

# Creative Telescoping

## 5.2 Holonomic Functions Demo

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Recent Trends in Computer Algebra  
Special Week @ Institut Henri Poincaré



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## Execute Closure Properties of D-Finite Functions

Some D-finite and some non-D-finite functions:

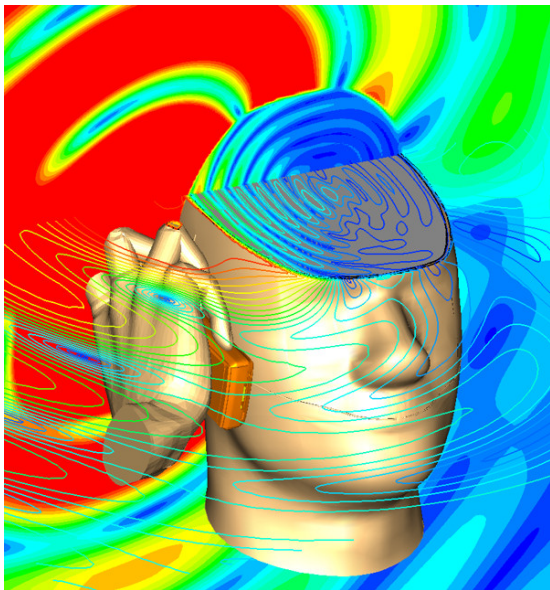
$$\operatorname{erf}(\sqrt{x+1})^2 + \exp(\sqrt{x+1})^2 \quad \checkmark$$

$$\left( (\sinh(x))^2 + (\sin(x))^{-2} \right) \cdot \left( (\cosh(x))^2 + (\cos(x))^{-2} \right) \quad \times$$

$$\frac{\log(\sqrt{1-x^2})}{\exp(\sqrt{1-x^2})} \quad \checkmark$$

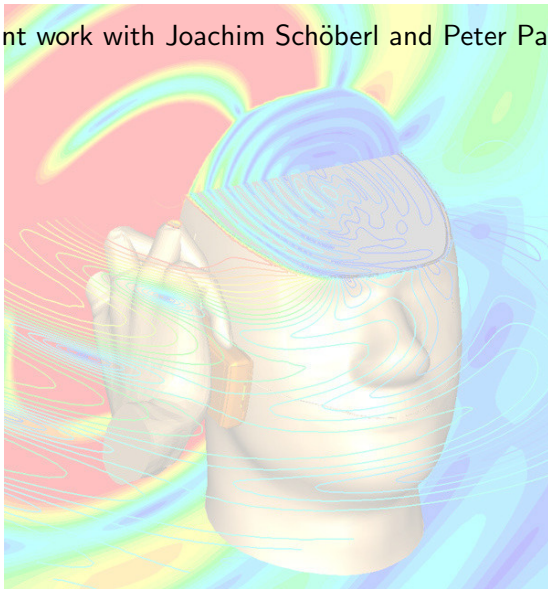
$$\arctan(e^x) \quad \times$$

# Finite Element Methods



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(joint work with Joachim Schöberl and Peter Paule)



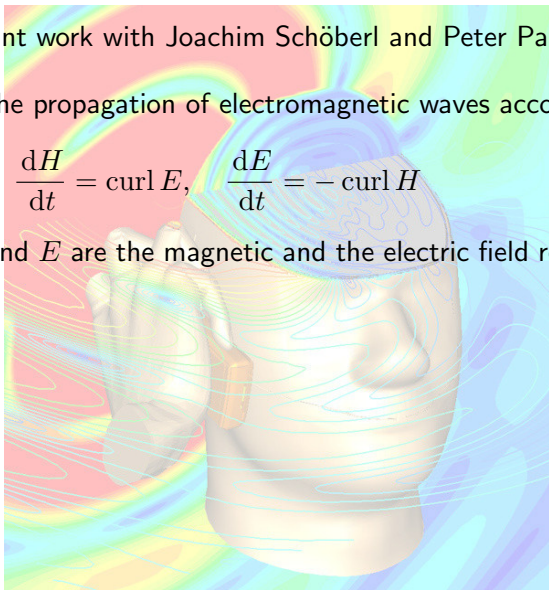
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Simulate the propagation of electromagnetic waves according to

$$\frac{dH}{dt} = \text{curl } E, \quad \frac{dE}{dt} = -\text{curl } H \quad (\text{Maxwell})$$

where  $H$  and  $E$  are the magnetic and the electric field respectively.



