Creative Telescoping

1.1 Introduction

Shaoshi Chen, Manuel Kauers, Christoph Koutschan

Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

Monday, 27.11.2023
Recent Trends in Computer Algebra
Special Week @ Institut Henri Poincaré

AMSS Academy of Mathematics and Systems Science,CAS

JKU JOHANNES KEPLER UNIVERSITY LINZ

ÖAW RICAM

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Many of them are **hypergeometric**: $\dfrac{f(n+1)}{f(n)} \in \mathbb{Q}(n)$.

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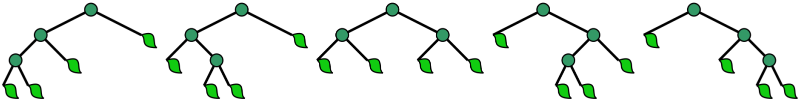

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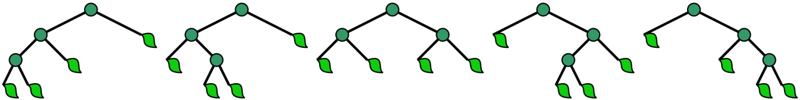

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{k+n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{k+n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

Combinatorial Identities

Many such hypergeometric summation identities can nowadays be proven in an automatic and mechanical way:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Special Functions

- arise in mathematical analysis and in real-world phenomena

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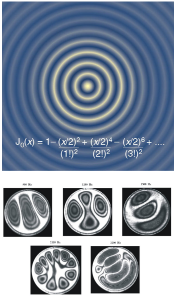

Bessel function

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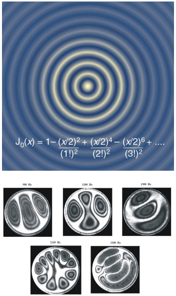
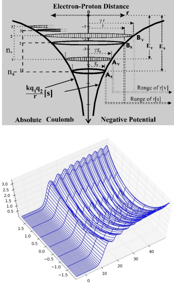


Bessel function



Coulomb function

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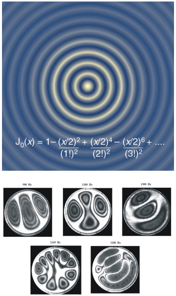

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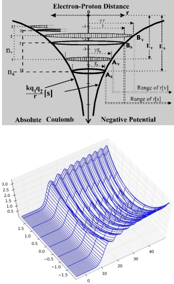

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Special Functions

- arise in mathematical analysis and in real-world phenomena
- are solutions to certain differential equations
- cannot be expressed in terms of the usual elementary functions ($\sqrt{\ }$, exp, log, sin, cos, ...)

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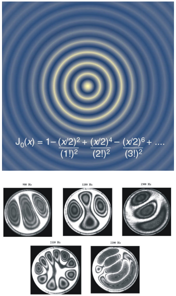

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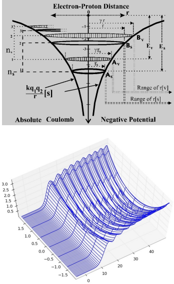

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4. $$\int_0^1 x^\nu K_\nu(ax)\,dx = 2^{\nu-1}a^{-\nu}\pi^{\frac{1}{2}}\,\Gamma\left(\nu+\tfrac{1}{2}\right)\left[K_\nu(a)\,\mathbf{L}_{\nu-1}(a)+\mathbf{L}_\nu(a)\,K_{\nu-1}(a)\right]$$ $[\operatorname{Re}\nu > -\tfrac{1}{2}]$

5. $$\int_0^1 x^{\nu+1} J_\nu(ax)\,dx = a^{-1} J_{\nu+1}(a)$$ $[\operatorname{Re}\nu > -1]$

6. $$\int_0^1 x^{\nu+1} Y_\nu(ax)\,dx = a^{-1} Y_{\nu+1}(a) + 2^{\nu+1}a^{-\nu-2}\pi^{-1}\,\Gamma(\nu+1)$$ $[\operatorname{Re}\nu > -1]$

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8. $$\int_0^1 x^{\nu+1} K_\nu(ax)\,dx = 2^\nu a^{-\nu-2}\,\Gamma(\nu+1) - a^{-1} K_{\nu+1}(a)$$ $[\operatorname{Re}\nu > -1]$

9. $$\int_0^1 x^{1-\nu} J_\nu(ax)\,dx = \frac{a^{\nu-2}}{2^{\nu-1}\,\Gamma(\nu)} - a^{-1} J_{\nu-1}(a)$$

10. $$\int_0^1 x^{1-\nu} Y_\nu(ax)\,dx = \frac{a^{\nu-2}\cot(\nu\pi)}{2^{\nu-1}\,\Gamma(\nu)} - a^{-1} Y_{\nu-1}(a)$$ $[\operatorname{Re}\nu < 1]$

11. $$\int_0^1 x^{1-\nu} I_\nu(ax)\,dx = a^{-1} I_{\nu-1}(a) - \frac{a^{\nu-2}}{2^{\nu-1}\,\Gamma(\nu)}$$

12. $$\int_0^1 x^{1-\nu} K_\nu(ax)\,dx = 2^{-\nu}a^{\nu-2}\,\Gamma(1-\nu) - a^{-1} K_{\nu-1}(a)$$ $[\operatorname{Re}\nu < 1]$

13.7 $$\int_0^1 x^\mu J_\nu(ax)\,dx = \frac{2^\mu\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\ldots}$$

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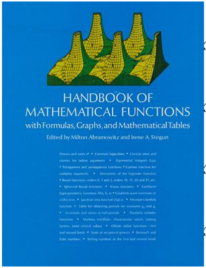

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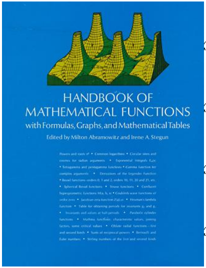

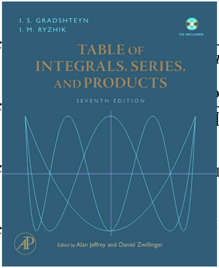

$x \ldots \nu) \; - a^{-1} J_{\nu-1}(a)$

$x \ldots \mathrm{t}(\nu\pi) / \Gamma(\nu) \; - a^{-1} Y_{\nu-1}(a) \qquad [\operatorname{Re}\nu < 1]$

