

Creative Telescoping

5.2 HolonomicFunctions Demo

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Recent Trends in Computer Algebra
Special Week @ Institut Henri Poincaré



AMSS

Academy of Mathematics and Systems Science,CAS



ÖAW RICAM

Execute Closure Properties of D-Finite Functions

Some D-finite and some non-D-finite functions:

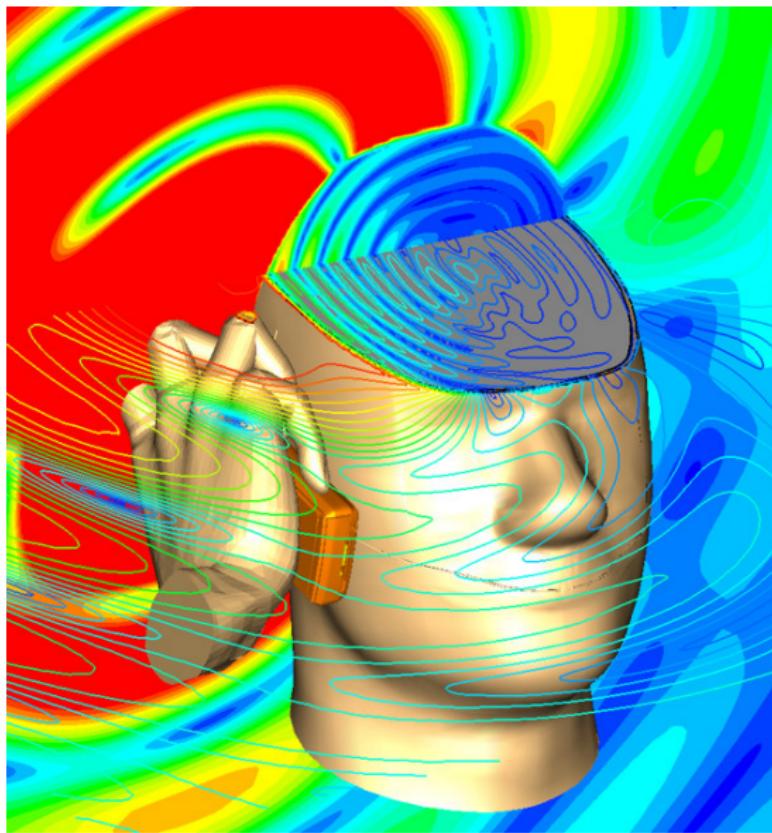
$$\operatorname{erf}(\sqrt{x+1})^2 + \exp(\sqrt{x+1})^2 \quad \checkmark$$

$$\left((\sinh(x))^2 + (\sin(x))^{-2} \right) \cdot \left((\cosh(x))^2 + (\cos(x))^{-2} \right) \quad \times$$

