-\\
&\quad\quad 172j^3n-60j^4n-8j^5n+27n^2+78jn^2+72j^2n^2+28j^3n^2+4j^4n^2),\\
&9\,(-1+j-n)\,(1+n)\,(3+j+n)\,(3+2n)\,(-1+3n)\,(1+3n)^2\,(2+3n)\,(4+3n)\\
&\quad(24j+88j^2+126j^3+88j^4+30j^5+4j^6-27n-108jn-168j^2n-\\
&\quad\quad 132j^3n-52j^4n-8j^5n+18n^2+60jn^2+72j^2n^2+40j^3n^2+8j^4n^2)\,S_n^2-\\
&\quad 6\,(-1+3n)\,(1+3n)\,(-7776j-37\,152j^2-67\,608j^3-46\,704j^4+20\,400j^5+62\,688j^6+\\
&\quad\quad 50\,568j^7+20\,784j^8+4416j^9+384j^{10}+7452n-11\,664jn-171\,684j^2n-419\,220j^3n-\\
&\quad\quad 423\,960j^4n-95\,880j^5n+199\,776j^6n+217\,644j^7n+99\,624j^8n+22\,416j^9n+\\
&\quad\quad 2016j^{10}n+45\,846n^2+137\,178jn^2-49\,626j^2n^2-698\,252j^3n^2-1\,160\,368j^4n^2-\\
&\quad\quad 829\,556j^5n^2-162\,976j^6n^2+150\,088j^7n^2+113\,888j^8n^2+30\,744j^9n^2+3024j^{10}n^2+\\
&\quad\quad 92\,826n^3+444\,129jn^3+709\,672j^2n^3+196\,482j^3n^3-783\,488j^4n^3-1\,136\,566j^5n^3-\\
&\quad\quad 696\,552j^6n^3-193\,240j^7n^3-4800j^8n^3+8856j^9n^3+1296j^{10}n^3+67\,662n^4+
\end{aligned}
$$

$$
448 944 j n^{4} + 1 099 060 j^{2} n^{4} + 1 296 068 j^{3} n^{4} + 682 760 j^{4} n^{4} - 45 596 j^{5} n^{4} - 265 128 j^{6} n^{4} -
145 072 j^{7} n^{4} - 33 104 j^{8} n^{4} - 2592 j^{9} n^{4} - 2142 n^{5} + 92 601 j n^{5} + 425 078 j^{2} n^{5} +
781 090 j^{3} n^{5} + 755 604 j^{4} n^{5} + 411 194 j^{5} n^{5} + 117 540 j^{6} n^{5} + 11 968 j^{7} n^{5} - 672 j^{8} n^{5} -
10 836 n^{6} - 61 056 j n^{6} - 108 784 j^{2} n^{6} - 51 056 j^{3} n^{6} + 46 096 j^{4} n^{6} + 67 744 j^{5} n^{6} +
29 248 j^{6} n^{6} + 4224 j^{7} n^{6} + 18 504 n^{7} + 34 464 j n^{7} - 20 256 j^{2} n^{7} - 79 264 j^{3} n^{7} - 66 208 j^{4} n^{7} -
19 840 j^{5} n^{7} - 1536 j^{6} n^{7} + 18 288 n^{8} + 56 640 j n^{8} + 58 752 j^{2} n^{8} + 23 360 j^{3} n^{8} -
1472 j^{4} n^{8} - 1920 j^{5} n^{8} + 4320 n^{9} + 14 400 j n^{9} + 17 280 j^{2} n^{9} + 9600 j^{3} n^{9} + 1920 j^{4} n^{9}\Big) S_{n} -
12 (1 + j) (4 + 2 j - n) (5 + 2 j - n) (1 + 3 n) (4 + 3 n) (-1 + 4 n) (1 + 4 n)
\Big(-54 - 288 j - 576 j^{2} - 464 j^{3} + 100 j^{4} + 488 j^{5} + 380 j^{6} + 128 j^{7} + 16 j^{8} - 135 n - 648 j n -
1284 j^{2} n - 1408 j^{3} n - 912 j^{4} n - 336 j^{5} n - 56 j^{6} n - 108 n^{2} - 456 j n^{2} - 788 j^{2} n^{2} -
752 j^{3} n^{2} - 428 j^{4} n^{2} - 144 j^{5} n^{2} - 24 j^{6} n^{2} - 27 n^{3} - 96 j n^{3} - 128 j^{2} n^{3} - 80 j^{3} n^{3} - 20 j^{4} n^{3}\Big)
S_{j} + 4 (1 + 2 j + n) (4 + 3 n) (-1 + 4 n) (1 + 4 n)
\Big(-3240 - 16 956 j - 37 044 j^{2} - 39 072 j^{3} - 10 224 j^{4} + 23 592 j^{5} + 32 280 j^{6} + 19 848 j^{7} +
6792 j^{8} + 1248 j^{9} + 96 j^{10} - 11 988 n - 63 612 j n - 148 950 j^{2} n - 191 700 j^{3} n -
129 264 j^{4} n - 15 168 j^{5} n + 48 816 j^{6} n + 43 164 j^{7} n + 17 304 j^{8} n + 3504 j^{9} n +
288 j^{10} n - 6642 n^{2} - 34 137 j n^{2} - 93 402 j^{2} n^{2} - 180 616 j^{3} n^{2} - 246 480 j^{4} n^{2} -
225 316 j^{5} n^{2} - 132 048 j^{6} n^{2} - 46 924 j^{7} n^{2} - 9072 j^{8} n^{2} - 720 j^{9} n^{2} + 162 n^{3} +
8463 j n^{3} + 48 482 j^{2} n^{3} + 112 656 j^{3} n^{3} + 137 240 j^{4} n^{3} + 94 480 j^{5} n^{3} + 36 088 j^{6} n^{3} +
6768 j^{7} n^{3} + 432 j^{8} n^{3} - 1350 n^{4} - 12 231 j n^{4} - 34 770 j^{2} n^{4} - 45 488 j^{3} n^{4} -
29 376 j^{4} n^{4} - 7544 j^{5} n^{4} + 696 j^{6} n^{4} + 472 j^{7} n^{4} + 1170 n^{5} + 4113 j n^{5} + 2932 j^{2} n^{5} -
4516 j^{3} n^{5} - 8416 j^{4} n^{5} - 5068 j^{5} n^{5} - 1024 j^{6} n^{5} + 1728 j n^{6} + 5760 j^{2} n^{6} + 6912 j^{3} n^{6} +
3840 j^{4} n^{6} + 768 j^{5} n^{6} - 576 n^{7} - 1920 j n^{7} - 2304 j^{2} n^{7} - 1280 j^{3} n^{7} - 256 j^{4} n^{7}\Big) \Big\}
$$

