Galois group for large steps walks

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with Pierre Bonnet (Labri-Bordeaux)

Computer Algebra for Functional Equations in Combinatorics and Physics,
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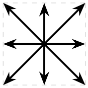

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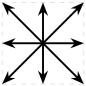

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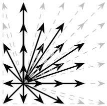

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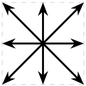

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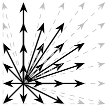

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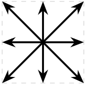

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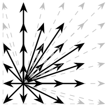

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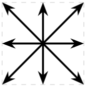

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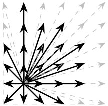

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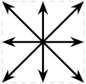

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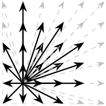

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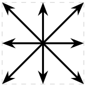

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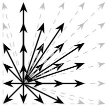

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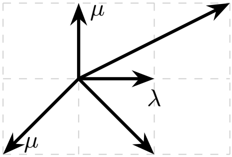

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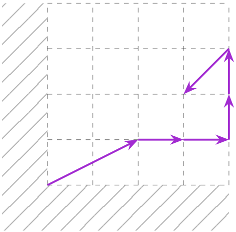

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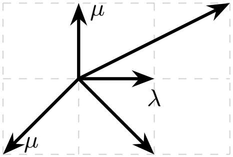

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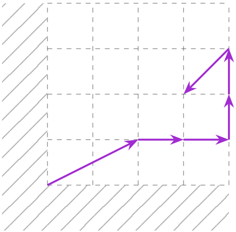

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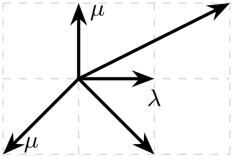

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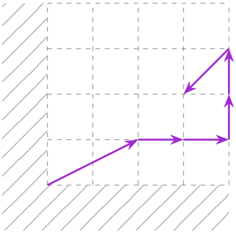

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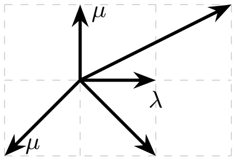

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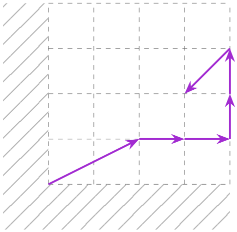

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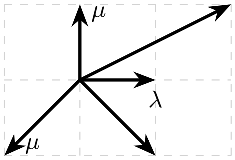

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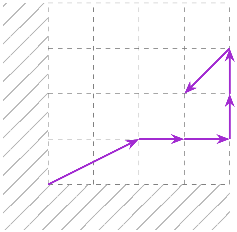

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Prop.: $Q(X, Y, t)$ is X, Y-D-finite over $\mathbb{Q}(X, Y, t) \Leftrightarrow A(X)$ is X-D-finite over $\mathbb{Q}(X, t) \Leftrightarrow B(Y)$ is Y-D-finite over $\mathbb{Q}(Y, t)$.

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Bernardi-Bousquet-Mélou-Raschel (2021)

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Classification results-small steps walks (II)

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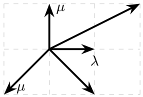
Algebricity proof for the weighted model $\mathcal{G}_3^{\lambda,\mu}$



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Computation of the coordinates of the orbit via resultants.

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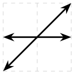

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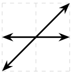

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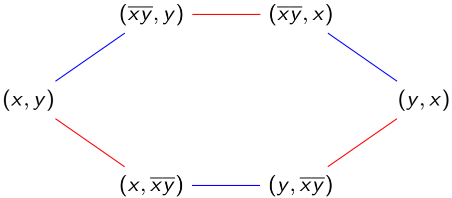

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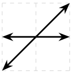

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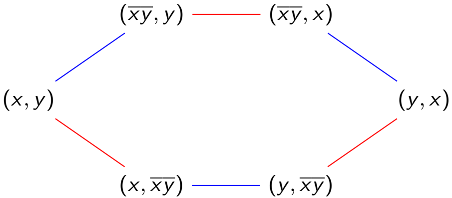

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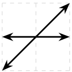