$x \ldots (a) - \dfrac{a^{\nu-2}}{2^{\nu-1}\,\Gamma(\nu)}$

$x \ldots {}^{2}\,\Gamma(1-\nu) - a^{-1} K_{\nu-1}(a)$

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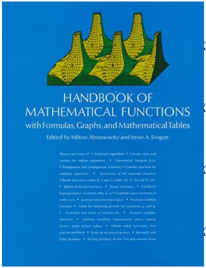

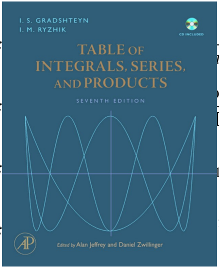

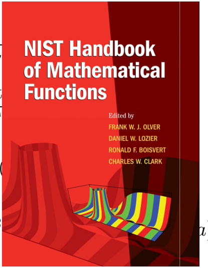

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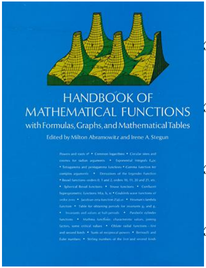

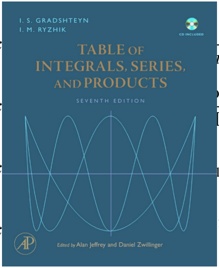

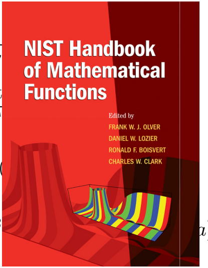

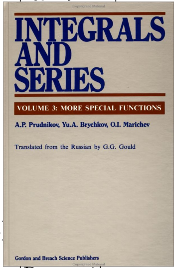

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The Holonomic Systems Approach

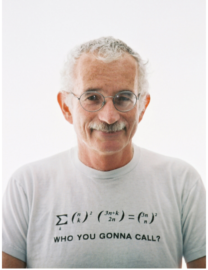
Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Received 14 November 1989

Abstract: We observe that many special functions are solutions of so-called holonomic systems. Bernstein's deep theory of holonomic systems is then invoked to show that any identity involving sums and integrals of products of these special functions can be verified in a finite number of steps. This is partially substantiated by an algorithm that proves terminating hypergeometric series identities, and that is given both in English and in MAPLE.



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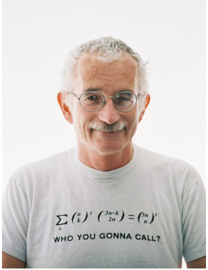

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On May 18, 2005, Frank Olver, the mathematics editor of DLMF, sent the following email to Peter Paule:

> *"The writing of DLMF Chapter BS Leonard Maximon and myself is now largely complete [...] However, a problem has arisen in connection with about a dozen formulas from Chapter 10 of Abramowitz and Stegun for which we have not yet tracked down proofs, and the author of this chapter, Henry Antosiewiecz, died about a year ago. Since it is the editorial policy for the DLMF not to state formulas without indications of proofs, I am hoping that you will be willing to step into the breach and supply verifications by computer algebra methods [...] I will fax you the formulas later today."*

Digital Library of Mathematical Functions

$$\frac{1}{z}\sin\sqrt{z^2+2zt} = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}y_{n-1}(z)$$

$$\frac{1}{z}\cos\sqrt{z^2-2zt} = \sum_{n=0}^{\infty}\frac{t^n}{n!}j_{n-1}(z)$$

$$\left[\frac{\partial}{\partial\nu}j_\nu(z)\right]_{\nu=0} = \frac{1}{z}(\mathrm{Ci}(2z)\sin z - \mathrm{Si}(2z)\cos z)$$

$$\left[\frac{\partial}{\partial\nu}j_\nu(z)\right]_{\nu=-1} = \frac{1}{z}(\mathrm{Ci}(2z)\cos z + \mathrm{Si}(2z)\sin z)$$

$$\left[\frac{\partial}{\partial\nu}y_\nu(z)\right]_{\nu=0} = \frac{1}{z}(\mathrm{Ci}(2z)\cos z + [\mathrm{Si}(2z)-\pi]\sin z)$$

$$\left[\frac{\partial}{\partial\nu}y_\nu(z)\right]_{\nu=-1} = -\frac{1}{z}(\mathrm{Ci}(2z)\sin z - [\mathrm{Si}(2z)-\pi]\cos z)$$

Digital Library of Mathematical Functions

$$J_0(z\sin\theta) = \sum_{n=0}^{\infty}(4n+1)\frac{(2n)!}{2^{2n}n!^2}j_{2n}(z)P_{2n}(\cos\theta)$$

$$j_n(2z) = -n!z^{n+1}\sum_{k=0}^{n}\frac{2n-2k+1}{k!(2n-k+1)!}j_{n-k}(z)y_{n-k}(z)$$

$$\sum_{n=0}^{\infty}j_n^2(z) = \frac{\mathrm{Si}(2z)}{2z}$$

$$\frac{1}{z}\sinh\sqrt{z^2-2\mathrm{i}zt} = \sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}\sqrt{\tfrac{1}{2}\pi/z}\,I_{-n+\frac{1}{2}}(z)$$

$$\frac{1}{z}\cosh\sqrt{z^2+2\mathrm{i}zt} = \sum_{n=0}^{\infty}\frac{(\mathrm{i}t)^n}{n!}\sqrt{\tfrac{1}{2}\pi/z}\,I_{n-\frac{1}{2}}(z)$$

Digital Library of Mathematical Functions

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=1/2} = -\frac{1}{\sqrt{2\pi z}}\left(\mathrm{Ei}(2z)\mathrm{e}^{-z} + \mathrm{E}_1(2z)\mathrm{e}^{z}\right)$$

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=-1/2} = \frac{1}{\sqrt{2\pi z}}\left(\mathrm{Ei}(2z)\mathrm{e}^{-z} - \mathrm{E}_1(2z)\mathrm{e}^{z}\right)$$

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=\pm 1/2} = \pm\sqrt{\frac{\pi}{2z}}\mathrm{E}_1(2z)\mathrm{e}^{z}$$

Digital Library of Mathematical Functions

$$\left[\frac{\partial}{\partial\nu} I_\nu(z)\right]_{\nu=1/2} = -\frac{1}{\sqrt{2\pi z}}\left(\mathrm{Ei}(2z)\mathrm{e}^{-z} + \mathrm{E}_1(2z)\mathrm{e}^{z}\right)$$

$$\left[\frac{\partial}{\partial\nu} I_\nu(z)\right]_{\nu=-1/2} = \frac{1}{\sqrt{2\pi z}}\left(\mathrm{Ei}(2z)\mathrm{e}^{-z} - \mathrm{E}_1(2z)\mathrm{e}^{z}\right)$$

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Within two weeks, all identities were proven with computer algebra, by the members of the algorithmic combinatorics group of RISC.

