Creative Telescoping

5.2 HolonomicFunctions Demo

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Friday, 01.12.2023
Recent Trends in Computer Algebra
Special Week @ Institut Henri Poincaré

AMSS
Academy of Mathematics and Systems Science,CAS

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ÖAW RICAM

Execute Closure Properties of D-Finite Functions

Some D-finite and some non-D-finite functions:

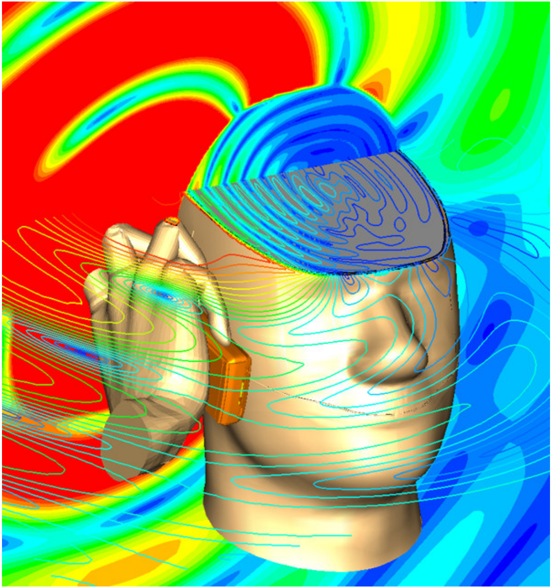
$$\operatorname{erf}\left(\sqrt{x+1}\right)^2 + \exp\left(\sqrt{x+1}\right)^2 \qquad ✓$$

$$\left(\left(\sinh(x)\right)^2 + \left(\sin(x)\right)^{-2}\right) \cdot \left(\left(\cosh(x)\right)^2 + \left(\cos(x)\right)^{-2}\right) \qquad ✗$$

$$\frac{\log\left(\sqrt{1-x^2}\right)}{\exp\left(\sqrt{1-x^2}\right)} \qquad ✓$$

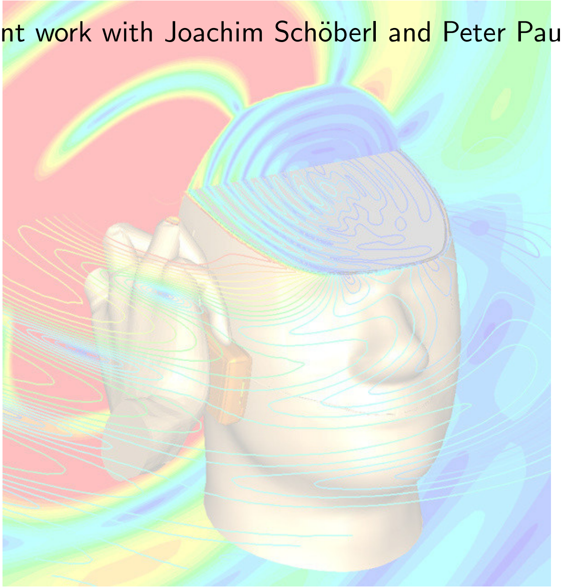
$$\arctan\left(\mathrm{e}^x\right) \qquad ✗$$

Finite Element Methods



Finite Element Methods

(joint work with Joachim Schöberl and Peter Paule)



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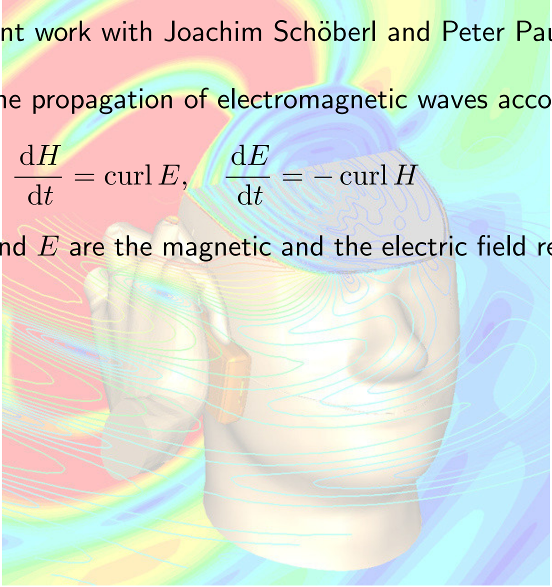