$$\frac{\log(\sqrt{1-x^2})}{\exp(\sqrt{1-x^2})} \quad \checkmark$$

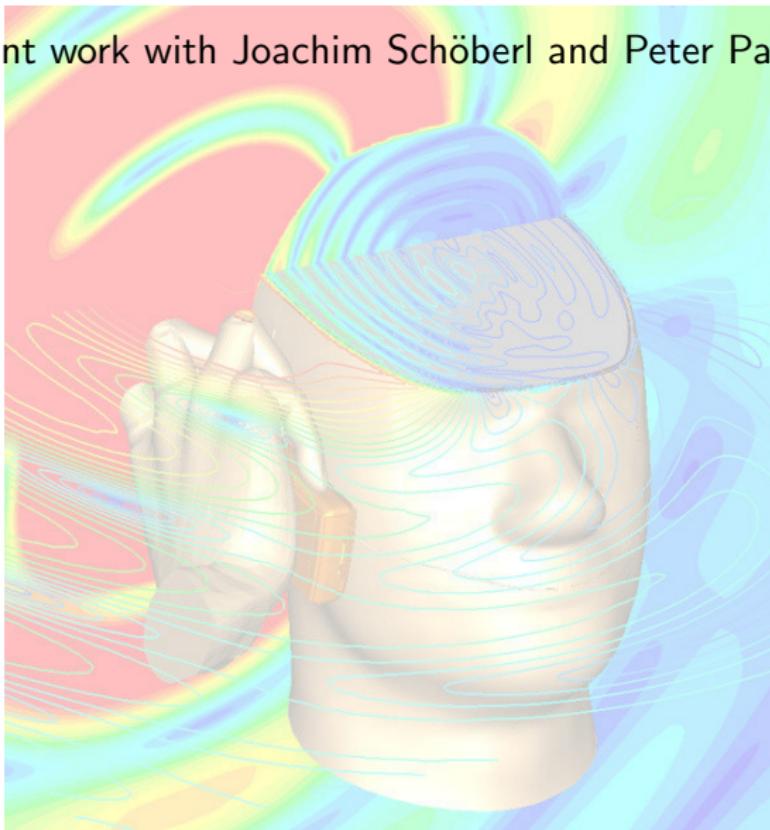
$$\arctan(e^x) \quad \times$$

Finite Element Methods



Finite Element Methods

(joint work with Joachim Schöberl and Peter Paule)



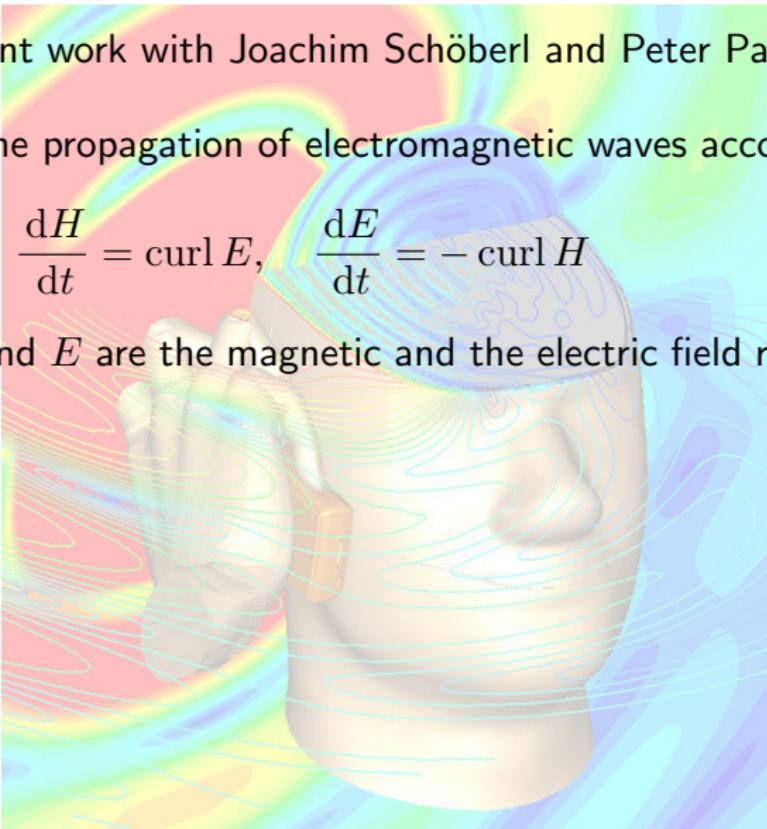
Finite Element Methods

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Simulate the propagation of electromagnetic waves according to

$$\frac{dH}{dt} = \operatorname{curl} E, \quad \frac{dE}{dt} = -\operatorname{curl} H \quad (\text{Maxwell})$$

where H and E are the magnetic and the electric field respectively.



