D-finite functions (univariate case)

Def: A function $f(x)$ is called D-finite (differentiably finite) | or: holonomic
if it satisfies a LODE with polynomial coeffs:
$$p_r(x) f^{(r)}(x) + \dots + p_1(x) f'(x) + p_0(x) f(x) = 0$$
with $p_0, \dots, p_r \in K[x]$ (not all zero)

Examples: x^n, const., sin, exp, ...
not D-finite: tan, $\Gamma(x)$

Features:
- finitely many initial conds. $\rightsquigarrow$ fin. amount of data
- rich class (show slide)
- closure properties

Thm: If $f(x)$ and $g(x)$ are D-finite, then also $f(x)+g(x) =: h(x)$ is DF.

Proof: Let r, s denote the orders of f, g, resp. To show:
$\exists u \in \mathbb{N}, p_0, \dots, p_u \in K[x]$: $p_u(x) h^{(u)}(x) + \dots + p_0(x) h(x) = 0$.
$\Leftrightarrow p_u(x)\left(f^{(u)}(x) + g^{(u)}(x)\right) + \dots + p_0(x)\left(f(x)+g(x)\right) = 0$

Using the LODE for f: $f^{(r)}(x) = \{\} \cdot f^{(r-1)}(x) + \dots + \{\} \cdot f'(x) + \{\} \cdot f(x)$
(and analogously for g) – works for any derivative

$\Rightarrow \ell_0(p_0, \dots, p_u) \cdot f(x) + \ell_1(p_0, \dots, p_u) \cdot f'(x) + \dots + \ell_{r-1}(p_0, \dots, p_u) f^{(r-1)}(x)$
$+ \ell_r(p_0, \dots, p_u) \cdot g(x) + \dots + \ell_{r+s-1}(p_0, \dots, p_u) \cdot g^{(s-1)}(x) = 0$

where each ℓ_j is a linear poly in the p_i: $\ell_j = \sum_{i=0}^{u} c_{i,j}(x) \cdot p_i(x)$.

The linear system $\ell_0 = \dots = \ell_{r+s-1} = 0$ is guaranteed to have a nontrivial solution if $u := r+s$ is chosen. $\square$

Hint: there may exist a solution for smaller u, hence try and loop.

Thm: If $f(x)$ and $g(x)$ are D-finite, then also
- $f(x) \cdot g(x)$ (proof: analogous, exercise!)
- $f'(x)$ (or more general: any diff. poly in $f, f', \dots$)
- $f(a(x))$ where $a(x)$ is an algebraic function: $p(x,a) = 0$ for $p \in K[x,y]$
- $\int f(x)\,dx$ (proof: replace $f^{(i)}$ by $f^{(i+1)}$)

AE2DE $\longrightarrow$ next page

Operator notation: • Let D_x denote the differentiation w.r.t. x, i.e.,

$$D_x(f(x)) = f'(x), \quad D_x^2(f(x)) = f''(x), \text{ etc.} \quad D_x^0(f(x)) = f(x)$$

- Let $K(x)\langle D_x\rangle$ denote the polynomial ring in D_x with coeffs in $K(x)$. It is not commutative $D_x \cdot x = x \cdot D_x + 1$ (Leibniz rule)

  more general: $D_x \cdot r(x) = r(x) \cdot D_x + r'(x)$

- $f(x)$ is D-finite $\Leftrightarrow \exists L \in K(x)\langle D_x\rangle \setminus \{0\} : L(f(x)) = 0$

Fact: Let $L_i \in K(x)\langle D_x\rangle\setminus\{0\}$ annihilate f, g resp., i.e., $L_1(f(x))=0$. Then $L_2(g(x))=0$

- $L \cdot D_x$ annihilates $\int f(x)\,dx$
- $\underbrace{\mathrm{lclm}(L_1, L_2)}_{=:L}$ annihilates $f+g$ (actually $c_1 f + c_2 g$ for arb. const. c_1, c_2)

  Proof: $L = M_1 L_1 = M_2 L_2$ for certain $M_1, M_2 \in K(x)\langle D_x\rangle$
- If f satisfies $L(f) = g$ for some D-finite g, then f is D-finite.

  Proof: Assume $M(g)=0$. Then $(M \cdot L)(f) = 0$.