(joint work with Stefan Gerhold, Manuel Kauers, Peter Paule, Carsten Schneider, and Burkhard Zimmermann)

Proof of the Irrationality of $\zeta(3)$

In Roger Apéry's proof (1978) a crucial step is to show that

$$b_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfies the second-order recurrence:

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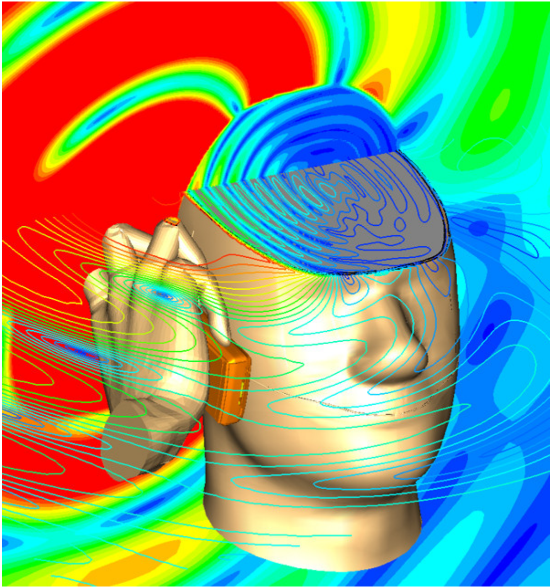
Alternative approach by Frits Beukers via the integral

$$\int_0^1 \int_0^1 \int_0^1 \frac{\big(x(1-x)y(1-y)z(1-z)\big)^n}{(1-z+xyz)^{n+1}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

Finite Element Methods

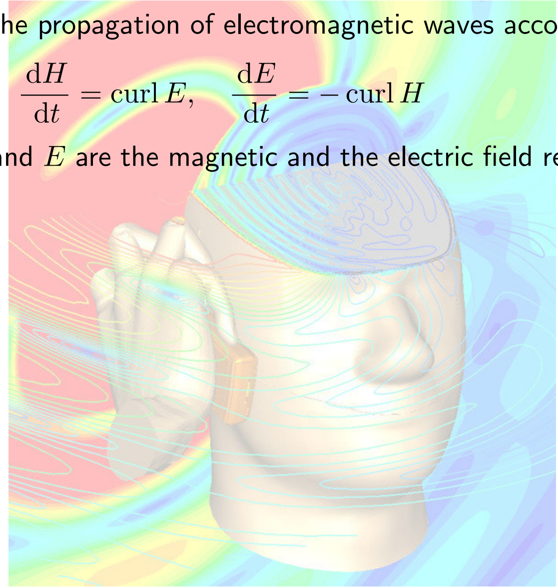


Finite Element Methods

Simulate the propagation of electromagnetic waves according to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \operatorname{curl} E, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = -\operatorname{curl} H \qquad \text{(Maxwell)}$$

where H and E are the magnetic and the electric field respectively.



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Define basis functions (2D case)

$$\varphi_{i,j}(x,y) := (1-x)^i P_j^{(2i+1,0)}(2x-1) P_i\left(\frac{2y}{1-x} - 1\right)$$

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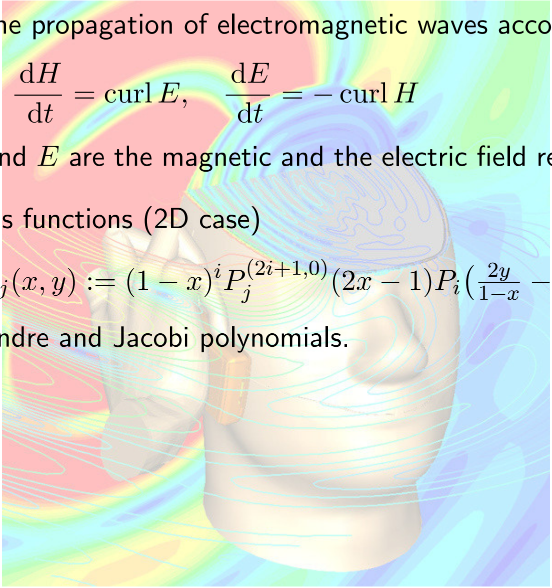

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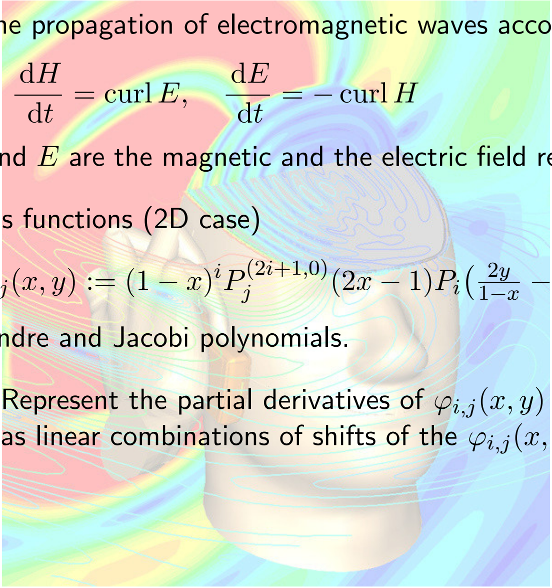