Identity (1)

```
In[44]:= anncnn = DFiniteSubstitute[annc, {j → n - 1}]
```

Out[44]=

$$
\Big\{ \Big(-457 228 800 - 6 193 454 400 n - 34 256 548 512 n^{2} - 91 453 484 844 n^{3} -
71 610 106 320 n^{4} + 294 537 737 277 n^{5} + 1 150 402 875 975 n^{6} + 2 108 420 980 272 n^{7} +
2 473 494 205 392 n^{8} + 2 006 558 429 715 n^{9} + 1 153 247 210 145 n^{10} + 469 113 764 184 n^{11} +
132 075 673 956 n^{12} + 24 455 298 060 n^{13} + 2 674 044 900 n^{14} + 130 491 000 n^{15}\Big) S_{n}^{3} +
\Big(11 256 537 600 + 152 571 522 240 n + 844 980 178 752 n^{2} + 2 263 142 886 444 n^{3} +
1 806 739 084 176 n^{4} - 7 164 763 851 273 n^{5} - 28 266 831 632 103 n^{6} - 52 055 982 476 364 n^{7} -
61 354 797 103 440 n^{8} - 50 044 063 651 515 n^{9} - 28 956 496 088 481 n^{10} - 11 879 474 942 568 n^{11} -
3 381 162 694 788 n^{12} - 634 895 662 380 n^{13} - 70 696 073 700 n^{14} - 3 532 923 000 n^{15}\Big) S_{n}^{2} +
\Big(-10 799 308 800 - 143 762 791 680 n - 784 148 124 672 n^{2} - 2 094 083 257 248 n^{3} -
1 845 933 271 344 n^{4} + 5 499 535 863 276 n^{5} + 22 726 398 444 464 n^{6} + 41 836 311 601 180 n^{7} +
48 956 015 319 040 n^{8} + 39 581 355 566 120 n^{9} + 22 697 889 390 912 n^{10} + 9 231 698 699 664 n^{11} +
2 606 360 389 408 n^{12} + 485 782 434 080 n^{13} + 53 730 675 200 n^{14} + 2 669 248 000 n^{15}\Big) S_{n} +
\Big(-2 615 276 160 n - 26 575 505 568 n^{2} - 77 606 144 352 n^{3} + 110 804 293 488 n^{4} +
1 370 690 250 720 n^{5} + 4 390 030 311 664 n^{6} + 8 111 249 894 912 n^{7} + 9 925 287 579 008 n^{8} +
8 456 149 655 680 n^{9} + 5 105 359 487 424 n^{10} + 2 178 662 478 720 n^{11} +
642 726 631 424 n^{12} + 124 657 930 240 n^{13} + 14 291 353 600 n^{14} + 733 184 000 n^{15}\Big) \Big\}
$$

In[45]:=
```
ApplyOreOperator[anncnn, 1]
```
Out[45]= {0}

In[46]:=
```
OreReduce[anncnn, Annihilator[1, S[n]]]
```
Out[46]= {0}

Identity (2)

