Lecture 2 (Shaoshi CHEN) IHP / 2023/11/28

Summation problems

Let R be a ring and $\sigma: R \to R$ be an automorphism. The pair (R, σ) is called a <u>difference ring</u> and $r \in R$ is called a <u>constant</u> if $\sigma(r) = r$. Denote by C_R the set $\{r \in R \mid \sigma(r) = r\}$, which is a subring of R.

<u>Problem</u> (Indefinite summation problem)

Given $f \in R$, which is a difference ring with automorphism σ, decide whether there exists $g \in R$ s.t.

$$f = \sigma(g) - g \triangleq \Delta_\sigma(g)$$

If such a g exists, we say that f is σ-summable in R.

Let $f \in R$. Denote

$$f^{\underline{m}} = f \sigma^{-1}(f) \cdots \sigma^{-m+1}(f) \quad \text{Falling } \sigma\text{-factorial}$$

$$f^{\overline{m}} = f \sigma(f) \cdots \sigma^{m-1}(f) \quad \text{Rising } \sigma\text{-factorial}$$

In particular, let $R = K[x]$ with $\sigma(f(x)) = f(x+1)$

Then

$$x^{\underline{m}} = x(x-1)\cdots(x-m+1)$$

$$\begin{aligned}
\Delta_\sigma(x^{\underline{m}}) &= \sigma(x^{\underline{m}}) - x^{\underline{m}} \\
&= (x+1)\underbrace{x(x-1)\cdots(x-m+2)} - \underbrace{x(x-1)\cdots}(x-m+1) \\
&= x(x-1)\cdots(x-m+2)(x+1-x+m-1) \\
&= m x^{\underline{m-1}} \quad ①
\end{aligned}$$

[Think about the formula: $(x^m)' = m x^{m-1}$]

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(2.1) Summation of polynomials

Note that $\deg(x^{\underline{m}}) = m$, then

$1, x^{\underline{1}}, x^{\underline{2}}, \ldots, x^{\underline{m}}, \ldots$

form a basis of $K[x]$, as a vector space over K.

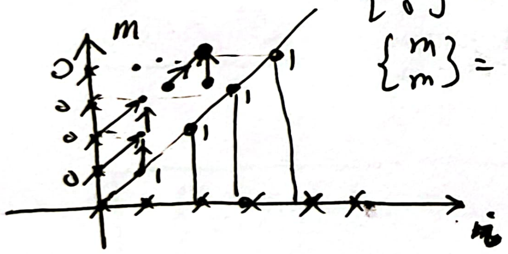
Then
$$x^m = \sum_{i=0}^{m} \left\{ {m \atop i} \right\} x^{\underline{i}}$$

Where $\left\{ {m \atop i} \right\}$ is called the Stirling numbers of the second kind which counts the number of partitions of the set $\{1, 2, \ldots, n\}$ into i nonempty subsets.

FACT $\left\{ {m \atop i} \right\} = 0$ if $i > m$, $\left\{ {m \atop 0} \right\} = 0$ for $m \geq 1$

$\left\{ {0 \atop 0} \right\} = 1$ and $\left\{ {m \atop i} \right\} = \left\{ {m-1 \atop i-1} \right\} + i \left\{ {m-1 \atop i} \right\}$ (*)

$\left\{ {m \atop m} \right\} = 1$ $(m \geq i > 0)$



Proof of (*)

1. $x^m = x \cdot x^{m-1} = x \cdot \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} x^{\underline{i}} = \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} x \cdot x^{\underline{i}}$

$$= \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} (x-i) x^{\underline{i}} + \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} i x^{\underline{i}}$$

$$= \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} x^{\underline{i+1}} + \sum_{0 \leq i < m} \left\{ {m-1 \atop i} \right\} i x^{\underline{i}}$$

$$= x^{\underline{m}} + \sum_{1 \leq i < m} \left(\left\{ {m-1 \atop i-1} \right\} + i \left\{ {m-1 \atop i} \right\} \right) x^{\underline{i}}$$

$$\Rightarrow \left\{ {m \atop i} \right\} = \left\{ {m-1 \atop i-1} \right\} + i \left\{ {m-1 \atop i} \right\}.$$

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Theorem For any $f = \sum_{i=0}^{d} a_i x^i$, $a_i \in K$

Then $\exists\, g \in K[x]$ s.t. $f = \sigma(g) - g$, where

$= g(x+1) - g(x)$

$$g = \sum_{0 \le j \le i \le d} a_i {i \brace j} \frac{x^{\underline{j+1}}}{j+1}$$

proof Since $\Delta(x^{\underline{m}}) = m x^{\underline{m-1}}$

$$\Rightarrow x^{\underline{m}} = \Delta\left(\frac{1}{m+1} x^{\underline{m+1}}\right) \quad m \ge 0$$

$$f = \sum_{i=0}^{d} a_i x^i = \sum_{i=0}^{d} a_i \left(\sum_{j=0}^{i} {i \brace j} x^{\underline{j}} \right)$$

$$= \sum_{i=0}^{d} \sum_{j=0}^{i} a_i {i \brace j} \Delta\left(\frac{1}{j+1} x^{\underline{j+1}}\right)$$

$$= \Delta\left(\sum_{i=0}^{d} \sum_{j=0}^{i} a_i {i \brace j} \frac{1}{j+1} x^{\underline{j+1}} \right)$$

