Creative Telescoping

3.3 Zeilberger's Algorithm

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Johann Radon Institute for Computational and Applied Mathematics
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Recent Trends in Computer Algebra
Special Week @ Institut Henri Poincaré

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ÖAW RICAM

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- Have to choose two parameters r and s (order w.r.t. n and k).

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$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+n)}}_{=:\, F_n} \rightsquigarrow (n+2)^2 F_{n+2} = (n+1)(2n+3)F_{n+1} - n(n+1)F_n$$

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Method for doing integrals and sums
(aka Feynman's differentiating under the integral sign)

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Summing from a to b yields a recurrence for $F(n)$:

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Consider the following integration problem: $F(x) := \int_a^b f(x,y)\,\mathrm{d}y$

Telescoping: find g such that $f(x,y) = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y)$.

Then $F(n) = \int_a^b \left(\frac{\mathrm{d}}{\mathrm{d}y} g(x,y)\right)\mathrm{d}y \qquad = g(x,b) - g(x,a)$.

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$$c_r(x)\frac{\mathrm{d}^r}{\mathrm{d}x^r} f(x,y) + \cdots + c_0(x) f(x,y) \ = \frac{\mathrm{d}}{\mathrm{d}y} g(x,y).$$

Integrating from a to b yields a differential equation for $F(x)$:

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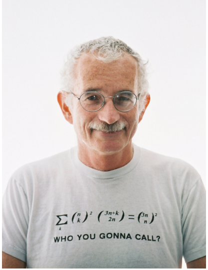
Zeilberger's (Fast) Algorithm



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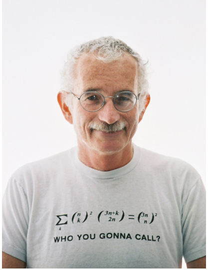

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Communication

A fast algorithm for proving terminating hypergeometric identities

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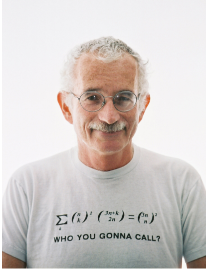

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J. Symbolic Computation (1991) **11**, 195–204

The Method of Creative Telescoping

DORON ZEILBERGER

Department of Mathematics and Computer Science, Temple University, Philadelphia, PA 19122, *USA*

In memory of John Riordan, master of ars combinatorica

(*Received* 1 *June* 1989)

An algorithm for definite hypergeometric summation is given. It is based, in a non-obvious way, on Gosper's algorithm for definite hypergeometric summation, and its theoretical justification relies on Bernstein's theory of holonomic systems.

Bivariate Hypergeometric Terms

Definition: A bivariate term $f(n,k)$ is called hypergeometric (w.r.t. n and k) if

$$\frac{f(n+1,k)}{f(n,k)} \qquad \text{and} \qquad \frac{f(n,k+1)}{f(n,k)}$$

are both rational functions in n and k.

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- Get $q(n)S(n+1) - p(n)S(n) = 0$ with $S(n) := \sum_k f(n,k)$.
- Check that $h(0) = S(0)$. Hence $S(n) = h(n)$ for all n.

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The original identity follows.

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- Try order $r = 0, 1, \ldots$ until success.
- Write recurrence with undetermined coefficients $p_i \in K(n)$:

$$p_r(n)S(n+r) + \cdots + p_1(n)S(n+1) + p_0(n)S(n) = 0.$$

Apply Gosper's algorithm to $p_r(n)f(n+r,k) + \cdots + p_0(n)f(n,k)$.

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Apply the parametrized Gosper algorithm to the **hypergeometric** term

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Let's prove this...

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Examples:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=-n}^{n} (-1)^k \binom{2n}{n+k}^2 = \frac{(2n)!}{(n!)^2}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad \rightsquigarrow \quad \text{second-order recurrence}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{dk}{n} = (-d)^n$$

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Then there exist polynomials $p_0(n), \ldots, p_r(n)$, not all zero, and $q(n,k) \in K(n,k)$ such that $g(n,k) := q(n,k) f(n,k)$ satisfies

$$\sum_{i=0}^{r} p_i(n) f(n+i,k) = g(n,k+1) - g(n,k).$$