Computer algebra in a combinatorialist's life

Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux, France

In this talk

Computer algebra in the solution of a counting problem

I. From objects to numbers

II. Guess

III. Prove

IV. Simplify

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I. From objects to numbers

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Examples

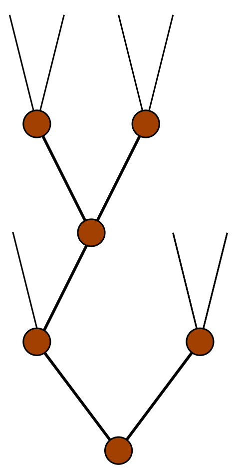
Questions

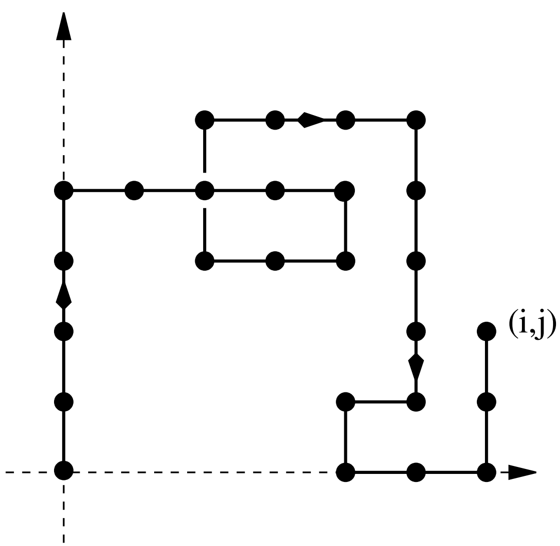
Three objectives

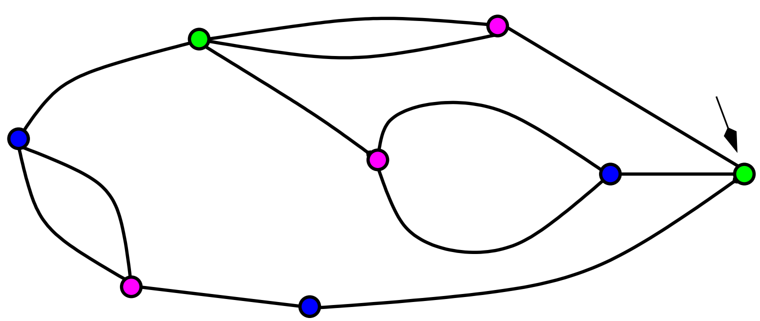
I. From objects to numbers

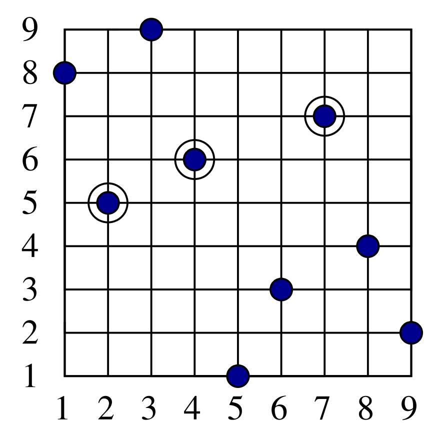
Setting

Let A be a set of discrete objects, equipped with an integer size such that the number a(n) of objects of size n is finite for any n.