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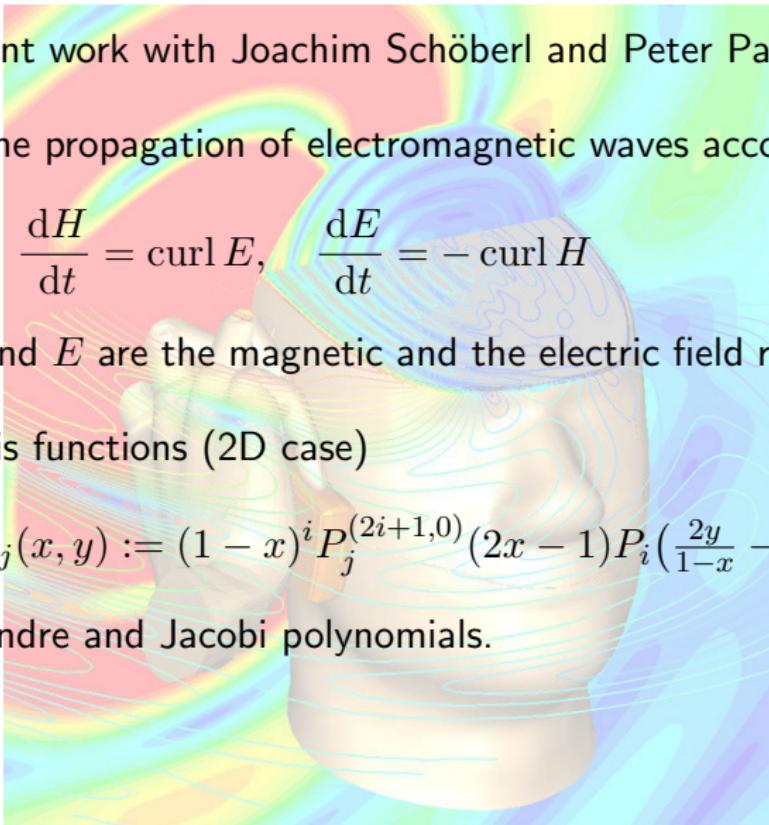
$$\frac{dH}{dt} = \operatorname{curl} E, \quad \frac{dE}{dt} = -\operatorname{curl} H \quad (\text{Maxwell})$$

where H and E are the magnetic and the electric field respectively.

Define basis functions (2D case)

$$\varphi_{i,j}(x, y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x}-1\right)$$

using Legendre and Jacobi polynomials.



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using Legendre and Jacobi polynomials.

Problem: Represent the partial derivatives of $\varphi_{i,j}(x, y)$ in the basis (i.e., as linear combinations of shifts of the $\varphi_{i,j}(x, y)$ itself).

Find Certain Operators in Annihilator Ideals

Ansatz: One needs a relation of the form

$$\sum_{(k,l) \in A} a_{k,l}(i,j) \frac{d}{dx} \varphi_{i+k,j+l}(x,y) = \sum_{(m,n) \in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x,y),$$

that is free of x and y (and similarly for $\frac{d}{dy}$).

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that is free of x and y (and similarly for $\frac{d}{dy}$).

Result: With our holonomic methods, we find the relation

$$\begin{aligned} & (2i + j + 3)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+1}(x,y) + \\ & 2(2i + 1)(i + j + 3) \frac{d}{dx} \varphi_{i,j+2}(x,y) - \\ & (j + 3)(2i + 2j + 5) \frac{d}{dx} \varphi_{i,j+3}(x,y) + \\ & (j + 1)(2i + 2j + 7) \frac{d}{dx} \varphi_{i+1,j}(x,y) - \\ & 2(2i + 3)(i + j + 3) \frac{d}{dx} \varphi_{i+1,j+1}(x,y) - \\ & (2i + j + 5)(2i + 2j + 5) \frac{d}{dx} \varphi_{i+1,j+2}(x,y) + \\ & 2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x,y) + \\ & 2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x,y) = 0. \end{aligned}$$

Prove Special Function Identities

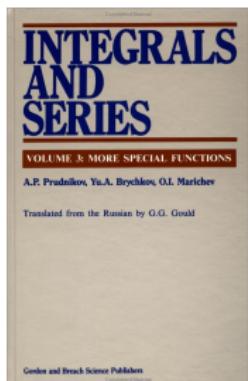
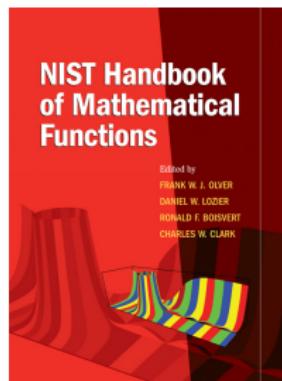
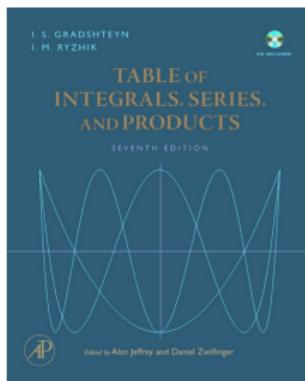
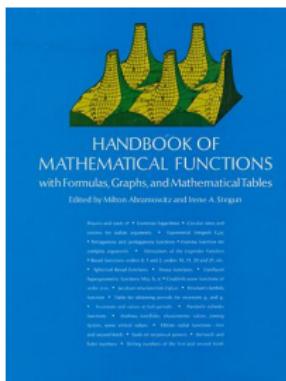