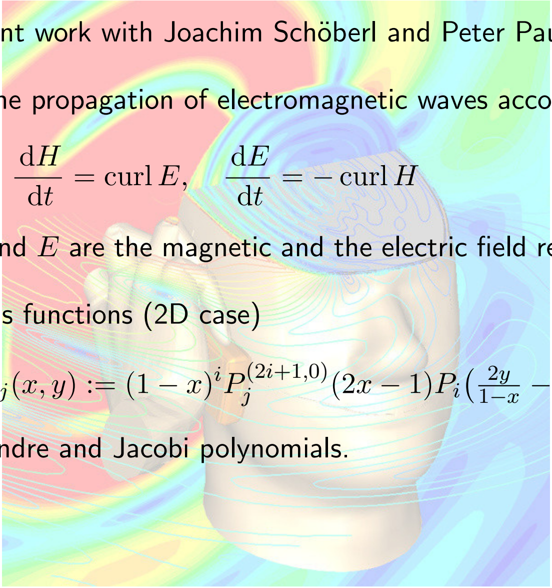
(joint work with Joachim Schöberl and Peter Paule)

Simulate the propagation of electromagnetic waves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H \qquad \text{(Maxwell)}$$

where H and E are the magnetic and the electric field respectively.

Finite Element Methods



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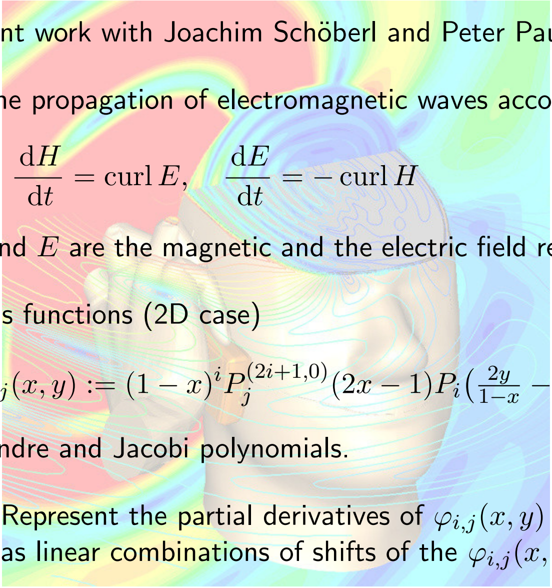
where H and E are the magnetic and the electric field respectively.

Define basis functions (2D case)

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\big(\tfrac{2y}{1-x} - 1\big)$$

using Legendre and Jacobi polynomials.

Finite Element Methods



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using Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x,y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x,y)$ itself).

Find Certain Operators in Annihilator Ideals

Ansatz: One needs a relation of the form

$$\sum_{(k,l)\in A} a_{k,l}(i,j)\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+k,j+l}(x,y) \;=\; \sum_{(m,n)\in B} b_{m,n}(i,j)\varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

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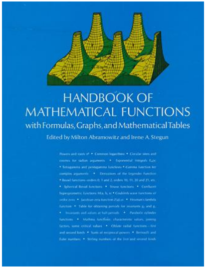
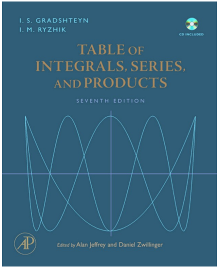
that is free of x and y (and similarly for $\frac{\mathrm{d}}{\mathrm{d}y}$).

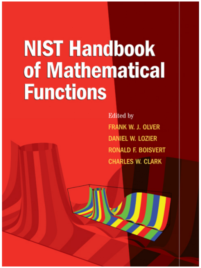
Result: With our holonomic methods, we find the relation

$$\begin{aligned}
&(2i+j+3)(2i+2j+7)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+1}(x,y)+\\
&2(2i+1)(i+j+3)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+2}(x,y)-\\
&(j+3)(2i+2j+5)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j+3}(x,y)+\\
&(j+1)(2i+2j+7)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j}(x,y)-\\
&2(2i+3)(i+j+3)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+1}(x,y)-\\
&(2i+j+5)(2i+2j+5)\,\tfrac{\mathrm{d}}{\mathrm{d}x}\varphi_{i+1,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i,j+2}(x,y)+\\
&2(i+j+3)(2i+2j+5)(2i+2j+7)\varphi_{i+1,j+1}(x,y)=0.
\end{aligned}$$