$\boxed{\text{AE2DE}}$ → Corollary of previous Thm (with $f(x)=x$)

Thm.: Let $f(x)$ be an algebraic function. Then f is D-finite.

Proof: Let ~~...~~ $m \in K[x,y]$ be the minimal polynomial of f, i.e., $m(x, f(x)) = 0$ and m irreducible.

$$\hookrightarrow a_d(x)(f(x))^d + a_{d-1}(x)(f(x))^{d-1} + \dots + a_1(x) f(x) + a_0(x) = 0$$

Differentiate: $a_d' f^d + d \cdot a_d f^{d-1} f' + \dots + a_1' f + a_1 f' + a_0' = 0$

$$\Rightarrow f' = \frac{-(a_d' f^d + \dots + a_1' + a_0')}{d \cdot a_d f^{d-1} + \dots + a_1} =: \frac{q(x,f)}{r(x,f)}$$

Note: $\gcd(m, r) = 1$, hence by EEA $\exists u, v \in K(x)[f] : u \cdot m + v \cdot r = 1$

We get $v(x,f) \cdot r(x,f) = 1 \pmod{m(x,f)}$, hence

$$f' = \frac{q(x,f) \cdot v(x,f)}{r(x,f) \cdot v(x,f)} = q(x,f) \cdot v(x,f) = c_{1,d-1} f^{d-1} + \dots + c_{1,1} f + c_{1,0} \pmod m$$

Differentiate again: $\quad f'' = c_{2,d-1} f^{d-1} + \dots + c_{2,1} f + c_{2,0}$, etc.

It follows that f satisfies a LODE with polynomial coeffs. of order at most d.

Def.: A sequence $(a_n)_{n\in\mathbb{N}}$ is called P-recursive if it satisfies a linear recurrence equation with polynomial coefficients: | or: holonomic | or: D-finite

$$p_r(n)\cdot a_{n+r} + \dots + p_1(n)\, a_{n+1} + p_0(n)\, a_n = 0$$

with $p_0,\dots,p_r \in K[n]$ (not all zero).

Examples: Fibonacci, polynomials, hypergeometric, H_n, ortho. polys

not P-rec: prime numbers

Features:
- finitely many initial values $\rightsquigarrow$ finite amount of data
- rich class (show slide)
- closure properties

Thm: If a_n and b_n are P-recursive, then also the following are P-rec.
- $a_n + b_n$
- $a_n \cdot b_n$
- a_{cn+d} for integers c,d
- $\sum_n a_n$ (indefinite sum, i.e., c_n s.t. $c_{n+1} - c_n = a_n$)

Operator notation:
- S_n denotes the forward shift operator, i.e., $S_n(a_n) = a_{n+1}$, $S_n^2(a_n) = a_{n+2}$
- $K(n)\langle S_n\rangle$: ring of all linear recurrence operators with coeffs in $K(n)$

Commutation rule: $S_n \cdot n = (n+1)\cdot S_n$, more gen.: $S_n \cdot r(n) = r(n+1)\cdot S_n$

Facts: see D-finite

Thm: A function $f(x)$ is D-finite if and only if the sequence of its Taylor coefficients is P-recursive. Stated differently: A sequence is P-rec. iff its generating function is D-finite.

Proof: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $f'(x) = \sum_{n=1}^{\infty} n\cdot a_n \cdot x^{n-1}$, $f^{(i)}(x) = \sum_{n=i}^{\infty} (n-i+1)_i\, a_n x^{n-i}$

$= \sum_{n=0}^{\infty} (n+1)_i\, a_{n+i}\, x^n$

LODE: $\sum_{i=0}^{r}\sum_{j=0}^{d} p_{i,j}\, x^j f^{(i)}(x) = 0$

$$\Rightarrow \sum_{i=0}^{r}\sum_{j=0}^{d}\sum_{n=0}^{\infty} p_{i,j}\,(n+1)_i\, a_{n+i}\, x^{n+j} = 0$$

$$\Rightarrow \sum_{i=0}^{r}\sum_{j=0}^{d}\sum_{n=j}^{\infty} p_{i,j}\,(n-j+1)_i\, a_{n-j+i}\, x^n \Longrightarrow \sum_{i=0}^{r}\sum_{j=0}^{d} p_{i,j}\,(n-j+1)_i\, a_{n-j+i} = 0$$

for $n \geq d$