Example $\sum_{k=0}^{n-1} k^2 = \sum_{k=0}^{n-1} (k^{\underline{2}} + k^{\underline{1}})$ ${2 \brace 0} = 0$, ${2 \brace 1} = 1$, ${2 \brace 2} = 1$

$= \sum_{k=0}^{n-1} \Delta(k^{\underline{3}}/3 + k^{\underline{2}}/2)$

$$\boxed{f = \Delta(g) = g(x+1) - g(x) \Rightarrow \sum_{k=0}^{n-1} f(k) = g(n) - g(0)}$$

$$= \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} = \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2}$$

$$= \frac{n(n-1)(2n-1)}{6}$$

$$\sum_{0 \le k < n} k^3 = \sum_{0 \le k < n} (k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}) = \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}$$

${3 \brace 0} = 0$ ${3 \brace 2} = 3$

$$= \frac{n^2(n-1)^2}{4}.$$

${3 \brace 1} = 1$ ${3 \brace 3} = 1$

2.2 Summation of Rational Functions

Let $p \in k[x] \setminus k$, we call the integer

$$\max \{ i \in \mathbb{Z} \mid \gcd(p, \sigma^i(p)) \neq 1 \}$$

the dispersion of p w.r.t. σ, where $\sigma(p(x)) = p(x+1)$

denoted by $\mathrm{disp}_\sigma(p)$.

Example $\quad p = x(x+3)(x-\sqrt{2})(x+\sqrt{2})$

Then $\mathrm{disp}_\sigma(p) = 3$

$p \in k[x] \setminus k$ is called a shift-free polynomial if $\mathrm{disp}_\sigma(p) = 0$

Lemma Let $f = \frac{a}{b} \in k(x)$ with $b \notin k$. Then

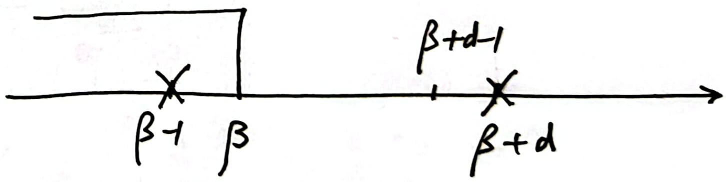
$$\Delta(f) = \frac{P}{Q} \quad \text{with } \gcd(P, Q) = 1$$

and $\mathrm{disp}_\sigma(Q) = \mathrm{disp}_\sigma(b) + 1$

In particular If $\mathrm{disp}_\sigma(b) = 0$ and $\gcd(a,b) = 1$, then ($f = \Delta(g)$ for some $g \in k(x)$ $\Leftrightarrow$ $a = 0$)

Proof Let $d = \mathrm{disp}_\sigma(b)$

$\exists \beta \in \bar{k}$ s.t. $b(\beta) = b(\beta + d) = 0$



$$\Delta(f) = \frac{a(x+1)}{b(x+1)} - \frac{a(x)}{b(x)} = \frac{P(x)}{Q(x)}$$

$$= \frac{a(x+1)b(x) - a(x)b(x+1)}{b(x)\,b(x+1)}$$

Claim (see picture)

$a(x+1)b(x) - a(x)b(x+1)$ are not zero at $\beta-1$ and $\beta+d$ but $b(x)b(x+1)$ vanishes at $\beta-1$ and $\beta+d$

claim $\Rightarrow \mathrm{disp}_\sigma(Q) \geq d+1$

but $\mathrm{disp}_\sigma(b(x)b(x+1)) \leq d+1$

$\Rightarrow \mathrm{disp}_\sigma(Q) = d+1$.

Corollary Let $f = \frac{a}{b}$ be such that $a, b \in K[x]$ with

(*) (1) $\gcd(a,b) = 1$, (2) $\deg(a) < \deg(b)$, (3) b shift-free

Then $f = \Delta(g)$ for some $g \in K(x)$

$\iff a = 0$

Decomposition problem: Given $f \in K(x)$, compute $g, r \in K(x)$ s.t. $f = \sigma(g) - g + r$

where $r = \frac{a}{b}$ with the conditions in (*)