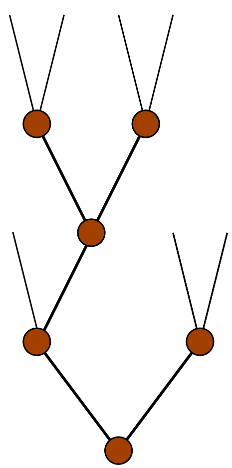


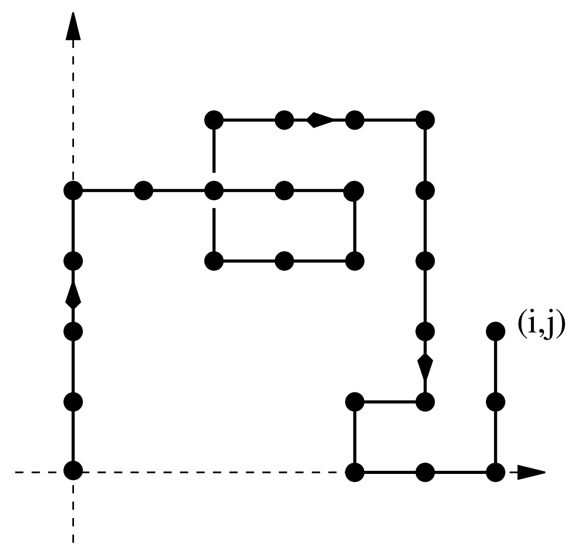


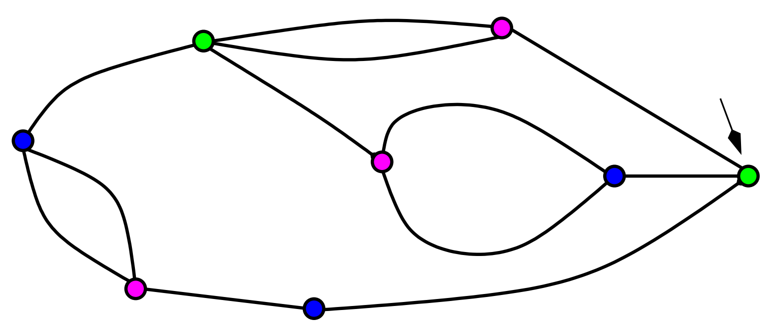
Setting

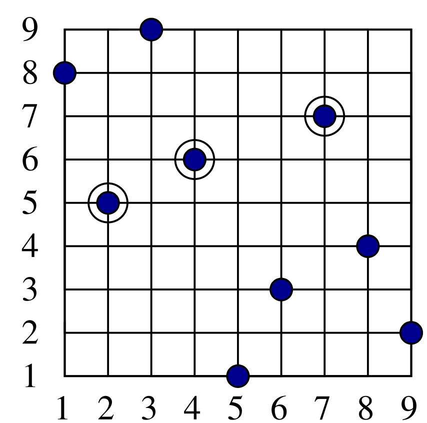
Let A be a set of discrete objects, equipped with an integer size such that the number a(n) of objects of size n is finite for any n.

Objective: generate a(1), a(2), ... , a(N) for N large.