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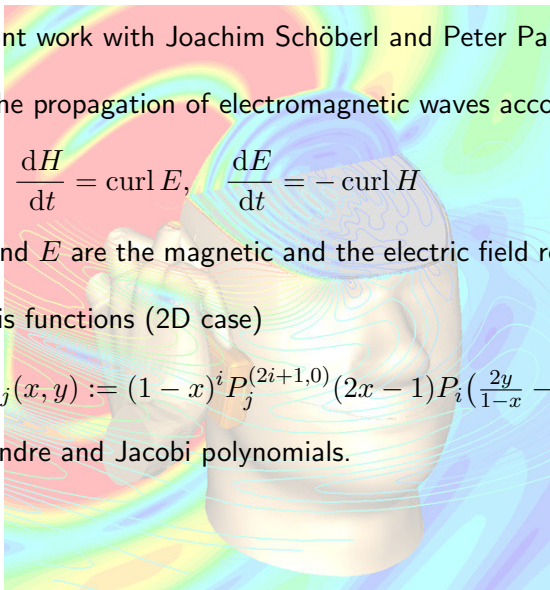
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where  $H$  and  $E$  are the magnetic and the electric field respectively.

Define basis functions (2D case)

$$\varphi_{i,j}(x, y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

using Legendre and Jacobi polynomials.



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**Problem:** Represent the partial derivatives of  $\varphi_{i,j}(x, y)$  in the basis (i.e., as linear combinations of shifts of the  $\varphi_{i,j}(x, y)$  itself).

## Find Certain Operators in Annihilator Ideals

**Ansatz:** One needs a relation of the form

$$\sum_{(k,l) \in A} a_{k,l}(i,j) \frac{d}{dx} \varphi_{i+k,j+l}(x,y) = \sum_{(m,n) \in B} b_{m,n}(i,j) \varphi_{i+m,j+n}(x,y),$$

that is free of  $x$  and  $y$  (and similarly for  $\frac{d}{dy}$ ).



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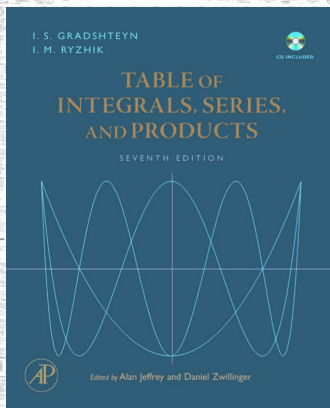
**Result:** With our holonomic methods, we find the relation

$$\begin{aligned} & (2i + j + 3)(2i + 2j + 7) \frac{d}{dx} \varphi_{i,j+1}(x,y) + \\ & 2(2i + 1)(i + j + 3) \frac{d}{dx} \varphi_{i,j+2}(x,y) - \\ & (j + 3)(2i + 2j + 5) \frac{d}{dx} \varphi_{i,j+3}(x,y) + \\ & (j + 1)(2i + 2j + 7) \frac{d}{dx} \varphi_{i+1,j}(x,y) - \\ & 2(2i + 3)(i + j + 3) \frac{d}{dx} \varphi_{i+1,j+1}(x,y) - \\ & (2i + j + 5)(2i + 2j + 5) \frac{d}{dx} \varphi_{i+1,j+2}(x,y) + \\ & 2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i,j+2}(x,y) + \\ & 2(i + j + 3)(2i + 2j + 5)(2i + 2j + 7) \varphi_{i+1,j+1}(x,y) = 0. \end{aligned}$$