Abramov Reduction $\frac{a}{b^m} = \left(\frac{u}{b^{m-1}}\right)' + \frac{v}{b}$

$$\frac{a}{\sigma^m(b)} = \frac{a}{\sigma^m(b)} - \frac{\sigma^{-1}(a)}{\sigma^{m-1}(b)} + \frac{\sigma^{-1}(a)}{\sigma^{m-1}(b)}$$

$$= \Delta\left(\frac{\sigma^{-1}(a)}{\sigma^{m-1}(b)}\right) + \frac{\sigma^{-1}(a)}{\sigma^{m-1}(b)}$$

$$= \Delta\left(\frac{\sigma^{-1}(a)}{\sigma^{m-1}(b)} + \frac{\sigma^{-2}(a)}{\sigma^{m-2}(b)} + \cdots + \frac{\sigma^{-(m-1)}(a)}{b}\right) + \frac{\sigma^{-m}(a)}{b}$$

$$= \Delta(g) + \frac{\sigma^{-m}(a)}{b}$$

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{l=0}^{t_{ij}} \frac{a_{ijl}}{\sigma^l(b_i)^j}$$

where for $i \neq j$ $\sigma^k(b_i) \neq b_j$ for any $k \in \mathbb{Z}$ and b_i's are irreducible

$$= \Delta(g) + \sum_i \sum_j \frac{\hat{a}_{ij}}{b_i^j}$$

$$= \Delta(g) + \frac{a}{b}$$

We can avoid Irreducible Factorization of $\mathrm{den}(f)$ by using GFF (see Paule's JSC 1995)

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Hypergeometric Summation: before Gosper's Algorithm.

1. Binomial coefficients and Combinatorial Identities

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad n! = 1\times 2\times 3\times\cdots\times n$$

▶ Choose k apples (different size) from n apples (different size)

Basic (relations / properties / identities) of binomial coefficients:

① $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ $\quad k\neq 0 \qquad \binom{n}{k} = \binom{n}{n-k}$

Pascal - Yang

② $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad \binom{n}{k} = (-1)^k\binom{k-n-1}{k}$

$$\Rightarrow \text{(2.1)}\ \binom{n+1}{m+1} = \binom{0}{m} + \binom{1}{m} + \cdots + \binom{n}{m} = \sum_{k=0}^{n}\binom{k}{m}$$

③ $(x+y)^n = \sum_{k=0}^{n} \underline{\binom{n}{k}}\, x^k y^{n-k}$

④ $\Rightarrow \begin{cases} 2^n = \sum_{k=0}^{n}\binom{n}{k} \\ 0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k} \end{cases}$

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

⑤ (2.2) $\sum_{0\le k\le n}\binom{r+k}{k} = \binom{r+n+1}{n}$

⑥ Chu-Vandermonde's Identity:

$$\sum_{k}\binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n} \qquad \underline{\text{Integers } m, n}$$

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Chu-Vandermonde's Identity: A Combinatorial proof.

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$$

By $k \to k-m$, $n \to n-m$, we get

$$\sum_k \binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

$\binom{r+s}{n}$ is the number of ways to choose n people from among r men and s women. On the left, each term of the sum is the number of ways to choose k of the men and $n-k$ of the women.

▶ How to show identities by using basic identities:

problem $\displaystyle\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = ??$ $n \geq m \geq 0$

$m=2$ $n=4$

$$\binom{2}{0}/\binom{4}{0} + \binom{2}{1}/\binom{4}{1} + \binom{2}{2}/\binom{4}{2} = 1+\frac{1}{2}+\frac{1}{6} = \frac{5}{3}$$

claim $\displaystyle\sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{n}{k}} = \frac{n+1}{n+1-m}$ (≥ 1)

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$$\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{n}{k}} \overset{\text{Magic}}{=} \sum_{k=0}^{m}\frac{\binom{n-k}{m-k}}{\binom{n}{m}}$$

$$\frac{m!\;k!\;(n-k)!}{n!\;k!\;(m-k)!} = \frac{\dfrac{(n-k)!}{(m-k)!\,(n-m)!}}{\dfrac{n!}{m!\,(n-m)!}}$$

$$= \frac{\sum_{k=0}^{m}\binom{n-k}{m-k}}{\binom{n}{m}}$$

$$= \frac{\sum_{m\geq k}\binom{n-(m-k)}{m-(m-k)}}{\binom{n}{m}}$$

$$\sum_{k=0}^{n}\binom{r+k}{k}=\binom{r+n+1}{n}$$

$$= \frac{\sum_{m\geq k}\binom{n-m+k}{k}}{\binom{n}{m}} \overset{\text{Identity 5}}{=} \frac{\binom{n-m+m+1}{m}}{\binom{n}{m}}$$

$$= \frac{\binom{n+1}{m}}{\binom{n}{m}} = \frac{n+1}{n+1-m}$$

Identities:

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k} = 4^n$$

$$S_n = \sum_{k=0}^{n}\binom{2n-2k}{n-k}^2\binom{2k}{k}^2 = ??$$

$$n^3 S_n = 16\left(n-\tfrac{1}{2}\right)(2n^2-2n+1)S_{n-1} - 256(n-1)^3 S_{n-2}$$

Knuth's Foreword to «A=B»:

" Science is what we understand well enough to explain to a computer. Art is everything else we do. During the past several years an important part of Mathematics has been transformed from an Art to a science: No longer do we need a brilliant insight 8 in order to evaluate sum of binomial coeffic

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