Lecture 5 (shaoshi CHEN) IHP / 2023/12/01

Hypergeometric (first-order)	D-finite (High-order)
Summation: Gosper's Algorithm	Integration: Abramov-Van Hoeij's Algorithm
Creative Telescoping: Zeilberger's Algorithm	Creative Telescoping: Chyzak's Algorithm

1. Abramov - Van Hoeij's Algorithm (1997)

<u>problem</u> Given a D-finite function $f(y)$ of order n, decide whether there exists another D-finite function $g(y)$ of order n s.t.

$$f = D_y(g).$$

Let $L = \sum_{i=0}^{r} a_i D_y^i \in C(y)[D_y]$ be the <u>minimal-order</u> operator such that $L(f) = 0$. We call $L^* = \sum_{i=0}^{r} (-1)^i D_y^i a_i$ the <u>adjoint operator</u> of L.

<u>Remark</u> 1) $(L^*)^* = L$

2) Lagrange's identity: $\forall$ functions u, v, we have

$$u L(v) - v L^*(u) = D_y(M(u,v))$$

where $M \in C(y)[x, x', \dots; y, y', \dots]$ is a differential polynomial of order in x and y at most $\mathrm{ord}(L)$.
This identity can be viewed as a high-order extension of the Leibniz' rule: $u D_y(v) + v D_y(u) = D_y(uv)$
with $L = D_y$, $L^* = -D_y$

<u>Claim</u> If $L^*(T) = 1$ and $L(f) = 0$, then

$f = D_y(T \cdot f)$ for some $T \in C(y)[D_y]$ with $\mathrm{ord}(T) < \mathrm{ord}(L)$.

Abramov - Van Hoeij's Algorithm

$$L = \sum_{i=0}^{n} l_i D_y^i$$

Input: f, a D-finite function defined by the minimal operator $L \in C(y)[D_y]$ with $\deg_{D_y}(L) = n$.

Output: $g = T(f)$ with $T \in C(y)[D_y]$ s.t. $f = D_y(g)$

Otherwise Return <u>NO</u>

<u>Step 1</u> Compute L^*

<u>Step 2</u> Find a rational solution $r(y)$ in $C(y)$ of the equation: $L^*(Z(y)) = 0$

If no such a solution exists, return <u>No</u>

otherwise return $T(f)$ where $rL + D_y T = 1$.

<u>Theorem</u> Let $f(y)$ be a D-finite function of order n. Then TFAE:

1) $f = D_y(g)$ for some D-finite function g of the same order n

2) $f = D_y(T(f))$ for some $T \in C(y)[D_y]$ of order $\leq n-1$

3) $L^*(Z(y)) = 1$ has a rational solution in $C(y)$.

<u>proof</u> 1) $\Rightarrow$ 2) Let P be the minimal operator of order n for g, i.e. $P(g) = 0$. Since $L D_y(g) = 0$, we have $P \mid L D_y$

Note that $C(y)[D_y]$ is a left Euclidean domain. Then $P = \bar{P} D_y + r$ with $r \in C(y)$ and $\operatorname{ord}(\bar{P}) < \operatorname{ord}(P)$. If $r = 0$, then $\bar{P} D_y(g) = \bar{P}(f) = 0$, which contradicts that L is the minimal operator. Then $r \neq 0$, which implies that

$$0 = P(g) = \bar{P} D_y(g) + r \cdot g \Rightarrow g = \tfrac{1}{r}\bar{P}(f) \quad \underline{\text{Take } T = \tfrac{1}{r}\bar{P}}$$

2) $\Rightarrow$ 3) If $f = D_y(T(f))$ for some $T \in C(y)[D_y]$ of order $\leq n-1$

then $L \mid 1 - D_y T \Rightarrow \exists r \in C(y)$ $rL = 1 - D_y T \Rightarrow 1 = rL + D_y T$

$$\Rightarrow 1 = L^* \cdot r + T^*(-D_y) \xrightarrow[D_y(1)=0]{\text{evaluating at } 1} L^*(r) = 1, \text{ i.e. } r \in C(y) \text{ is a rational solution of } L^*(Z(y)) = 1.$$

3) $\Rightarrow$ 1) If $L^*(z(y)) = 1$ has a rational solution $r \in C(y)$, then $L^*(r) = 1$

In the Lagranger identity $uL(v) - vL^*(u) = D_y(M(u,v))$, we can choose $v = f$ and $u = r$. Then $r\underset{0}{\underset{\shortparallel}{L(f)}} - fL^*(r) = -f = D_y(T\cdot f)$

Take $g = -T(f)$. It is clear that g satisfies an operator of order $\leq n$.
If $\mathrm{ord}(g) < n$, then $f = D_y(g)$ will also have order $< n$ $\rightarrow\leftarrow$.

2. <u>Chyzak's Algorithm</u> for Creative telescoping for D-finite function

Let $f(x,y)$ be a D-finite function over $C(x,y)$. Then

$$\dim_{C(x,y)}\left(C(x,y)[D_x,D_y]/I_f \right) < +\infty$$

The algebra $\overset{\mathcal{D} \triangleq}{C(x,y)[D_x,D_y]}$ is quite close to the usual polynomial ring.
and any (left) ideal $I \trianglelefteq \mathcal{D}$ has a Gröbner basis. $G = \{g_1, \cdots, g_r\}$.
and the quotient module $\mathcal{D}/I$ has a finite basis $\{D_x^i D_y^j\}_{(i,j)\in\Omega}$
$|\Omega| < +\infty$ if I is D-finite over $C(x,y)$.

Chyzak's Algorithm:

Input: a basis B for the annihilating ideal I_f of $f(x,y)$

Output: a pair (P, Q) s.t. $P(x, D_x)(f) = D_y(Q(f))$

<u>Step 1</u> Compute a Gröbner basis G of B. And get the finite basis $\{D_x^i D_y^j\}_{(i,j)\in\Omega}$ with $|\Omega| < +\infty$. of $\mathcal{D}/I_f$.

<u>Step 2</u> For $r = 0, 1, \cdots$

2.1) Make an ansatz: $P = \sum_{i=0}^{r} p_i D_x^i$ and $Q = \sum_{(i,j)\in\Omega} q_{i,j} D_x^i D_y^j$ and rewrite $D_y Q - P$ in the basis of I_f by reduction w.r.t. G

Abramov's Algorithm or Barkatou's Algorithm. $\longleftarrow$ 2.2) Solve the corresponding system of first order linear differential equations for all solutions $p_i \in C(x)$ and $q_{ij} \in C(x,y)$

2.3) if solvable, return (P, Q); otherwise loop.

Examples: <u>see Koutschan's Lecture</u>

Reference 1) S.A. Abramov, M. van Hoeij. A method for the integration of solutions of Ore Equations. Proceedings of ISSAC'97, 172-175. 1997

2) S.A. Abramov, M. van Hoeij. Integration of Solutions of Linear Functional Equations. Integral Transformation and special Functions. Vol.8. No.1-2 pp.3-12, 1999

3) F. Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Mathematics. 217: 115-134, 2000.