In[∘]:=
```
annci = OreGroebnerBasis[Append[annc, S[i] - 1], OreAlgebra[S[n], S[j], S[i]]];
annSmnd1 = DFiniteTimes[
   Annihilator[2^i * Binomial[i + 2 j + 1, 2 j + 1], {S[n], S[j], S[i]}], annci];
annSmnd2 = DFiniteTimes[Annihilator[Binomial[i - 1, 2 j + 1],
    {S[n], S[j], S[i]}], annci];
```

In[∘]:=
```
#[annSmnd1] & /@ {ByteCount, UnderTheStaircase}
#[annSmnd2] & /@ {ByteCount, UnderTheStaircase}
```
Out[∘]= {579 032, {1, S_j, S_n}}

Out[∘]= {578 960, {1, S_j, S_n}}

```
Timing[id2fct1 = FindCreativeTelescoping[annSmnd1, S[j] - 1];]
```
Out[∘]= {1342.33, Null}

In[∘]:=
```
#[id2fct1[[1]]] & /@ {ByteCount, UnderTheStaircase}
```
Out[∘]= {1 381 848, {1, S_i, S_n, S_i^2, $S_n S_i$, S_n^2}}

```
Timing[id2fct2 = FindCreativeTelescoping[annSmnd2, S[j] - 1];]
```
Out[∘]= {1382.94, Null}

In[∘]:=
```
#[id2fct2[[1]]] & /@ {ByteCount, UnderTheStaircase}
```
Out[∘]= {1 382 192, {1, S_i, S_n, S_i^2, $S_n S_i$, S_n^2}}

In[∘]:=
```
GBEqual[id2fct1[[1]], id2fct2[[1]]]
```
Out[∘]= True

In[∘]:=
```
AnnihilatorSingularities[id2fct1[[1]], {0, 0}]
```
Out[∘]= {{{i → 0, n → 0}, True}, {{i → 0, n → 1}, True},
{{i → 0, n → 2}, True}, {{i → 1, n → 0}, True}, {{i → 1, n → 1}, True},
{{i → 1, n → 2}, True}, {{i → 2, n → 0}, True}, {{i → 2, n → 1}, True},
{{i → 2, n → 2}, True}, {{i → 3, n → 0}, True}, {{i → 4, n → 0}, True}}

Identity (3)

In[47]:=
```
annSmnd1 =
  DFiniteTimes[Annihilator[2^(n - 1) * Binomial[n + 2 j, 2 j + 1], {S[n], S[j]}], annc];
annSmnd2 = DFiniteTimes[Annihilator[Binomial[n - 2, 2 j + 1], {S[n], S[j]}], annc];
```

In[49]:=
```
#[annSmnd1] & /@ {ByteCount, UnderTheStaircase}
#[annSmnd2] & /@ {ByteCount, UnderTheStaircase}
```

Out[49]= $\{244432, \{1, S_j, S_n\}\}$

Out[50]= $\{263440, \{1, S_j, S_n\}\}$

```
Timing[id3fct1 = FindCreativeTelescoping[annSmnd1, S[j] - 1];]
```

Out[•]= {1004.7, Null}

```
Timing[id3fct2 = FindCreativeTelescoping[annSmnd2, S[j] - 1];]
```

Out[•]= {830.27, Null}

In[61]:=
```
rec = DFinitePlus[id3fct1[[1]], id3fct2[[1]]];
Support[rec]
```

Out[•]= $\{\{S_n^6, S_n^5, S_n^4, S_n^3, S_n^2, S_n, 1\}\}$

In[•]:=
```
With[{b = 2 * prod[2^(i - 1) * (4 i - 2)! / (n + 2 i - 1)!, {i, 1, n}]}, b / (b /. n → n - 1)]
```

Out[•]= $\text{prod}\left[\frac{2^{-1+i}\,(-2+4\,i)!}{(-1+2\,i+n)!}, \{i, 1, n\}\right] \Big/ \text{prod}\left[\frac{2^{-1+i}\,(-2+4\,i)!}{(-2+2\,i+n)!}, \{i, 1, -1+n\}\right]$

In[•]:=
```
% /. prod[a_, {i, 1, n}] :> (a /. i → n) * prod[a, {i, 1, n - 1}] /.
  prod[a1_, b_] / prod[a2_, b_] :> prod[FunctionExpand[a1 / a2], b]
```

Out[•]= $\left(2^{-1+n}\,(-2+4\,n)!\,\text{prod}\left[\frac{1}{-1+2\,i+n}, \{i, 1, -1+n\}\right]\right) \Big/ (-1+3\,n)!$

In[•]:=
```
FunctionExpand[% /. prod → Product]
```

Out[•]= $\frac{\text{Gamma}\left[\frac{1}{2}+\frac{n}{2}\right]\,\text{Gamma}[-1+4\,n]}{\text{Gamma}[3\,n]\,\text{Gamma}\left[-\frac{1}{2}+\frac{3\,n}{2}\right]}$

In[•]:=
```
OreReduce[rec[[1]], Annihilator[%, S[n]]]
```

Out[•]= 0