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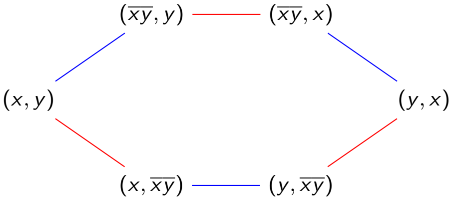

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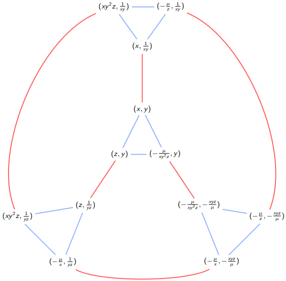

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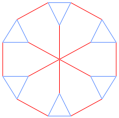

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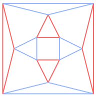

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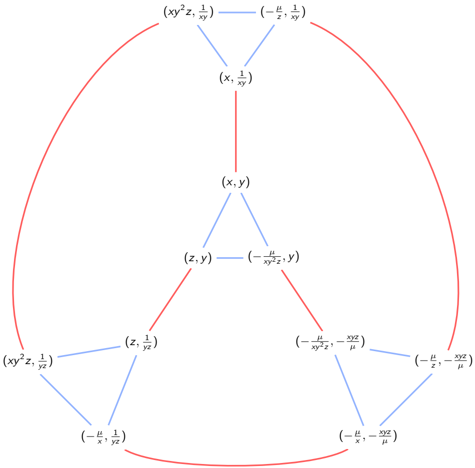
A sample of finite orbits

Orbit-types

Orbit of $\mathcal{G}_3^{\lambda,\mu}$



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The Galois extension and the group of the walk

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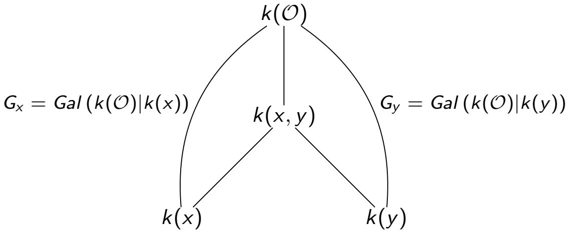

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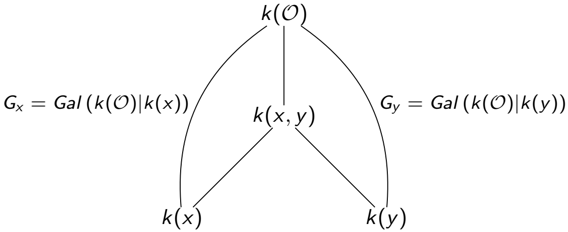

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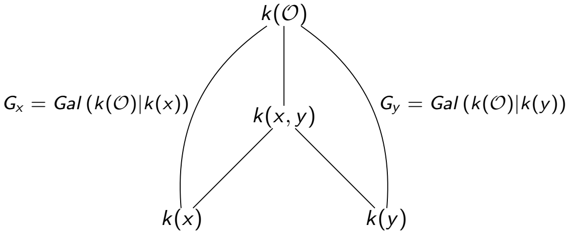

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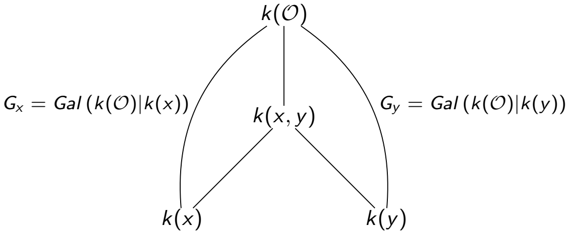

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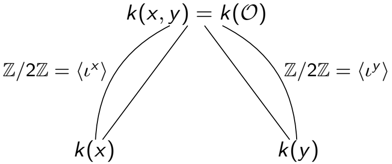

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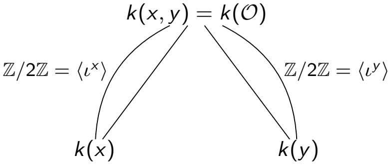

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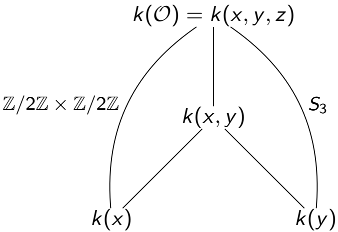
Geometric result $|G|$ finite $\Rightarrow |G| \leq 12$.