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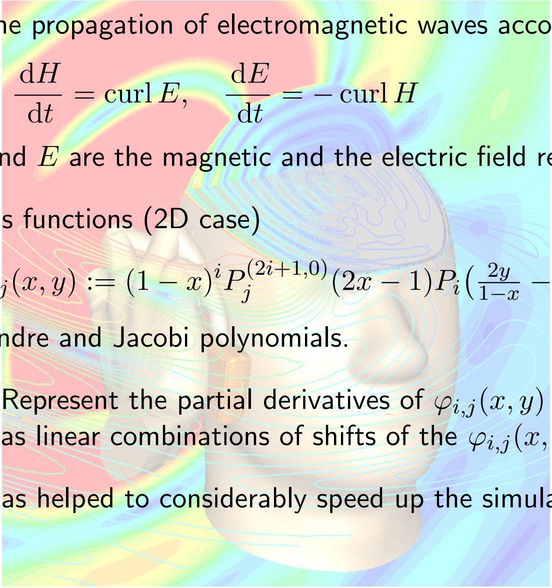

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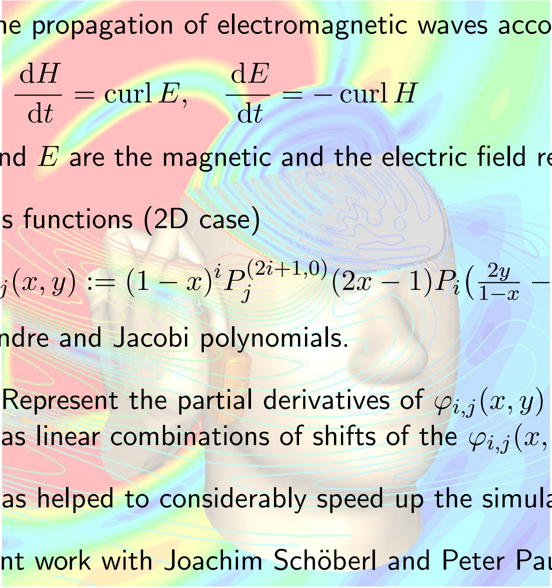

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(joint work with Joachim Schöberl and Peter Paule)

Symbolic Determinants via Holonomic Ansatz

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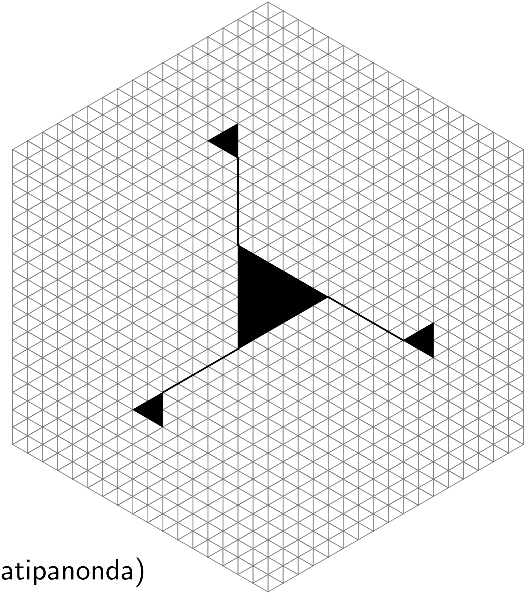
$$\det_{1\leqslant i,j\leqslant 2m+1}\left[\binom{\mu+i+j+2r}{j+2r-2}-\delta_{i,j+2r}\right]$$

$$= \frac{(-1)^{m-r+1}(\mu+3)(m+r+1)_{m-r}}{2^{2m-2r+1}\left(\frac{\mu}{2}+r+\frac{3}{2}\right)_{m-r+1}}\cdot\prod_{i=1}^{2m}\frac{(\mu+i+3)_{2r}}{(i)_{2r}}$$

$$\times\prod_{i=1}^{m-r}\frac{\left(\mu+2i+6r+3\right)_i^2\left(\frac{\mu}{2}+2i+3r+2\right)_{i-1}^2}{(i)_i^2\left(\frac{\mu}{2}+i+3r+2\right)_{i-1}^2}.$$

Rhombus Tilings



(joint work
with Hao Du,
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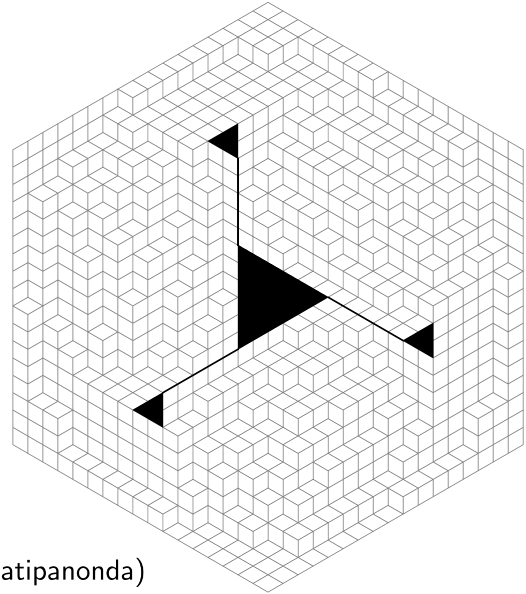

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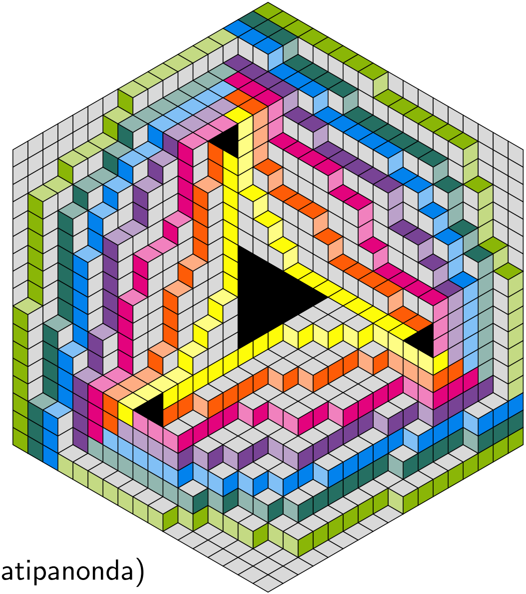