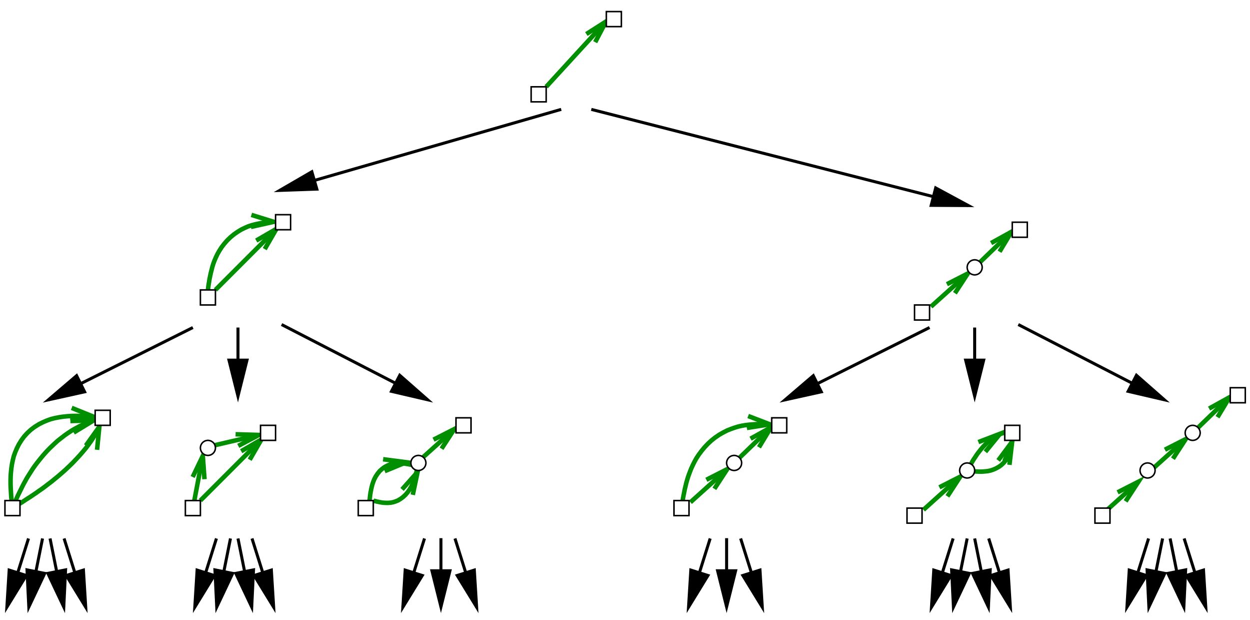


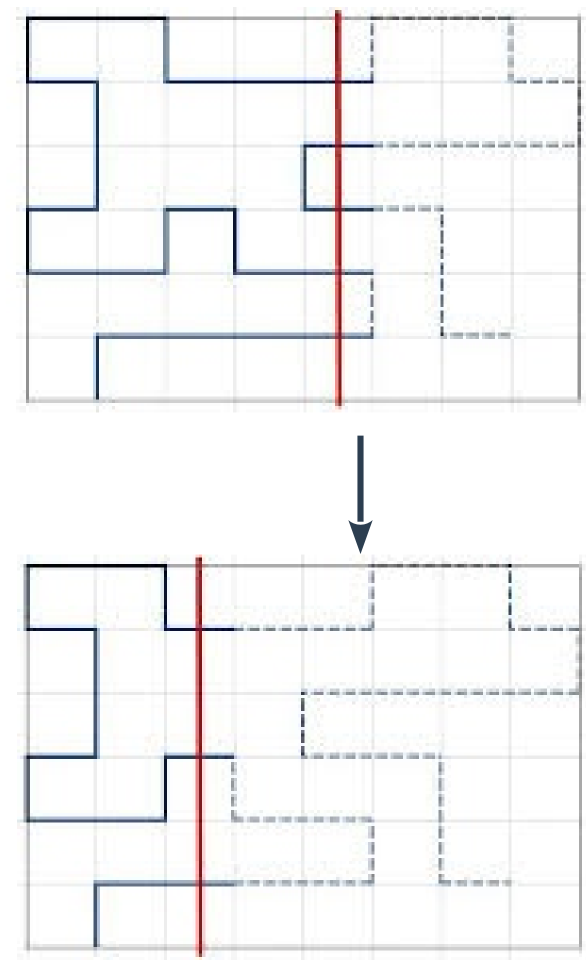


Case 1: when no recurrence relation is known

Generate numbers (and often objects) by any possible recursive construction

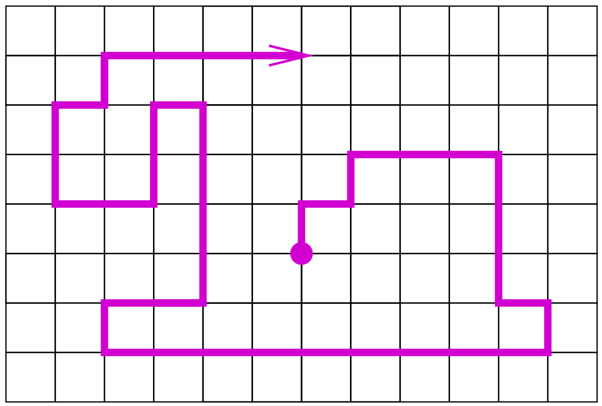
- **Generating trees:** add a step, an edge, a node...
- **Transfer matrices:** add a layer