# Table of Integrals by Gradshteyn and Ryzhik

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# Table of Integrals by Gradshteyn and Ryzhik

This image displays a comprehensive table of integrals, organized into a grid of columns and rows. Each entry typically consists of a mathematical expression involving variables, constants, and functions, followed by an equals sign and the integral result. The integrals are categorized by the type of function or variable involved, such as algebraic, trigonometric, exponential, and logarithmic. The table is densely packed with formulas, covering a wide range of mathematical topics. The layout is consistent, with each entry clearly separated from the others, allowing for easy reference and lookup of specific integral results.

# Table of Integrals by Gradshteyn and Ryzhik

7.20	7.21	7.22	7.23	7.24	7.25	7.26	7.27	7.28	7.29	7.30	7.31	7.32	7.33	7.34	7.35	7.36	7.37	7.38	7.39	7.40	7.41	7.42	7.43	7.44	7.45	7.46	7.47	7.48	7.49	7.50	7.51	7.52	7.53	7.54	7.55	7.56	7.57	7.58	7.59	7.60	7.61	7.62	7.63	7.64	7.65	7.66	7.67	7.68	7.69	7.70	7.71	7.72	7.73	7.74	7.75	7.76	7.77	7.78	7.79	7.80	7.81	7.82	7.83	7.84	7.85	7.86	7.87	7.88	7.89	7.90	7.91	7.92	7.93	7.94	7.95	7.96	7.97	7.98	7.99	8.00
7.20	7.21	7.22	7.23	7.24	7.25	7.26	7.27	7.28	7.29	7.30	7.31	7.32	7.33	7.34	7.35	7.36	7.37	7.38	7.39	7.40	7.41	7.42	7.43	7.44	7.45	7.46	7.47	7.48	7.49	7.50	7.51	7.52	7.53	7.54	7.55	7.56	7.57	7.58	7.59	7.60	7.61	7.62	7.63	7.64	7.65	7.66	7.67	7.68	7.69	7.70	7.71	7.72	7.73	7.74	7.75	7.76	7.77	7.78	7.79	7.80	7.81	7.82	7.83	7.84	7.85	7.86	7.87	7.88	7.89	7.90	7.91	7.92	7.93	7.94	7.95	7.96	7.97	7.98	7.99	8.00

# Table of Integrals by Gradshteyn and Ryzhik

## 7.319

$$1. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n}^\lambda(\gamma x^{1/2}) dx = (-1)^n \frac{\Gamma(\lambda+n)\Gamma(\mu)\Gamma(\nu)}{n!\Gamma(\lambda)\Gamma(\mu+\nu)} {}_3F_2\left(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > 0] \quad \text{ET II 191(41)a}$$

$$2. \int_0^1 (1-x)^{\mu-1} x^{\nu-1} C_{2n+1}^\lambda(\gamma x^{1/2}) dx = \frac{(-1)^n 2^\gamma \Gamma(\mu)\Gamma(\lambda+n+1)\Gamma(\nu+\frac{1}{2})}{n!\Gamma(\lambda)\Gamma(\mu+\nu+\frac{1}{2})} \\ \times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2\right) \\ [\operatorname{Re} \mu > 0, \operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 191(42)}$$

## 7.32 Combinations of Gegenbauer polynomials $C_n^\nu(x)$ and elementary functions

$$7.321 \int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{iax} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n!\Gamma(\nu)} a^{-\nu} J_{\nu+n}(a) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET II 281(7), MO 99a}$$

$$7.322 \int_0^{2a} [x(2a-x)]^{\nu-\frac{1}{2}} C_n^\nu\left(\frac{x}{a}-1\right) e^{-bx} dx = (-1)^n \frac{\pi \Gamma(2\nu+n)}{n!\Gamma(\nu)} \left(\frac{a}{2b}\right)^\nu e^{-ab} I_{\nu+n}(ab) \\ [\operatorname{Re} \nu > -\frac{1}{2}] \quad \text{ET I 171(9)}$$