Table of Integrals by Gradshteyn and Ryzhik

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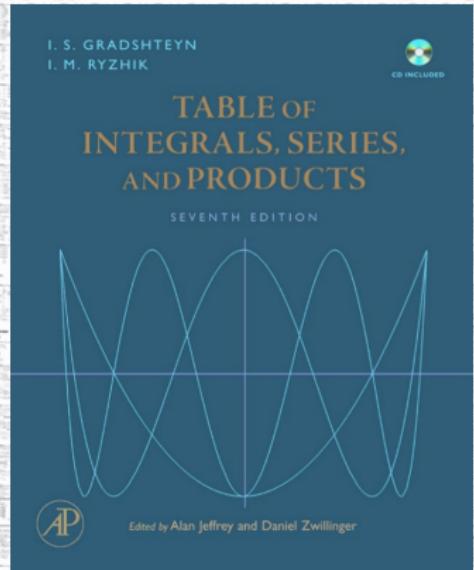


Table of Integrals by Gradshteyn and Ryzhik

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Table of Integrals by Gradshteyn and Ryzhik

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7.319

$$1. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda (\gamma x^{1/2}) dx = (-1)^n \frac{\Gamma(\lambda+n) \Gamma(\mu) \Gamma(\nu)}{n! \Gamma(\lambda) \Gamma(\mu+\nu)} {}_3F_2 \left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2 \right)$$

[Re $\mu > 0$, Re $\nu > 0$] ET II 191(41)a

$$2. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda (\gamma x^{1/2}) dx = \frac{(-1)^n 2\gamma \Gamma(\mu) \Gamma(\lambda+n+1) \Gamma(\nu + \frac{1}{2})}{n! \Gamma(\lambda) \Gamma(\mu+\nu+\frac{1}{2})} {}_3F_2 \left(-n, n+\lambda+1, \nu + \frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2 \right)$$

[Re $\mu > 0$, Re $\nu > -\frac{1}{2}$] ET II 191(42)

7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

$$7.321 \quad \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

[Re $\nu > -\frac{1}{2}$] ET II 281(7), MO 99a

$$7.322 \quad \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu \left(\frac{x}{a} - 1 \right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n! \Gamma(\nu)} \left(\frac{a}{2b} \right)^\nu e^{-ab} I_{\nu+n}(ab)$$

[Re $\nu > -\frac{1}{2}$] ET I 171(9)

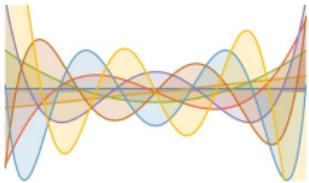
7.323

$$1. \int_0^\pi C_n^\nu (\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots]$$

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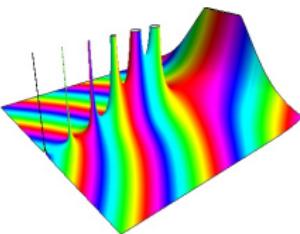
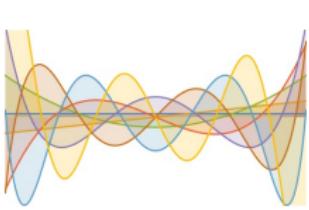
Table of Integrals by Gradshteyn and Ryzhik



Gegenbauer
polynomials $C_n^{(\alpha)}(x)$

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

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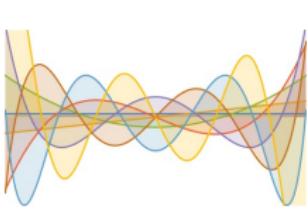