Prove Special Function Identities







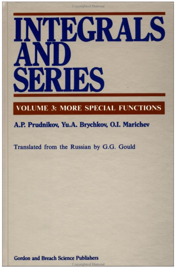
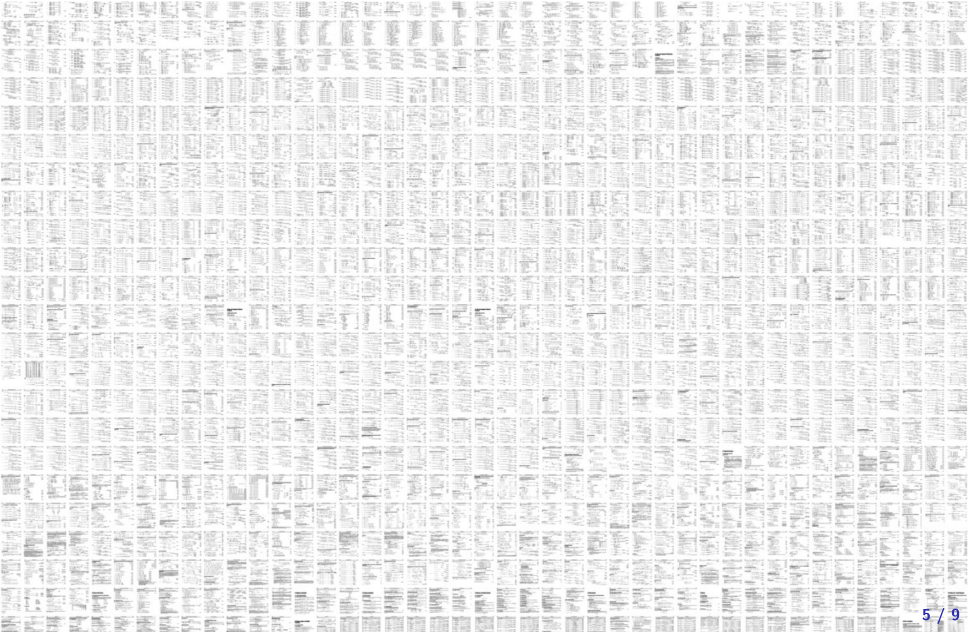


Table of Integrals by Gradshteyn and Ryzhik



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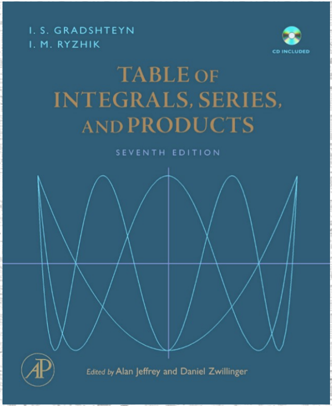

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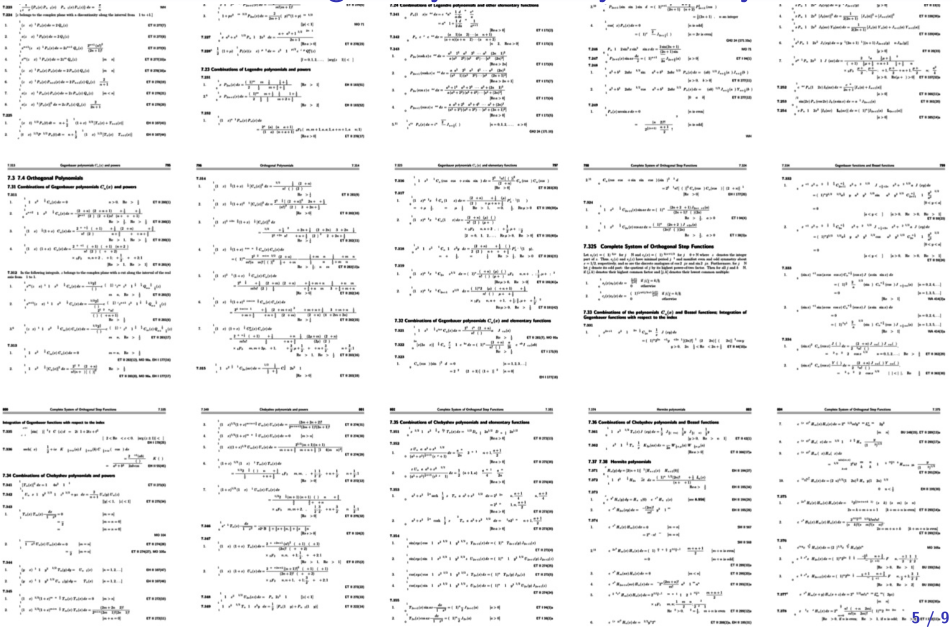


