Lecture 3 (Shaoshi CHEN) IHP/2023/11/29

Hypergeometric Summation: Gosper's Algorithm

A sequence $H(n)$ is hypergeometric over $K(n)$ if

$$\frac{H(n+1)}{H(n)} \in K(n).$$

<u>Examples</u>: $\frac{1}{n}$, n^2+1, 2^n, $n!$, $\Gamma(\alpha+n)$, ...

Two hypergeometric terms $H_1(n)$, $H_2(n)$ are said to be <u>similar</u> if $H_1/H_2 \in K(n)$, denoted by $H_1 \sim H_2$.

<u>problem</u> Given a hypergeometric $H(n)$, decide whether there exists another hypergeometric $G(n)$ s.t.

$$H(n) = G(n+1) - G(n) \triangleq \Delta_n(G(n)).$$

If such a $G(n)$ exists, we say that $H(n)$ is <u>hypergeometric summable</u>

<u>Remark</u> If $H(n) = \Delta_n(G(n))$ for some hypergeom. $G(n)$, then $G(n+1) = g(n)\, G(n)$ for some rational $g \in K(n)$. Thus $H(n) = (g(n) - 1)\, G(n)$ which means $H(n)$ and $G(n)$ are <u>similar</u>.

To decide the existence of $G(n)$, we make an ansatz

$G(n) = y(n) \cdot H(n)$ Assume $\frac{H(n+1)}{H(n)} = f(n) \in K(n)$

$H(n) = \Delta_n(G(n)) \Leftrightarrow H(n) = y(n+1)H(n+1) - y(n)H(n)$

$= (y(n+1)f(n) - y(n))H(n)$

$\Rightarrow_{(*)} 1 = f(n)\, y(n+1) - y(n)$ has a rational solution in $K(n)$

Gosper (1978) gave a nice way to guess the denominator of $y(n)$ and a degree bound for the numerator of $y(n)$.

<u>DEF</u> (Gosper form) Let $f \in K(n)$. We call the triple $(p, q, r) \in K[n]^3$ a <u>Gosper form</u> of f if

$$f = \frac{p(n+1)}{p(n)} \frac{q(n)}{r(n)}$$

where $\gcd(q(n), r(n+i)) = 1 \;\; \forall i \in \mathbb{N}$ (*)

<u>Theorem</u> Any rational function has a Gosper form.

<u>proof</u> Let $f = \frac{a}{b}$ with $\gcd(a,b)=1$. If

$g = \gcd(a(n), b(n+j_0)) \neq 1$ for some $j_0 \in \mathbb{N}$

then $g(n) \mid a(n)$ and $g(n) \mid b(n+j_0)$

② $\Rightarrow g(n-j_0) \mid b(n)$

Let $a = g(n)\bar{a}$ $\quad b = g(n-j_0)\bar{b}$

$$\frac{a(n)}{b(n)} = \frac{g(n)\bar{a}}{g(n-j_0)\bar{b}} = \frac{g(n)}{g(n-1)}\frac{g(n-1)}{g(n-2)}\cdots\frac{g(n-j_0+1)}{g(n-j_0)}\frac{\bar{a}}{\bar{b}}$$

$$= \frac{P_1(n+1)}{P_1(n)}\frac{\bar{a}}{\bar{b}} \quad \text{with } P_1 = g(n-1)\cdots g(n-j_0)$$

We can iterate the process for $\bar{a}/\bar{b}$ until it satisfies the GCD condition $(*)$.

• Equation to solve:

$$\boxed{1 = f(n)\,y(n+1) - y(n)}$$

Algorithm (Gosper 1978)

Input: A hypergeometric $H(n)$

Output: No or $\underset{\in K(n)}{y(n)}$ s.t. $H(n) = \Delta_n(y(n)H(n))$

Step 1: Compute a Gosper form of f

$$f = \frac{p(n+1)}{p(n)}\frac{q(n)}{r(n)}$$

Step 2: Make an ansatz: $y(n) = \frac{r(n-1)}{p(n)} z(n)$

Then $1 = f(n)\,y(n+1) - y(n) \Leftrightarrow 1 = \frac{p(n+1)}{p(n)}\frac{q(n)}{r(n)}\frac{r(n)}{p(n+1)} z(n+1) - \frac{r(n-1)}{p(n)} z(n)$

$$\Leftrightarrow \quad p(n) = q(n)\,z(n+1) - r(n-1)\,z(n)$$

$(**)$ Gosper's equation

Gosper's Key Lemma

The equation $1 = f(n)y(n+1) - y(n)$ has a rational solution $\Longleftrightarrow$ the Gosper equation

$$(**)\quad p(n) = q(n)z(n+1) - r(n-1)z(n)$$

has only a <u>polynomial solution</u>

Proof "$\Leftarrow$" clear

"$\Rightarrow$" If $1 = f(n)y(n+1) - y(n)$ has a rational solution then $(**)$ also has a rational solution, say

$$z(n) = \frac{a(n)}{b(n)} \in K(n) \text{ with } \gcd(a,b)=1$$

$$\Rightarrow\quad p(n) = q(n)\frac{a(n+1)}{b(n+1)} - r(n-1)\frac{a(n)}{b(n)}$$

Claim $b(n) \in K^*$. otherwise, assume that $b \notin K^*$

Then $p(n)b(n)b(n+1) = q(n)a(n+1)b(n) - r(n-1)a(n)b(n+1)$

Take any irreducible polynomial $u(n)$ s.t. $u(n) \mid b(n)$

Then $\exists$ maximal $j \in \mathbb{N}$ s.t.