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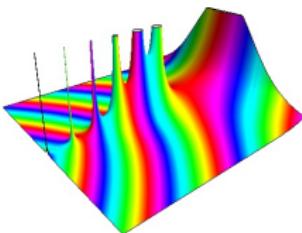
Gamma
function $\Gamma(x)$

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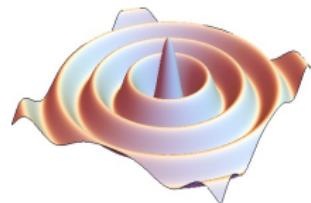
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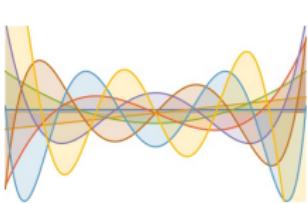
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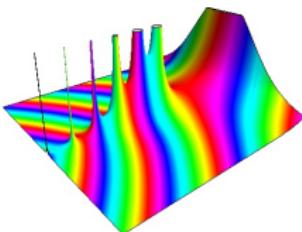
Bessel
function $J_\nu(x)$

$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$$

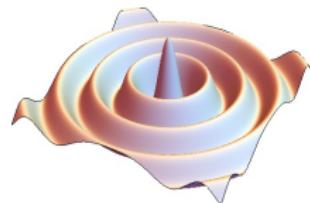
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Gegenbauer
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Gamma
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Bessel
function $J_\nu(x)$

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Let's prove this identity with creative telescoping...

Prove Special Function Identities

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad (1)$$

Prove Special Function Identities

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$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi P_m^{(m+\frac{1}{2}, -m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \quad (2)$$

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$$e^{-x} x^{a/2} n! L_n^a(x) = \int_0^\infty e^{-t} t^{\frac{a}{2}+n} J_a(2\sqrt{tx}) dt \quad (3)$$

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Prove Special Function Identities

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad (1)$$

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi P_m^{(m+\frac{1}{2}, -m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \quad (2)$$

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$$\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^{(\nu)}(x) dx = \frac{\pi i^n \Gamma(n+2\nu) J_{n+\nu}(a)}{2^{\nu-1} a^\nu n! \Gamma(\nu)} \quad (5)$$

Symbolic Determinants via Holonomic Ansatz

$$\det_{1 \leq i, j \leq n} \left(\frac{1}{i+j-1} \right) = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k+1)_{n-1}}$$

Symbolic Determinants via Holonomic Ansatz

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$$\det_{0 \leq i, j \leq n-1} \left(\sum_k \binom{i}{k} \binom{j}{k} 2^k \right) = 2^{n(n-1)/2}$$

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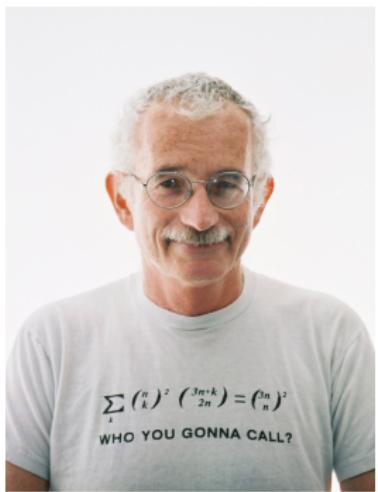
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$$\begin{aligned} & \det_{1 \leq i, j \leq 2m+1} \left(\binom{\mu+i+j+2r}{j+2r-2} - \delta_{i,j+2r} \right) \\ &= \frac{(-1)^{m-r+1} (\mu+3) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r + \frac{3}{2}\right)_{m-r+1}} \cdot \prod_{i=1}^{2m} \frac{(\mu+i+3)_{2r}}{(i)_{2r}} \\ & \times \prod_{i=1}^{m-r} \frac{\left(\mu+2i+6r+3\right)_i^2 \left(\frac{\mu}{2}+2i+3r+2\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2}+i+3r+2\right)_{i-1}^2}. \end{aligned}$$