7.319

1. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n}^{\lambda}\left(\gamma x^{1/2}\right) dx = (-1)^n \frac{\Gamma(\lambda+n)\,\Gamma(\mu)\,\Gamma(\nu)}{n!\,\Gamma(\lambda)\,\Gamma(\mu+\nu)}\; {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > 0]$ ET II 191(41)a

2. $$\int_0^1 (1-x)^{\mu-1} x^{\nu-1}\, C_{2n+1}^{\lambda}\left(\gamma x^{1/2}\right) dx = \frac{(-1)^n 2\gamma\,\Gamma(\mu)\,\Gamma(\lambda+n+1)\,\Gamma\left(\nu+\frac{1}{2}\right)}{n!\,\Gamma(\lambda)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)} \times {}_3F_2\left(-n, n+\lambda+1, \nu+\tfrac{1}{2}; \tfrac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right)$$

$[\operatorname{Re}\mu > 0, \quad \operatorname{Re}\nu > -\frac{1}{2}]$ ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^{\nu}(x)$ and elementary functions

7.321 $$\int_{-1}^1 \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

$[\operatorname{Re}\nu > -\frac{1}{2}]$ ET II 281(7), MO 99a

7.322 $$\int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}}\, C_n^{\nu}\left(\frac{x}{a}-1\right) e^{-bx}\, dx = (-1)^n \frac{\pi\,\Gamma(2\nu+n)}{n!\,\Gamma(\nu)} \left(\frac{a}{2b}\right)^{\nu} e^{-ab}\, I_{\nu+n}(ab)$$

$[\operatorname{Re}\nu > -\frac{1}{2}]$ ET I 171(9)

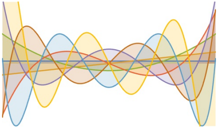
7.323

1. $$\int_0^{\pi} C_n^{\nu}(\cos\varphi)\,(\sin\varphi)^{2\nu}\, d\varphi = 0$$ $[n = 1, 2, 3, \ldots]$

Table of Integrals by Gradshteyn and Ryzhik

$$\int_{-1}^{1} \left(1 - x^2\right)^{\nu - \frac{1}{2}} e^{iax} \, C_n^{\nu}(x) \, dx = \frac{\pi 2^{1-\nu} i^n \, \Gamma(2\nu + n)}{n! \, \Gamma(\nu)} a^{-\nu} \, J_{\nu+n}(a)$$

Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer polynomials $C_n^{(\alpha)}(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

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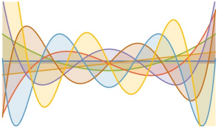

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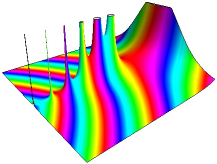

Gamma function $\Gamma(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

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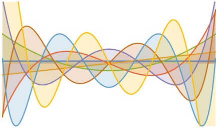

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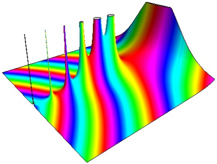

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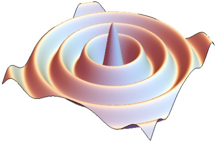

Bessel function $J_\nu(x)$

$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax}\, C_n^{\nu}(x)\, dx = \frac{\pi 2^{1-\nu} i^n\, \Gamma(2\nu+n)}{n!\,\Gamma(\nu)} a^{-\nu}\, J_{\nu+n}(a)$$

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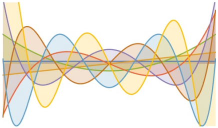
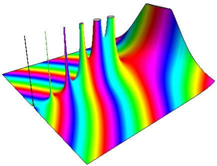


Gegenbauer polynomials $C_n^{(\alpha)}(x)$



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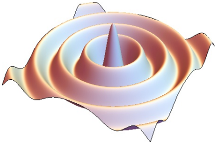

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Let's prove this identity with creative telescoping...