## 7.323

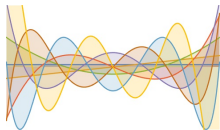
$$1. \int_0^\pi C_n^\nu(\cos \varphi) (\sin \varphi)^{2\nu} d\varphi = 0 \quad [n = 1, 2, 3, \dots]$$

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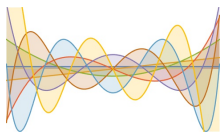
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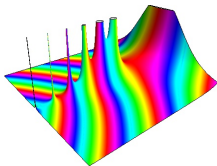
Gegenbauer  
polynomials  $C_n^{(\alpha)}(x)$


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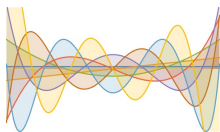
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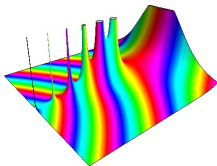
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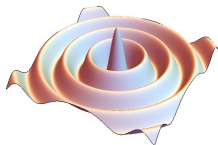
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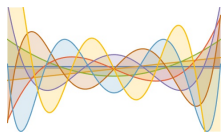
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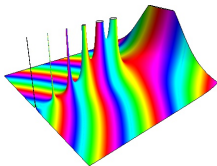
Bessel  
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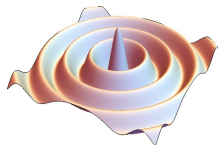
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Let's prove this identity with creative telescoping...

## Prove Special Function Identities

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad (1)$$

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$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi P_m^{(m+\frac{1}{2}, -m-\frac{1}{2})}(a)}{2^{m+\frac{3}{2}}(a+1)^{m+\frac{1}{2}}} \quad (2)$$

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## Symbolic Determinants via Holonomic Ansatz

$$\det_{1 \leq i, j \leq n} \left( \frac{1}{i + j - 1} \right) = \frac{1}{(2n - 1)!} \prod_{k=1}^{n-1} \frac{(k!)^2}{(k + 1)_{n-1}}$$

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$$\begin{aligned} & \det_{1 \leq i, j \leq 2m+1} \left( \binom{\mu+i+j+2r}{j+2r-2} - \delta_{i, j+2r} \right) \\ &= \frac{(-1)^{m-r+1} (\mu+3) (m+r+1)_{m-r}}{2^{2m-2r+1} \left(\frac{\mu}{2} + r + \frac{3}{2}\right)_{m-r+1}} \cdot \prod_{i=1}^{2m} \frac{(\mu+i+3)_{2r}}{(i)_{2r}} \\ & \times \prod_{i=1}^{m-r} \frac{(\mu+2i+6r+3)_i^2 \left(\frac{\mu}{2} + 2i + 3r + 2\right)_{i-1}^2}{(i)_i^2 \left(\frac{\mu}{2} + i + 3r + 2\right)_{i-1}^2}. \end{aligned}$$

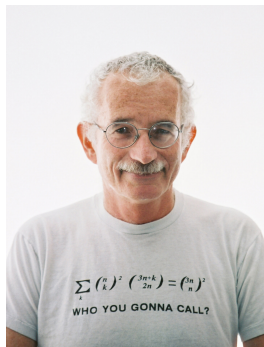
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**Problem:** Prove a determinantal identity of the form  $\det_{1 \leq i, j \leq n} (a_{i,j}) = b_n$