$$u(n+j) \mid b(n) \quad \text{but} \quad u(n+j+1) \nmid b(n)$$

$$\gcd(a(n), b(n)) = 1, \Rightarrow \begin{matrix} u(n) \mid b(n) \\ u(n+j) \mid b(n) \end{matrix} \Rightarrow \begin{matrix} u(n) \nmid a(n) \\ u(n+j) \nmid a(n) \end{matrix}$$

$$u(n+j) \mid b(n) \qquad \Rightarrow u(n+j+1) \nmid a(n+1)$$

$$\Rightarrow u(n+j+1) \mid b(n+1) \Rightarrow u(n+j+1) \mid q(n)a(n+1)b(n)$$

$$\Rightarrow \boxed{u(n+j+1) \mid q(n)}$$

On the other hand, $\exists$ maximal $i \in \mathbb{N}$ s.t.

$$u(n-i) \mid b(n) \quad \text{but} \quad u(n-i-1) \nmid b(n)$$

$$\Rightarrow u(n-i) \nmid b(n+1)$$

$$u(n-i) \mid b(n) \Rightarrow u(n-i) \mid r(n-1)\, a(n)\, b(n+1)$$

$$\Rightarrow \boxed{u(n-i) \mid r(n-1)} \quad \left(\underset{\text{since } \gcd(a(n), b(n)) = 1}{u(n-i) \nmid a(n)} \right)$$

So we now get

$$\left\{ \begin{array}{l} u(n+j+1) \mid q(n) \\ \\ u(n-i+1) \mid r(n) \end{array} \right. \Rightarrow \gcd\big(q(n), r(n+i+j)\big) \neq 1$$

$i+j \in \mathbb{N}$

$\rightarrow\!\cdot\!\leftarrow$ Contradiction!!!

Step 3 Find a polynomial solution of

$$p(n) = q(n)\, z(n+1) - r(n-1)\, z(n)$$

$$\Leftrightarrow \quad p(n) = q(n)\big(z(n+1) - z(n)\big) + \big(q(n) - r(n-1)\big) z(n)$$

$$= q(n)\, \Delta\big(z(n)\big) + \big(q(n) - r(n-1)\big) z(n)$$

Now we can estimate the degree bound of $z(n)$ and then apply the method of undetermined coefficients we finally leads to solving a linear system over $\mathbb{K}$.

Estimating the degree bound for the equation

$$a(n) = b(n)\Delta(Z(n)) + C(n)Z(n) \qquad (*)$$

$$Z = Z_d x^d + Z_{d-1} x^{d-1} + \cdots + Z_0$$

$$\Delta(Z) = d Z_d x^{d-1} + \text{lower terms}$$

Case 1 $\quad \deg_x b \leq \deg_x C$

then $\quad d = \deg_x a - \deg_x C$

Case 2 $\quad \deg_x b > \deg_x C + 1$

Then $\quad d = \deg_x a - \deg_x b + 1$

Case 3 $\quad \deg_x b = \deg_x C + 1 = \rho$

$$b = b_\rho x^\rho + \cdots + b_0 \qquad C = C_{\rho-1} x^{\rho-1} + \cdots + C_0$$

$$b\Delta(Z) + CZ = (b_\rho d Z_d + C_{\rho-1} Z_d)\, x^{\rho+d-1} + \text{lower terms}$$

Then either $\quad d = -\frac{C_{\rho-1}}{b_\rho}$ or $\rho + d - 1 = \deg_x a$

If $-\frac{C_{\rho-1}}{b_\rho} \notin \mathbb{N}$, then $d = \deg_x a - \rho - 1$

Otherwise $\quad d = \max\left\{-\frac{C_{\rho-1}}{b_\rho},\ \deg_x a - \rho - 1\right\}$

Example $H = \frac{\binom{m}{k}}{\binom{n}{k}}$

Claim $H = \Delta_k(G)$

$$G = \frac{n-k+1}{m-n+1} \cdot H(k)$$

We now compute G by Gosper's Algorithm.

Step 1 $\frac{H(k+1)}{H(k)} = \frac{m-k}{n-k} = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k)}$

$$\Rightarrow \quad p(k) = 1 \qquad q = m-k \qquad r = n-k$$

Step 2 Finding a polynomial solution of the equation

$$1 = (m-k)\, z(k+1) - (n-k+1)\, z(k)$$
$$= (m-k)\, \Delta_k(z(k)) + (m-n-1)\, z(k)$$
$$\Rightarrow \deg_k(z) = 0 \Rightarrow z(k) = \frac{1}{m-n-1}$$
$$\Rightarrow y(k) = \frac{r(k-1)}{p(k)} z(k) = \frac{n-k+1}{m-n-1}$$

Then $\sum_{k=0}^{m} H(k) = \frac{n+1}{n-m+1}$