The example of $\mathcal{G}_3^{\lambda,\mu}$



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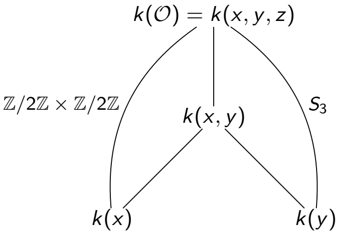

Here $\mathrm{Gal}(k(\mathcal{O}|k(x, y)) = \mathbb{Z}/2\mathbb{Z}$.

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Fried's theorem applied to walks

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The following statements are equivalent

- The orbit is finite
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Invariants for $\mathcal{G}_3^{\lambda,\mu}$

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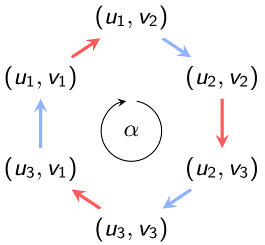

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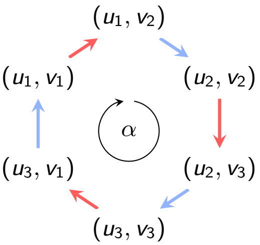

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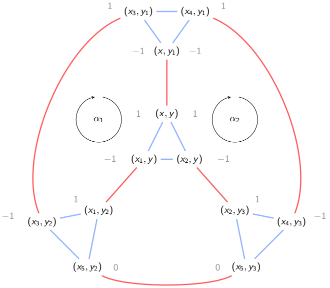


$\alpha = \alpha_1 + \alpha_2$ for $\mathcal{G}_3^{\lambda,\mu}$

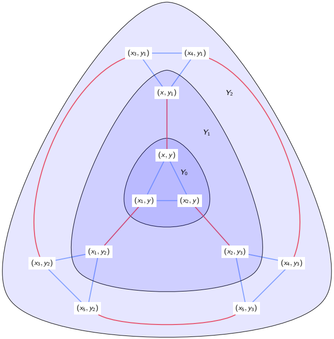
Graph theoretic formulas

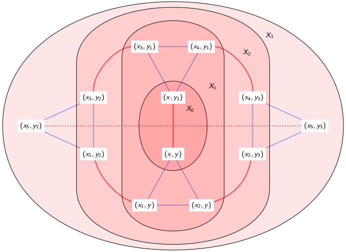
If the graph automorphisms of $\mathcal{O}$ acts transitively on level lines $\mathcal{X}$ and $\mathcal{Y}$ then one can express a decoupling as follows

$$\gamma_x = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\sum_{\substack{j\leq i\\ j \text{ odd}}} \frac{|\mathcal{X}_i|}{|\mathcal{X}_j|}X_j - \frac{|\mathcal{X}_i|}{|\mathcal{X}_{j-1}|}X_{j-1}$$

and

$$\gamma_y = -\frac{1}{|\mathcal{O}|}\sum_{i\geq 0}\sum_{\substack{j\leq i\\ j \text{ odd}}} \frac{|\mathcal{Y}_i|}{|\mathcal{Y}_j|}Y_j - \frac{|\mathcal{Y}_i|}{|\mathcal{Y}_{j-1}|}Y_{j-1} \quad .$$





Algebricity of $\mathcal{G}_3^{\lambda,\mu}$

XY decouples for $\mathcal{G}_3^{\lambda,\mu}$

$$XY = -\frac{3\lambda X^2 t - \mu\lambda t - 4X}{4t(X^2+\mu)} + \frac{-\lambda Y - 4}{4Y} - \frac{K(X,Y,t)}{(X^2+\mu)Yt} \quad \text{decouples for } \mathcal{G}_3^{\lambda,\mu}$$

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Assume that E is irreducible and smooth

Small steps

- The genus of E is 1
- The group of the walk can be seen as a subgroup of $\mathrm{Aut}(E)$.
- Bound for the finite order: 12

Large steps

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- When the group of the walk G is finite,
  G is a subgroup of automorphisms of a Riemann surface, finite cover of E.

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- Find a good notion of "weak invariants" for large steps walks.

Thank you for your attention!