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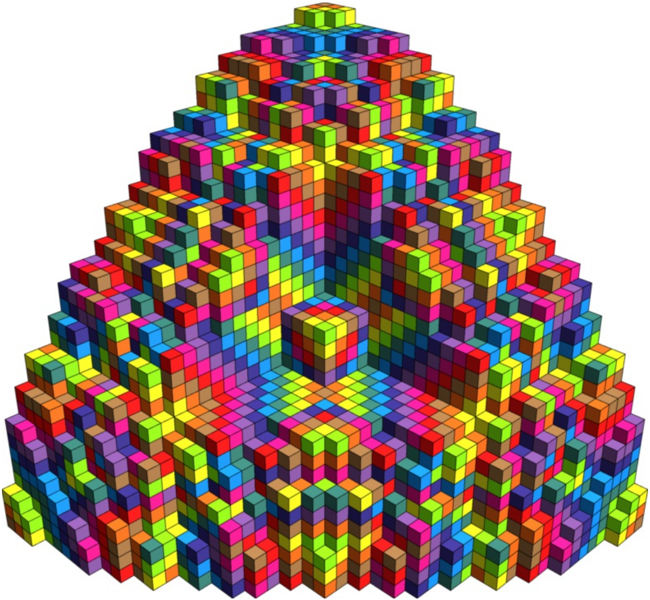
q-Enumeration of Totally Symmetric Plane Partitions



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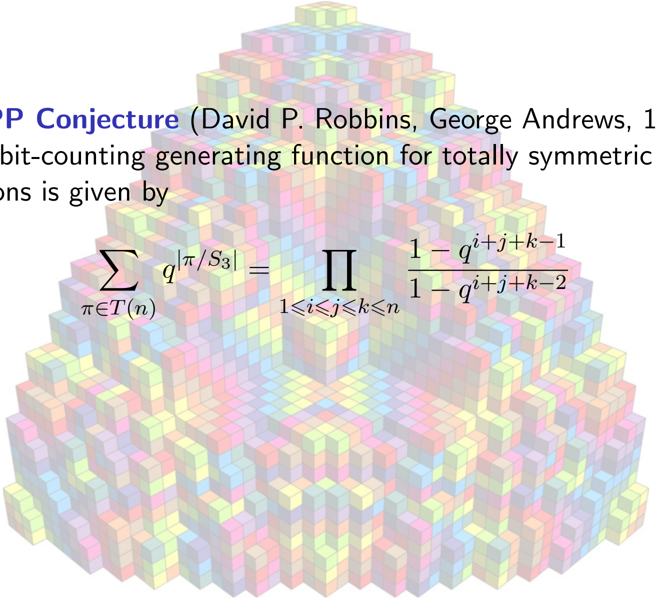

q-TSPP Conjecture (David P. Robbins, George Andrews, 1983)
The orbit-counting generating function for totally symmetric plane partitions is given by

$$\sum_{\pi \in T(n)} q^{|\pi/S_3|} = \prod_{1 \leqslant i \leqslant j \leqslant k \leqslant n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

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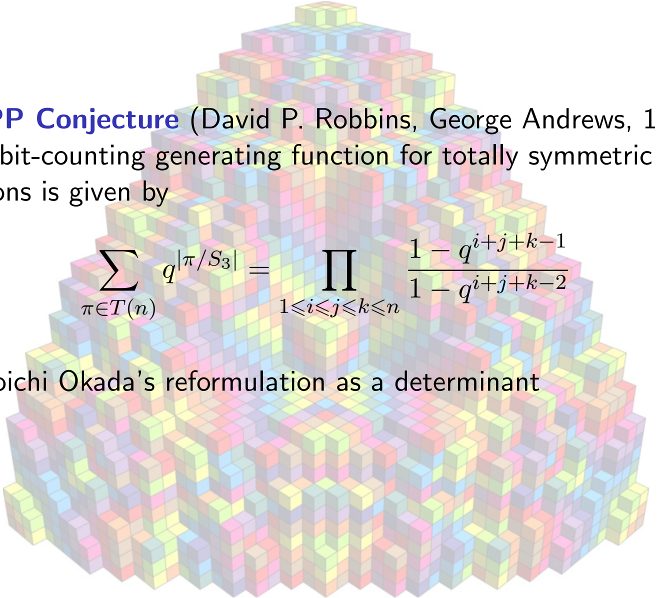

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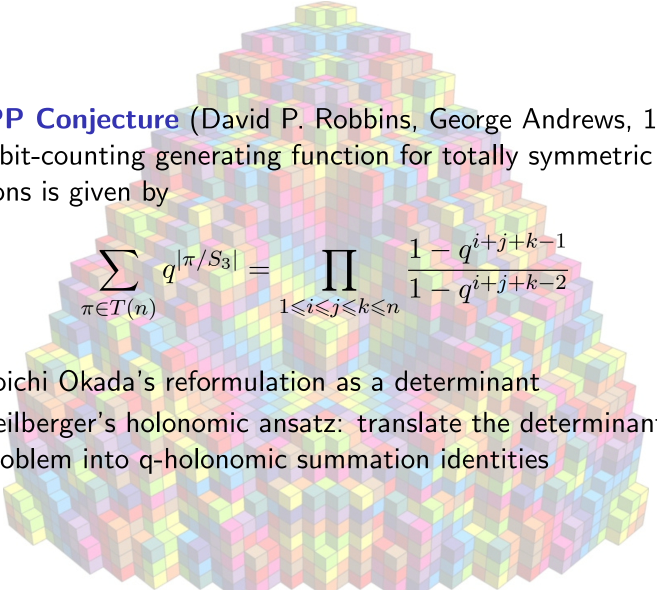