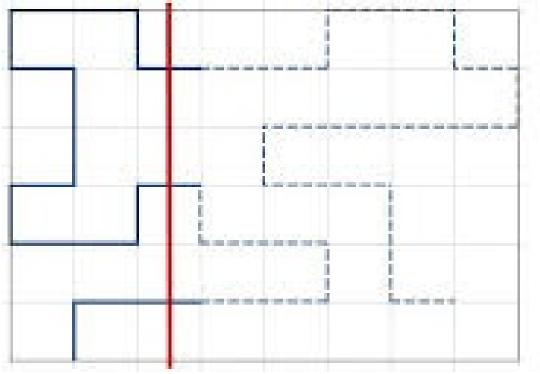


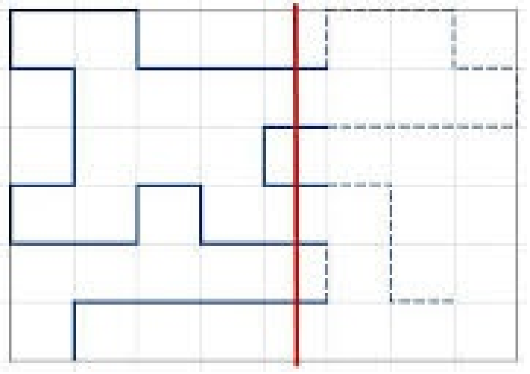
Case 1: when no recurrence relation is known

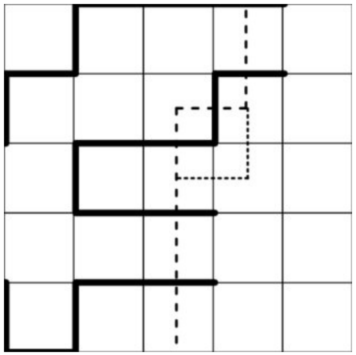
Self-avoiding walks



[Enting, Guttmann]







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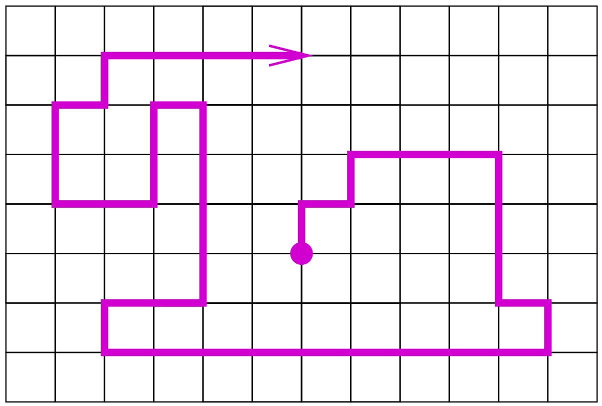
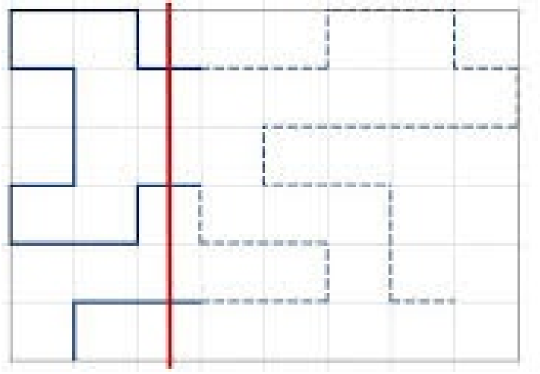
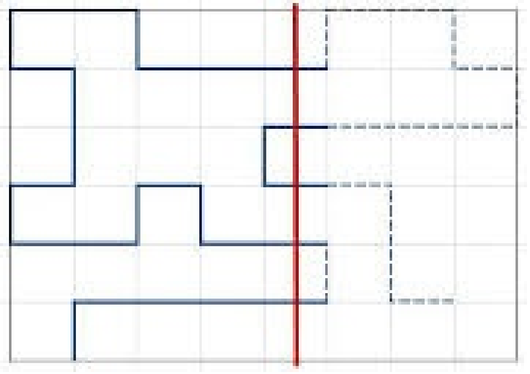


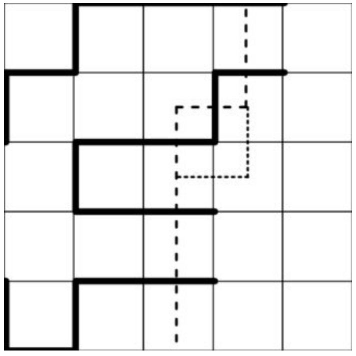
Question: is there a sub-exponential algorithm that computes the number of self-avoiding walks of length n?