Prove Special Function Identities

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{1}$$

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$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, \mathrm{d}x = \frac{\pi P_m^{\left(m+\frac{1}{2}, -m-\frac{1}{2}\right)}(a)}{2^{m+\frac{3}{2}} (a+1)^{m+\frac{1}{2}}} \tag{2}$$

Prove Special Function Identities

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$$e^{-x} x^{a/2} n! \, L_n^a(x) = \int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a\left(2\sqrt{tx}\right) \mathrm{d}t \tag{3}$$

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$$\int_{-1}^{1} \left(1-x^2\right)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x)\,\mathrm{d}x = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu\, n!\, \Gamma(\nu)} \qquad (5)$$

Symbolic Determinants via Holonomic Ansatz

$$\det_{1\leqslant i,j\leqslant n}\left(\frac{1}{i+j-1}\right)=\frac{1}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k!)^2}{(k+1)_{n-1}}$$

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$$\det_{0\leqslant i,j\leqslant n-1}\left(\binom{2i+2a}{j+b}\right)=2^{n(n-1)/2}\prod_{k=0}^{n-1}\frac{(2k+2a)!k!}{(k+b)!(2k+2a-b)!}$$

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$$\det_{1\leqslant i,j\leqslant 2m+1}\left(\binom{\mu+i+j+2r}{j+2r-2}-\delta_{i,j+2r}\right)$$

$$=\frac{(-1)^{m-r+1}\,(\mu+3)\,(m+r+1)_{m-r}}{2^{2m-2r+1}\left(\frac{\mu}{2}+r+\frac{3}{2}\right)_{m-r+1}}\cdot\prod_{i=1}^{2m}\frac{(\mu+i+3)_{2r}}{(i)_{2r}}$$

$$\times\prod_{i=1}^{m-r}\frac{\left(\mu+2i+6r+3\right)_i^2\left(\frac{\mu}{2}+2i+3r+2\right)_{i-1}^2}{(i)_i^2\left(\frac{\mu}{2}+i+3r+2\right)_{i-1}^2}.$$

The Holonomic Ansatz

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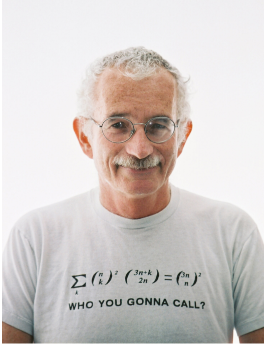
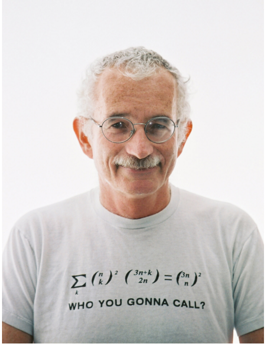


The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$



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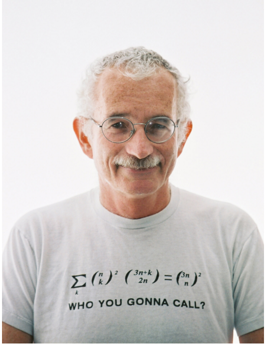
Problem: Prove a determinantal identity of the form $\det_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

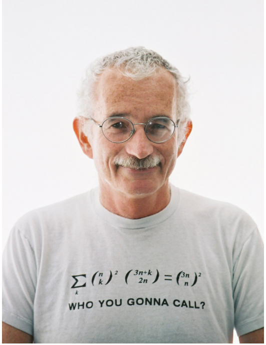
- $a_{i,j}$ is a holonomic sequence



The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det\limits_{1\leqslant i,j\leqslant n}(a_{i,j}) = b_n$, where

- $a_{i,j}$ is a holonomic sequence
- that does not depend on n



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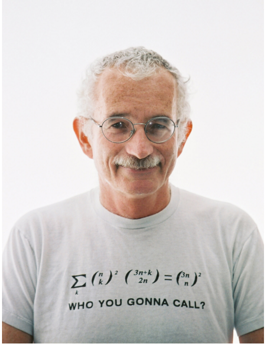
Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n}(a_{i,j}) = b_n$, where

- $a_{i,j}$ is a holonomic sequence
- that does not depend on n, and
- b_n is a closed form ($b_n \neq 0$ for all n).