The Holonomic Ansatz

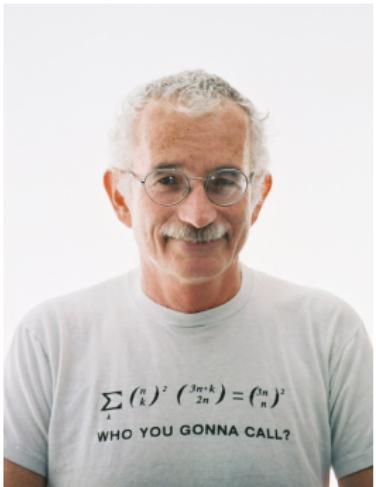
The Holonomic Ansatz



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Problem: Prove a determinantal identity of

$$\text{the form } \det_{1 \leq i, j \leq n} (a_{i,j}) = b_n$$

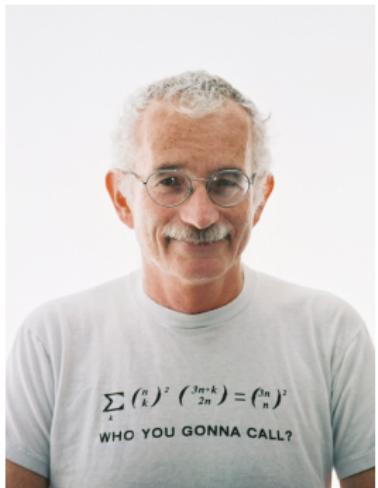


The Holonomic Ansatz

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- ▶ $a_{i,j}$ is a holonomic sequence

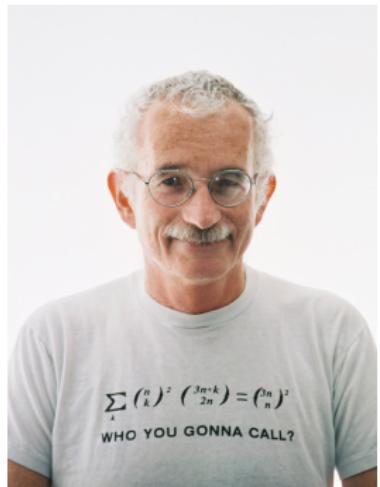


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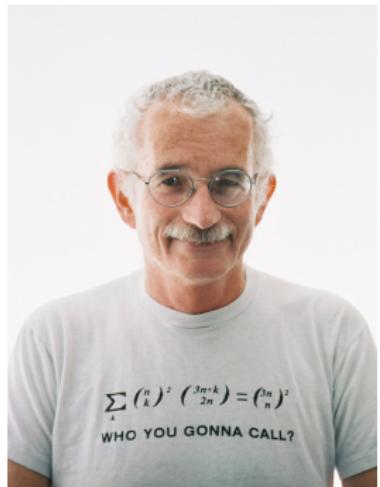


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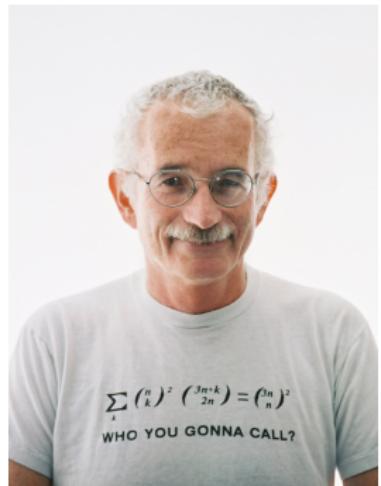
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$$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$$

WHO YOU GONNA CALL?

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Laplace expansion:

$$\det(\mathcal{A}_n) = a_{n,1}\text{Cof}_{n,1} + \dots + a_{n,n-1}\text{Cof}_{n,n-1} + a_{n,n}\det(\mathcal{A}_{n-1})$$

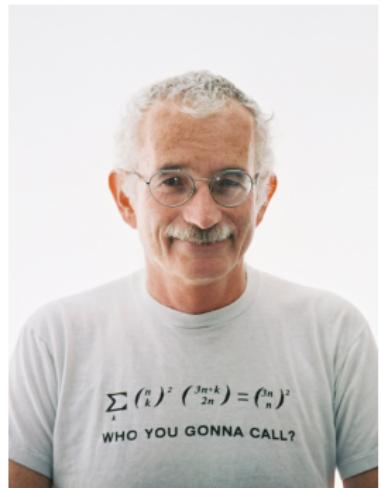
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$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1} \frac{\text{Cof}_{n,1}}{\det(\mathcal{A}_{n-1})} + \dots + a_{n,n-1} \frac{\text{Cof}_{n,n-1}}{\det(\mathcal{A}_{n-1})} + a_{n,n}$$

