Lecture 4 (shaoshi CHEN) IHP/2023/11/30

D-finite function in several variables

The ring of Linear differential operators in several variables

$$K = \mathbb{C}(x_1, \cdots x_n)$$

$$\delta_i = \frac{\partial}{\partial x_i} : K \to K \quad \text{paritial derivation in } x_i$$

$$\mathcal{D} \triangleq K[D_{x_1}, \cdots, D_{x_n}] = \Big\{ \sum_{1 \le i_1, \cdots, i_n \le d} f_{i_1, \cdots, i_n} D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} \,\Big|\, f_{i_1, \cdots, i_n} \in K \Big\}$$

$\forall f \in K$, we have

$$D_{x_i} f = f \cdot D_{x_i} + \delta_i(f)$$

$$D_{x_i} \cdot D_{x_j} = D_{x_j} \cdot D_{x_i} \quad \forall\ 1 \le i < j \le n$$

Let $M = \mathbb{C}[[x_1, \cdots, x_n]]$. and $L \in \mathcal{D}$. write

$$L = \sum_{i_1, \cdots, i_n} l_{i_1, \cdots, i_n} D_{x_1}^{i_1} \cdots D_{x_n}^{i_n}$$

$$L \cdot f = \sum_{i_1, \cdots, i_n} l_{i_1, \cdots, i_n} \cdot \frac{\partial^{i_1}}{\partial x_1} \cdots \frac{\partial^{i_n}}{\partial x_n} (f).$$

Then M becomes a $\mathcal{D}$-module.

<u>DEF</u> (D-finite power series)

Let $f \in \mathbb{C}[[x_1, \cdots, x_n]]$. Define

$$I_f = \{ L \in K[D_{x_1}, \cdots, D_{x_n}] \mid L \cdot f = 0 \}$$

I_f is a left ideal of $K[D_{x_1}, \cdots, D_{x_n}]$. and

$$\mathcal{D} \cdot f \cong \mathcal{D}/I_f$$

f is said to be D-finite over K if

$$\dim_K \left(K[D_{x_1}, \dots, D_{x_n}] / I_f \right) < +\infty$$

A sequence $T : \mathbb{N}^n \to K$ is said to be D-finite if

$$f(x_1, \dots, x_n) = \sum_{i_1, \dots, i_n} T(i_1, \dots, i_n)\, x_1^{i_1} \cdots x_n^{i_n}$$

is D-finite over K.

<u>Lemma</u> A series $f \in C[[x_1, \dots, x_n]]$ is D-finite

$\Longleftrightarrow$ $\forall\ i = 1, \dots, n$, $I_f \cap K[D_{x_i}] \neq \{0\}$.

<u>proof.</u> "$\Rightarrow$" If f is D-finite, then

$$d = \dim_K (D \cdot f) < +\infty$$

Thus for any $i = 1, 2, \dots, n$, the elements

$$f,\ D_{x_i} \cdot f,\ D_{x_i}^2 \cdot f,\ \dots,\ D_{x_i}^d \cdot f$$

are linearly dependent over K, which implies

$$I_f \cap K[D_{x_i}] \neq \{0\}.$$

"$\Leftarrow$" If $I_f \cap K[D_{x_i}] \neq \{0\}$ for any $i = 1, 2, \dots, n$,

then $\exists\ L_i \in I_f \setminus \{0\}$ s.t. $L_i = \sum_{j=0}^{d_i} \ell_{ij} D_{x_i}^j$.

$\Rightarrow$ $D_{x_1}^{i_1} \cdots D_{x_n}^{i_n} \cdot f$ can be rewritten into a K-linear combination of terms $D_{x_1}^{j_1} \cdots D_{x_n}^{j_n} \cdot f$ with $0 \leq j_s < d_i$

$\Rightarrow$ $\dim_K (D \cdot f) < +\infty$

Theorem Let $f, g \in \mathbb{C}[[x_1, \ldots, x_n]]$ be D-finite over K. Then

1) $f+g$ is D-finite
2) $L \cdot f$ is D-finite for any $L \in \mathcal{D}$
3) $f \cdot g$ is D-finite
4) If $\alpha_i \in \mathbb{C}[[y_1, \ldots y_m]]$ be algebraic for any $i=1, \ldots, n$ and $f(\alpha_1, \ldots, \alpha_n)$ is well-defined, then $f(\alpha_1, \ldots, \alpha_n)$ is D-finite over $\mathbb{C}(y_1, \ldots y_n)$

Proof. 1)

$$\dim_K (\mathcal{D} \cdot f) < +\infty$$
$$\dim_K (\mathcal{D} \cdot g) < +\infty$$
$$\Rightarrow \dim_K (\mathcal{D} \cdot f + \mathcal{D} \cdot g) < +\infty$$
$$\Rightarrow \dim_K (\mathcal{D} \cdot (f+g)) < +\infty \text{ since } \mathcal{D}(f+g) \subseteq \mathcal{D} \cdot f + \mathcal{D} \cdot g$$

2) Since $L \cdot f \in \mathcal{D} \cdot f$ and $P \cdot (L \cdot f) \in \mathcal{D} \cdot f \quad \forall P \in \mathcal{D}$

we have $\mathcal{D} \cdot (L \cdot f) \subseteq \mathcal{D} \cdot f$

$$\Rightarrow \dim_K (\mathcal{D}(L \cdot f)) \leq \dim_K (\mathcal{D} \cdot f) < +\infty$$

3) Since $I_f \cap K[D_{x_i}] \neq \{0\}$, $I_g \cap K[D_{x_i}] \neq \{0\}$

Using the closure property in the univariate case, we get

$I_{fg} \cap K[D_{x_i}] \neq \{0\} \Rightarrow fg$ is D-finite.