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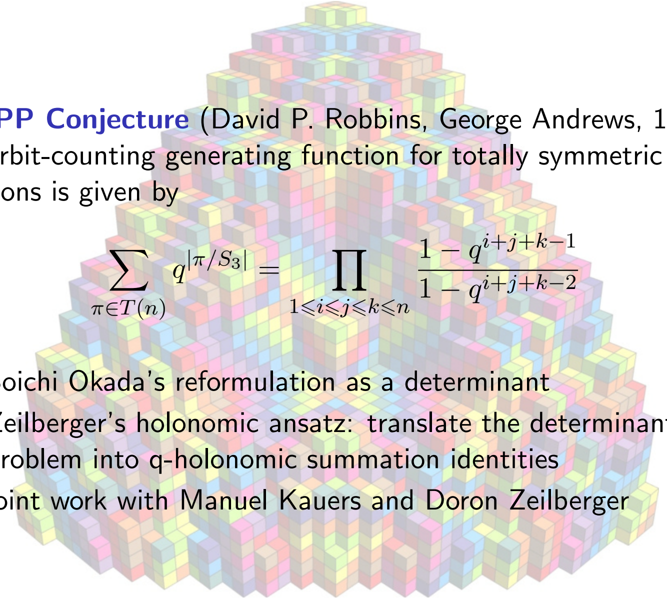

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- joint work with Manuel Kauers and Doron Zeilberger

Holonomic Description of the Cofactor Function

$$\bigcirc \cdot c_{n,j+4} = \bigcirc \cdot c_{n,j} + \bigcirc \cdot c_{n,j+1} + \bigcirc \cdot c_{n,j+2} + \bigcirc \cdot c_{n,j+3} + \bigcirc \cdot c_{n+2,j} + \bigcirc \cdot c_{n+2,j+1}$$

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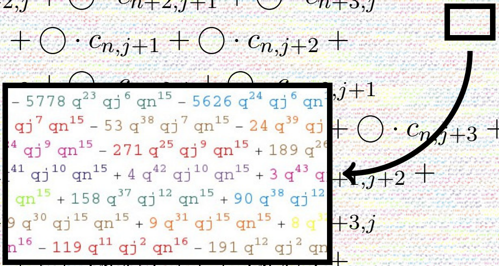

Series Acceleration Identities

Fast converging series for efficient computation of mathematical constants (Jesús Guillera & John Campbell)

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$$\frac{16\pi^2}{3} = \sum_{k=0}^{\infty} \left(\frac{27}{64}\right)^k \frac{\left(k!\right)^3 \left(\frac{5}{6}\right)_k \left(\frac{7}{6}\right)_k}{\left(\frac{3}{2}\right)_k^5} (74k^2 + 101k + 35)$$

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$$\pi^2 = \frac{128}{156279375} \sum_{k=0}^{\infty} \left(-\frac{1}{27}\right)^k \frac{\left(\frac{1}{2}\right)_k^2 \left(k!\right)^2 \left(\frac{3}{2}\right)_k^2 (2)_k \left(\frac{5}{2}\right)_k}{\left(\frac{7}{4}\right)_k \left(\frac{11}{6}\right)_k \left(\frac{13}{6}\right)_k \left(\frac{9}{4}\right)_k^2 \left(\frac{11}{4}\right)_k^2 \left(\frac{13}{4}\right)_k}$$
$$\times (1605632k^8 + 17633280k^7 + 83231232k^6 + 220523520k^5 + 358672608k^4 + 366633840k^3 + 229955938k^2 + 80885565k + 12211200)$$

Symbolic Summation in Particle Physics

- Complicated multi-sums that arise in the evaluation of Feynman integrals

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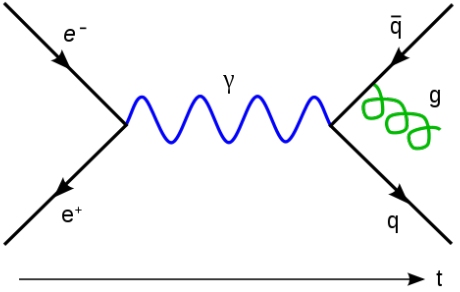

$$\int_0^1 \int_0^1 \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2} z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}} \left(1 - w^{n+1} - (1-w)^{n+1}\right) \,\mathrm{d}w\,\mathrm{d}z$$

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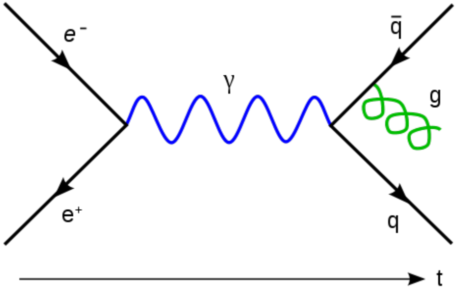

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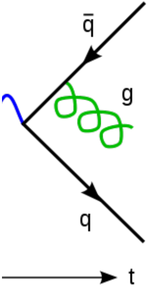
DESY 19–096, DO–TH 19/09, SAGEX-2019-13

Three loop heavy quark form factors and their asymptotic behavior

J. Ablinger[1], J. Blümlein[2], P. Marquard[2], N. Rana[2,3] and C. Schneider[1]

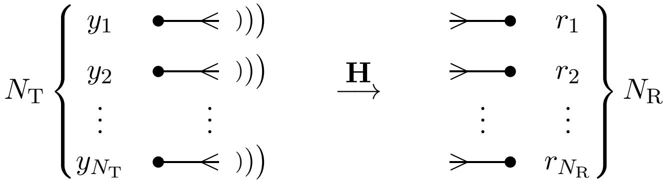
Abstract A summary of the calculation of the color–planar and complete light quark contributions to the massive three–loop form factors is presented. Here a novel calculation method for the Feynman integrals is used, solving general uni–variate first order factorizable systems of differential equations. We also present predictions for the asymptotic structure of these form factors.



$$\int_0^1 \int_0^1 \frac{w^{-1-\varepsilon/2}(1-z)^{\varepsilon/2} z^{-\varepsilon/2}}{(z+w-wz)^{1-\varepsilon}} \left(1 - w^{n+1} - (1-w)^{n+1}\right) \mathrm{d}w \, \mathrm{d}z$$

Creative Telescoping in Algebraic Statistics

MIMO Wireless Communication System:



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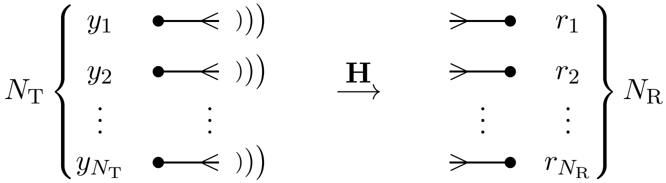
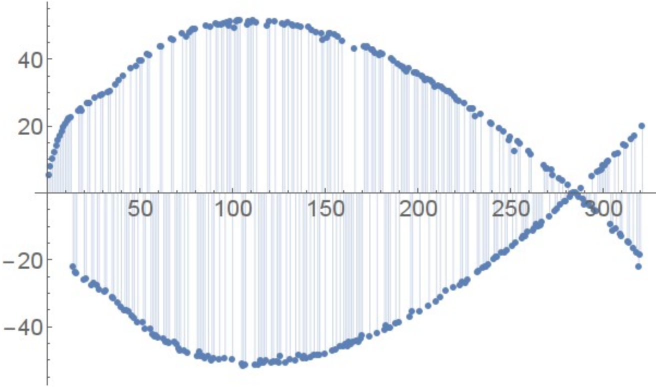


SNR probability density function $p(t; x_1, x_2)$:

$$\begin{aligned} p(t; x_1, x_2) &= \int_0^\infty e^{-st} M(s; x_1, x_2)\, \mathrm{d}s \\ &= e^{-x_2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{(N)_{n_1}}{(n_2 + N_{\mathrm{R}})_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{x_2^{n_2}}{n_2!} \\ &\quad \times \sum_{m_1=0}^{n_1} \binom{n_1}{m_1} \frac{(-1)^{m_1} t^{N+n_1-m_1-1} e^{-t/\Gamma_1}}{(N + n_1 - m_1 - 1)!\, \Gamma_1^{N+n_1-m_1}}. \end{aligned}$$

Difficulties in the Evaluation



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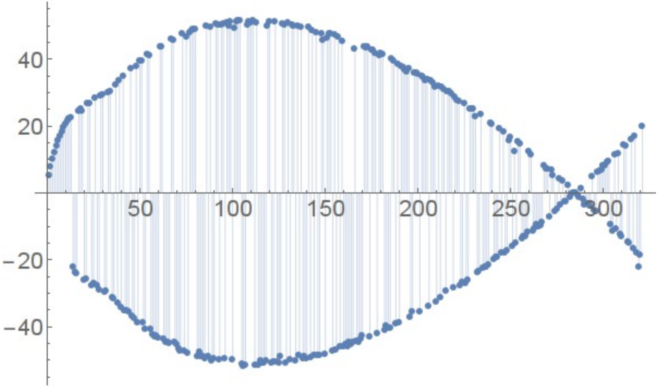

- Accuracy problems with standard floating-point arithmetic.
- Use arbitrary-precision in a computer algebra system.
  But this makes computations even slower.

Holonomic Gradient Method (HGM)

$\longrightarrow$ Methods for evaluating and optimizing certain expressions.
(Nakayama, Nishiyama, Noro, Ohara, Sei, Takayama, Takemura)

Input: $f(x_1, \ldots, x_s)$ holonomic, $(a_1, \ldots, a_s) \in \mathbb{R}^s$
Output: an approximation of $f(a_1, \ldots, a_s)$

1. Determine a holonomic system (set of differential equations) to which f is a solution, and let r be its holonomic rank.
2. Determine a suitable "basis" of derivatives $\mathbf{f} = \left(f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}\right)$ of $f(x_1, \ldots, x_s)$.
3. Convert the holonomic system into a set of Pfaffian systems, i.e., $\frac{\mathrm{d}}{\mathrm{d}x_i}\mathbf{f} = \mathbf{A}_i\mathbf{f}$ for each x_i.
4. Compute $f^{(\mathbf{m}_1)}, \ldots, f^{(\mathbf{m}_r)}$ at a suitably chosen point $(b_1, \ldots, b_s) \in \mathbb{R}^s$, for which this is easy to achieve.
5. Use your favourite numerical integration procedure (e.g., Euler, Runge-Kutta) to obtain $\mathbf{f}(a_1, \ldots, a_s)$.

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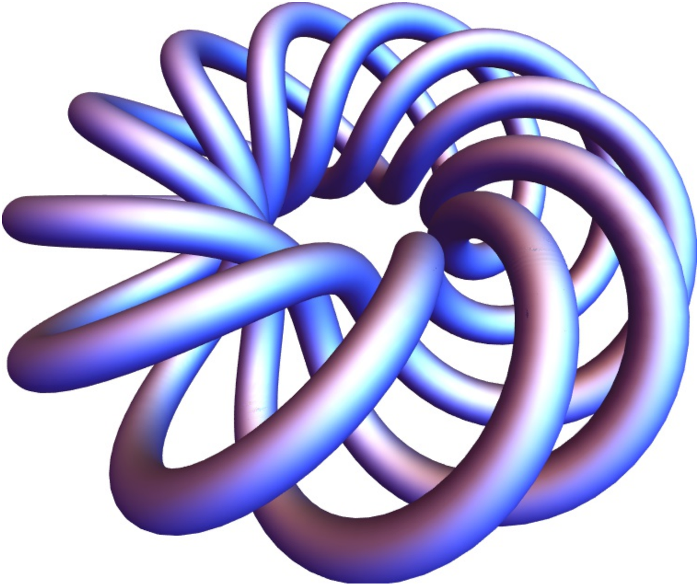
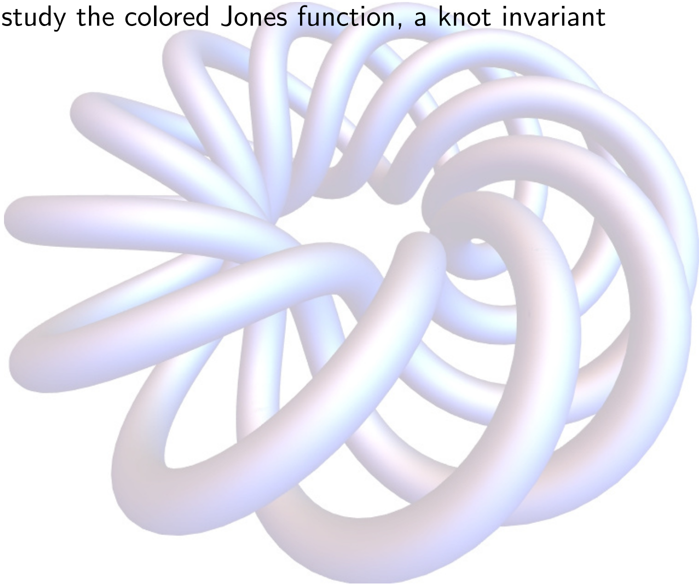
Creative Telescoping in Knot Theory