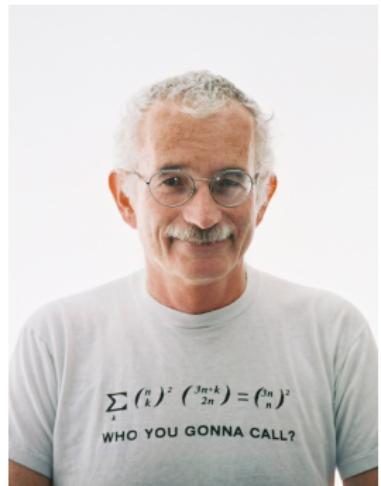
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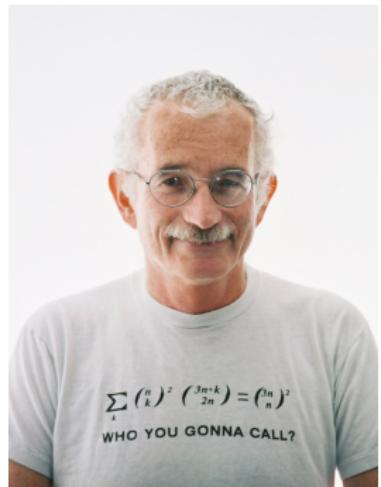
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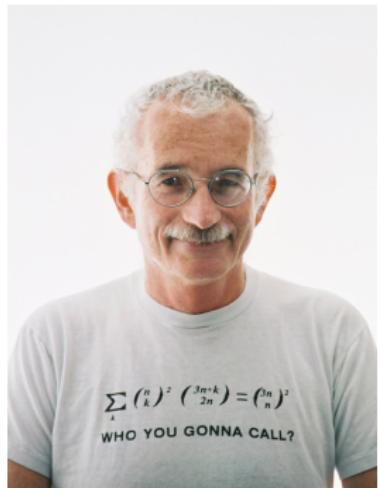
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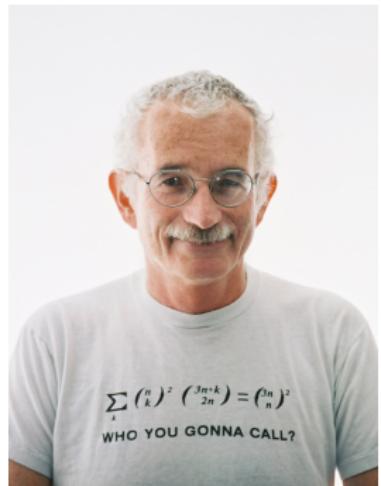
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$$0 = \sum_{j=1}^n a_{i,j} c_{n,j} \quad (1 \leq i < n)$$

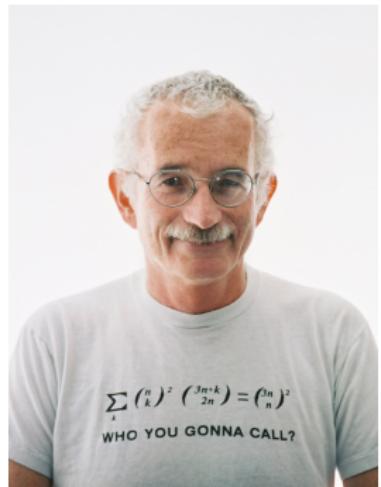
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Laplace expansion:

$$0 = \sum_{j=1}^n a_{i,j} c_{n,j} \quad (1 \leq i < n), \quad c_{n,n} = 1$$

Recipe

1. Guess a set of recurrences (holonomic description) for the normalized cofactors $c_{n,j}$.

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Conjecture (Di Francesco's determinant for 20V configurations):

$$\det_{0 \leq i, j < n} \left(2^i \binom{i+2j+1}{2j+1} - \binom{i-1}{2j+1} \right) = 2 \prod_{i=1}^n \frac{2^{i-1} (4i-2)!}{(n+2i-1)!}$$

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