4) Exercise :-)

DEF (Holonomic functions)

Let $A = \mathbb{C}[x_1, \dots, x_n][D_{x_1}, \dots, D_{x_n}]$ and f be an element of an A-module M.

Let $J \trianglelefteq A$ be a left ideal of A. I is called holonomic if for every subset $U \subseteq \{x_1, \dots, x_n, D_{x_1}, \dots, D_{x_n}\}$ with $|U| = n+1$ we have $J \cap \mathbb{C}[U] \neq \{0\}$

f is called holonomic if $J_f = \{L \in A \mid L \cdot f = 0\}$ is holonomic.

Theorem Let f be an element of a $\mathcal{D}$-module which can be viewed as an A-module. Then

$$f \text{ is holonomic} \iff f \text{ is D-finite}$$

Proof "$\Rightarrow$" $\forall i = 1, \dots, n$, we let

$$U_i = \{x_1, \dots, x_n, D_{x_i}\} \qquad |U_i| = n+1$$

Then $J_f \cap \mathbb{C}[U_i] \neq \{0\}$

$$\Rightarrow I_f \cap K[D_{x_i}] \neq \{0\} \quad \forall i = 1, \dots, n$$

$$\Rightarrow f \text{ is D-finite}$$

"$\Leftarrow$" If f is D-finite, then

$$r = \dim_K (\mathcal{D}\cdot f) < +\infty$$

Let $\{\overset{:=f}{b_1}, \dots, b_r\}$ be a basis of the vector space $\mathcal{D}\cdot f$. Then $\forall g \in \mathcal{D}\cdot f$

$$g = g_1 b_1 + \dots + g_r b_r = \vec{g} \cdot \vec{b}^T \quad \text{with } g_i \in K$$

We write for $i = 1, \dots, n$,

$$D_{x_i} \cdot g = A_i \cdot \vec{b}^T + \delta_i(\vec{g}) \cdot \vec{b}^T$$

Where $A_i \in K^{r\times r}$.

Let $q \in C[x_1, \dots, x_n]$ be a common denominator of all entries of A_i $(i = 1, \dots, n)$, and $d \geq 1$ be such that the total degree of q as well as the entries of the $q A_i$ $(i = 1, \dots, n)$ are less than d.

$$f = (1, 0, \dots, 0)\begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}$$

For every $k \in \mathbb{N}$, we have if $i_1 + \dots + i_n + j_1 + \dots + j_n \leq k$ then

$$x_1^{i_1} \cdots x_n^{i_n} D_{x_1}^{j_1} \cdots D_{x_n}^{j_n} \cdot f \in \overset{V_k \triangleq}{\left\{ \frac{1}{q^k}(P_1 b_1 + \dots + P_r b_r) \;\middle|\; \begin{array}{l} P_i \text{ are polynomials of} \\ \text{total degree} \leq kd \end{array} \right\}}$$

Let $W_{U,k}$ be the vector space over C generated by monimials of variables in $U \subseteq \{x_1, \dots, x_n, D_{x_1}, \dots, D_{x_n}\}$ with $|U| = n+1$

$$\phi_k: W_{U,k} \longrightarrow V_k \qquad \mathbb{C}\text{-linear map.}$$
$$L \longmapsto L\cdot f$$

$$\dim_{\mathbb{C}}(W_{U,k}) = \binom{n+l+k}{k} = \binom{n+l+k}{n+l} \sim O(k^{n+l})$$

and $\dim_{\mathbb{C}}(V_k) \leq r\cdot \binom{n+kd}{kd} = r\cdot\binom{n+kd}{n}$
$$\sim O(k^n)$$

Then for large enough k, we have $\ker(\phi_k)$ is nontrivial, which implies that

$$J_f \cap \mathbb{C}[U] \neq \{0\}$$

Then f is holonomic.

<u>Corollary</u> Let $f(x,y)$ be D-finite over $\mathbb{C}(x,y)$. Then there exists $P \in \mathbb{C}(x)[D_x]$ and $Q \in \mathbb{C}(x,y)[D_x, D_y]$ s.t.

$$P\cdot f = D_y(Q\cdot f).$$

<u>proof</u> Since f is D-finite, there exist $L \in \mathbb{C}[x,D_x,D_y]$ s.t. $L\cdot f = 0$. Now we write

$$L = D_y^m(\overline{P}(x,D_x) + D_y\overline{Q})$$

By Wegschaider's trick, $y^m D_y^m = D_y R + m!$

$$\Rightarrow 0 = y^m L\cdot f = (D_y R + m!)(\overline{P}(x,D_x) + D_y\overline{Q}) = (P(x,D_x) + D_y Q)\cdot f$$

$\boxed{?}$