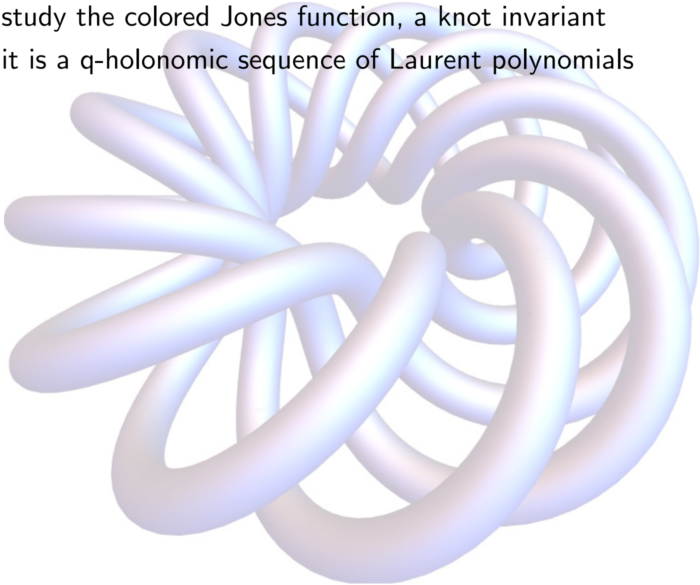
Creative Telescoping in Knot Theory

- study the colored Jones function, a knot invariant



Creative Telescoping in Knot Theory

- study the colored Jones function, a knot invariant
- it is a q-holonomic sequence of Laurent polynomials



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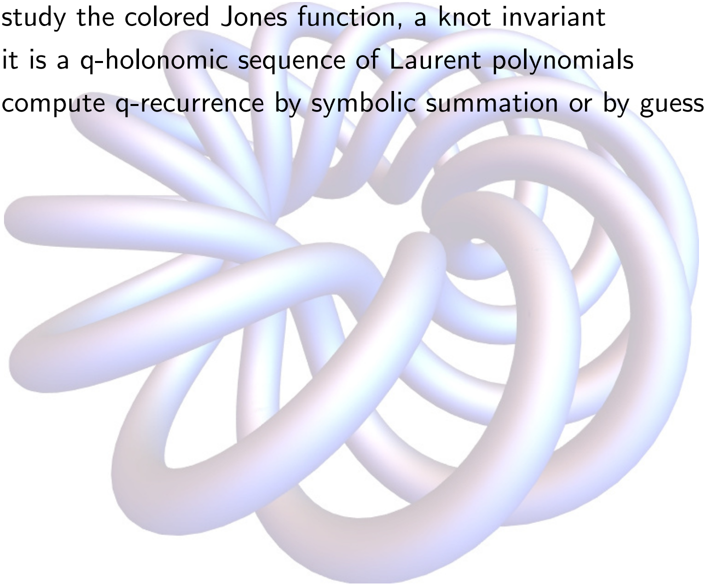

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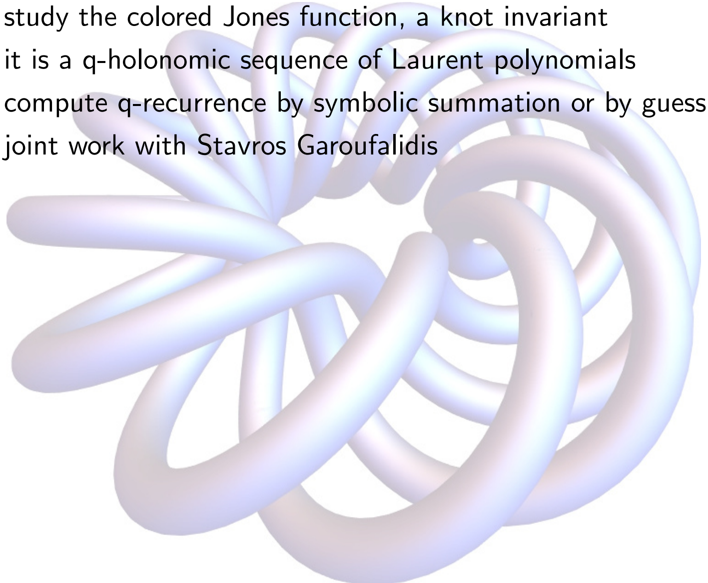

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Example: Colored Jones function of double twist knots $K_{p,p'}$:

$$J_{K_{p,p'},n}(q) = \sum_{k=0}^{n-1} (-1)^k c_{p,k}(q) c_{p',k}(q) q^{-kn-\frac{k(k+3)}{2}} (q^{n-1};q^{-1})_k (q^{n+1};q)_k$$

where the sequence $c_{p,n}(q)$ is defined by

$$c_{p,n}(q) = \sum_{k=0}^{n} (-1)^{k+n} q^{-\frac{k}{2}+\frac{k^2}{2}+\frac{3n}{2}+\frac{n^2}{2}+kp+k^2p} \frac{(1-q^{2k+1})(q;q)_n}{(q;q)_{n-k}(q;q)_{n+k+1}}.$$

Many More Applications of Creative Telescoping

- Hypergeometric expressions for generating functions of walks with small steps in the quarter plane (Alin Bostan, Frédéric Chyzak, Mark van Hoeij, Manuel Kauers, Lucien Pech)

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- Computing efficiently the n-dimensional volume of a compact semi-algebraic set, i.e., the solution set of multivariate polynomial inequalities, up to a prescribed precision 2^{-p} (Pierre Lairez, Marc Mezzarobba, Mohab Safey El Din)

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- Irrationality measures of mathematical constants such as elliptic L-values (Wadim Zudilin)

Plan of the Lecture

by

Shaoshi Chen	Manuel Kauers	Christoph Koutschan

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Fri	creative telescoping for D-finite functions	Example session: HolonomicFunctions	remarks on ongoing research topics