[Enting, Guttmann]

So far, n=79 [Jensen 13(a)]







Case 2: with a recurrence relation

... often encoded as a functional equation for the associated generating function:

$$A(t) \equiv A := \sum_{n \geq 0} a(n) t^n = \sum_{o \in \mathcal{A}} t^{|o|}$$

Multivariate enumeration: record additional statistics

$$A(t; x, y) \equiv A(x, y) := \sum_{n,i,j \geq 0} a(n; i, j) t^n x^i y^j$$

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A rich zoo of equations

Functional equations: our pet animals

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

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Several variables: one DE per variable

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Several variables: one DE per variable

More exotic animals

Substitutions: set partitions

$$A(t) = 1 + \frac{t}{1-t} A\left(\frac{t}{1-t}\right)$$

q-Equations: Dyck paths by length (t) and area (q)

$$A(t;q) = 1 + tqA(tq;q)A(t;q)$$

More exotic animals

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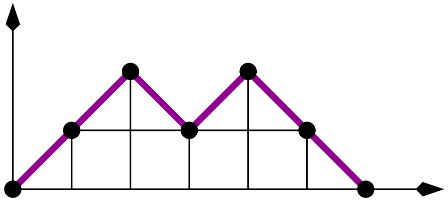

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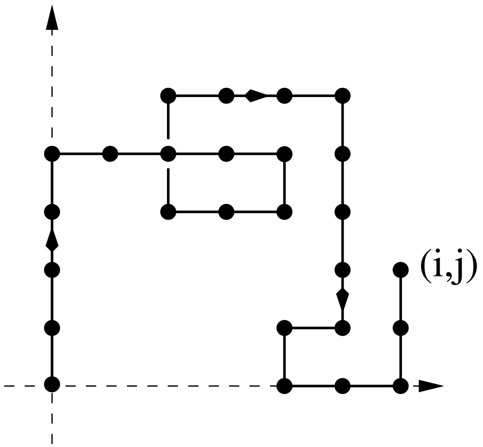

Discrete derivatives: quadrant walks

$$Q(x,y) = 1 + t(x+y)Q(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$

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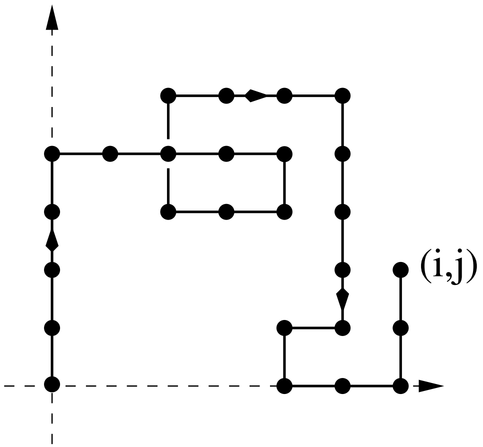

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or $$\left(1 - t\left(x + y + \frac{1}{x} + \frac{1}{y}\right)\right) xyQ(x,y) = xy - txQ(x,0) - tyQ(0,y)$$

x, y: *catalytic* variables

Hybrids

Discrete derivatives and q-equations: Tamari intervals on Dyck paths

[mbm, Fusy, Préville-Ratelle 11]

$$A(x, q) = 1 + tqA(x, q)\frac{A(xq, q) - A(1, q)}{xq - 1}$$

Hybrids

Discrete derivatives and q-equations: Tamari intervals on Dyck paths

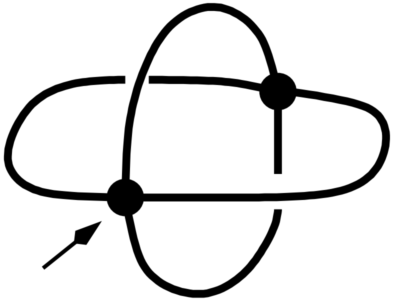
[mbm, Fusy, Préville-Ratelle 11]