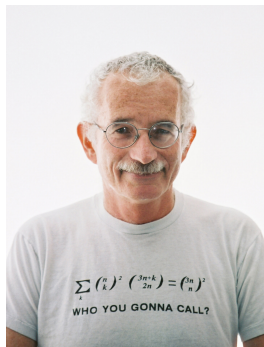




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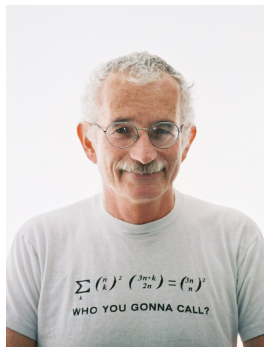
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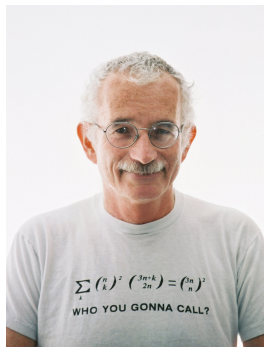
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- ▶  $b_n$  is a closed form ( $b_n \neq 0$  for all  $n$ ).



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$$\mathcal{A}_n = \left( \begin{array}{ccc|c} & & & \\ & \mathcal{A}_{n-1} & & \\ \hline & \cdots & & \\ a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array} \right)$$



**Laplace expansion:**

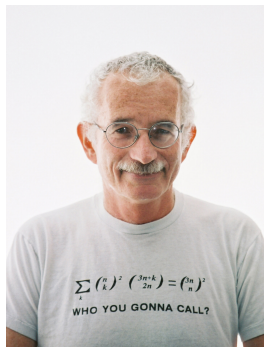
$$\det(\mathcal{A}_n) = a_{n,1} \text{Cof}_{n,1} + \dots + a_{n,n-1} \text{Cof}_{n,n-1} + a_{n,n} \det(\mathcal{A}_{n-1})$$

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**Laplace expansion:**

$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1} \frac{\text{Cof}_{n,1}}{\det(\mathcal{A}_{n-1})} + \dots + a_{n,n-1} \frac{\text{Cof}_{n,n-1}}{\det(\mathcal{A}_{n-1})} + a_{n,n}$$

## The Holonomic Ansatz

**Problem:** Prove a determinantal identity of the form  $\det_{1 \leq i, j \leq n} (a_{i,j}) = b_n$ , where

- ▶  $a_{i,j}$  is a holonomic sequence
- ▶ that does not depend on  $n$ , and
- ▶  $b_n$  is a closed form ( $b_n \neq 0$  for all  $n$ ).

$$\mathcal{A}_n = \left( \begin{array}{ccc|c} & & & \\ & \mathcal{A}_{n-1} & & \\ \hline & \cdots & & \\ a_{n,1} & \cdots & a_{n,n-1} & a_{n,n} \end{array} \right)$$



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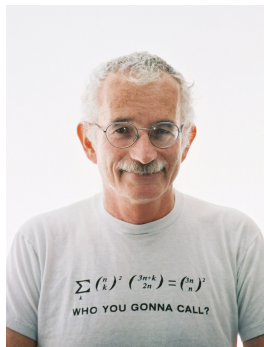
$$\frac{\det(\mathcal{A}_n)}{\det(\mathcal{A}_{n-1})} = a_{n,1}c_{n,1} + \dots + a_{n,n-1}c_{n,n-1} + a_{n,n}c_{n,n}$$

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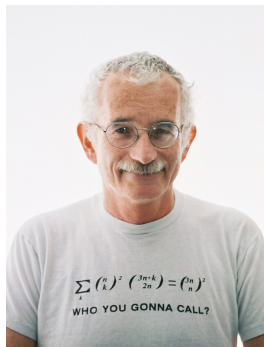
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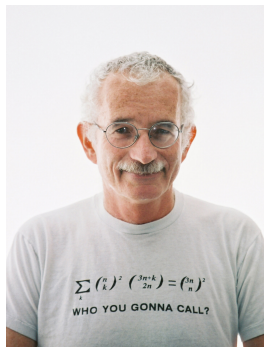


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**Conjecture** (Di Francesco's determinant for 20V configurations):

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