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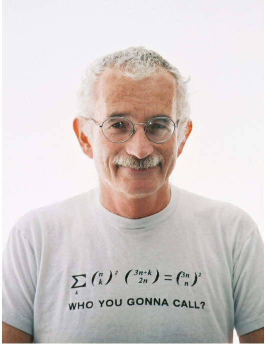
- $a_{i,j}$ is a holonomic sequence
- that does not depend on n, and
- b_n is a closed form ($b_n \neq 0$ for all n).

$$\mathcal{A}_n = \left(\begin{array}{ccc|c} & \mathcal{A}_{n-1} & & \\ \hline a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array}\right)$$



Laplace expansion:

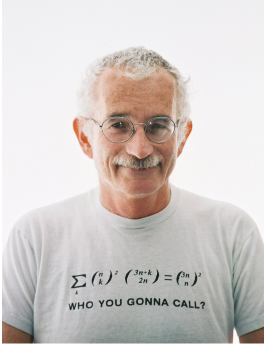
$$\det(\mathcal{A}_n) = a_{n,1}\mathrm{Cof}_{n,1} + \ldots + a_{n,n-1}\mathrm{Cof}_{n,n-1} + a_{n,n}\det(\mathcal{A}_{n-1})$$

The Holonomic Ansatz

Problem: Prove a determinantal identity of the form $\det_{1 \leqslant i,j \leqslant n} (a_{i,j}) = b_n$, where

- $a_{i,j}$ is a holonomic sequence
- that does not depend on n, and
- b_n is a closed form ($b_n \neq 0$ for all n).

$$\mathcal{A}_n = \left(\begin{array}{ccc|c} & \mathcal{A}_{n-1} & & \\ \hline a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array}\right)$$



Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1} \frac{\mathrm{Cof}_{n,1}}{\det(\mathcal{A}_{n-1})} + \ldots + a_{n,n-1} \frac{\mathrm{Cof}_{n,n-1}}{\det(\mathcal{A}_{n-1})} + a_{n,n}$$

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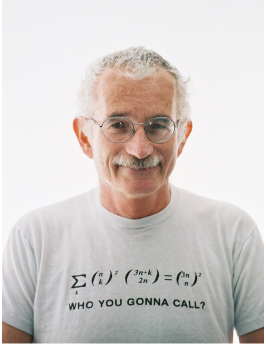

Laplace expansion:

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}c_{n,1} + \ldots + a_{n,n-1}c_{n,n-1} + a_{n,n}c_{n,n}$$

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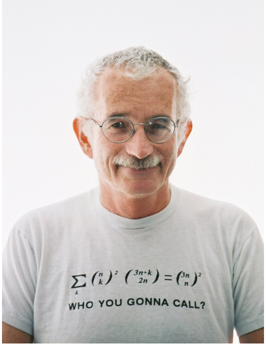

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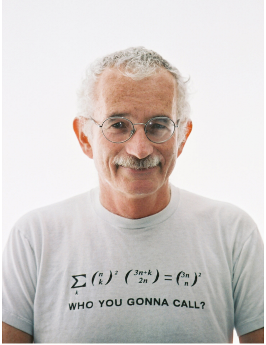

Laplace expansion:

$$0 = \sum_{j=1}^{n} a_{1,j} c_{n,j}$$

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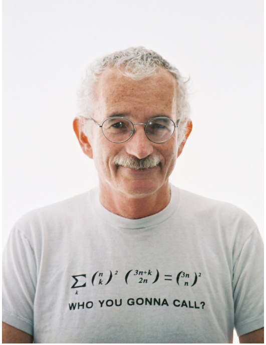

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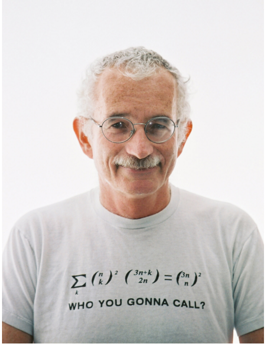

Laplace expansion:

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \quad (1 \leqslant i < n), \quad c_{n,n} = 1$$

Recipe

1. Guess a set of recurrences (holonomic description) for the normalized cofactors $c_{n,j}$.

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2. Use it to prove, via creative telescoping, the three identities

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$$\sum_{j=1}^{n} a_{i,j} c_{n,j} = 0 \qquad (1 \leqslant i < n) \tag{2}$$

$$\sum_{j=1}^{n} a_{n,j} c_{n,j} = \frac{b_n}{b_{n-1}} \qquad (1 \leqslant n) \tag{3}$$

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Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0 \leqslant i,j < n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^{n} \frac{2^{i-1}(4i-2)!}{(n+2i-1)!}$$

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Theorem (Di Francesco's determinant for 20V configurations):

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