$$A(x,q) = 1 + tqA(x,q)\frac{A(xq,q) - A(1,q)}{xq-1}$$

Substitutions in "catalytic" variables: bipartite quadrangulations by edges (t) and vertices (x), arbitrary genus [Louf 21]

$$2(1+2D)DA(x) = (A(x+1) + A(x-1) - 2A(x) - 2)(1+2D)A(x)$$

where D= t d/dt and A(x)=A(t,x).



With a recurrence relation/fixed point equation

- Coefficients in **polynomial time**
- Newton iteration [Pivoteau, Salvy & Soria 12]
- Work with the recurrence relation? With the functional equation?
- Work modulo primes?

Produce numbers: why?

- Predict asymptotic behaviour

Example: 1324-avoiding permutations [Conway & Guttmann 15]

$$a(n) \sim \kappa \alpha^n \beta^{\sqrt{n}} n^\gamma$$

(50 terms known)

$$\alpha \simeq 11.6 \quad \beta \simeq 0.04 \quad \gamma \simeq -1.1$$

- Conjecture (simpler) recurrence relations or functional equations

Interlude: Combinatorial exploration

An **automatized construction of recurrence relations** for some combinatorial classes.

"The Combinatorial Exploration framework produces rigorously verified combinatorial specifications for families of combinatorial objects. These specifications then lead to generating functions, counting sequence, polynomial-time counting algorithms, random sampling procedures, and more."

[Albert, Bean, Claesson, Nadeau, Pantone & Ulfarsson 22(a)]

Interlude: Combinatorial exploration

An automatized construction of recurrence relations for some combinatorial classes.

Ex. 1234-avoiding permutations

$$
\begin{aligned}
F_0(x) &= F_1(x) + F_2(x)\\
F_1(x) &= 1\\
F_2(x) &= F_{15}(x)\,F_3(x)\\
F_3(x) &= F_4(x,1)\\
F_4(x,y) &= F_1(x) + F_{16}(x,y) + F_5(x,y)\\
F_5(x,y) &= F_{10}(x,y)\,F_6(x,y)\\
F_6(x,y) &= F_7(x,1,y)\\
F_7(x,y,z) &= F_8(x,yz,z)\\
F_8(x,y,z) &= F_1(x) + F_{11}(x,y,z) + F_{13}(x,y,z) +\\
F_9(x,y,z) &= F_{10}(x,y)\,F_8(x,y,z)\\
F_{10}(x,y) &= yx\\
F_{11}(x,y,z) &= F_{10}(x,z)\,F_{12}(x,y,z)\\
F_{12}(x,y,z) &= \frac{-zF_7(x,1,z) + yF_7\left(x,\frac{y}{z},z\right)}{-z+y}\\
F_{13}(x,y,z) &= F_{14}(x,y,z)\,F_{15}(x)\\
F_{14}(x,y,z) &= \frac{zF_8(x,y,z) - F_8(x,y,1)}{-1+z}\\
F_{15}(x) &= x\\
F_{16}(x,y) &= F_{15}(x)\,F_{17}(x,y)\\
F_{17}(x,y) &= \frac{yF_4(x,y) - F_4(x,1)}{-1+y}
\end{aligned}
$$

[Albert, Bean, Claesson, Nadeau, Pantone & Ulfarsson 22(a)]

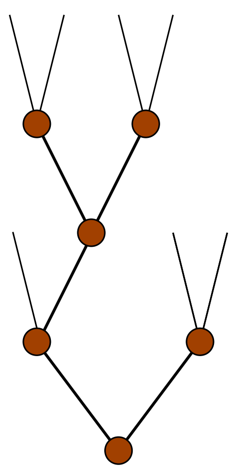
[PermPAL database]
Permutation Pattern Avoidance Library

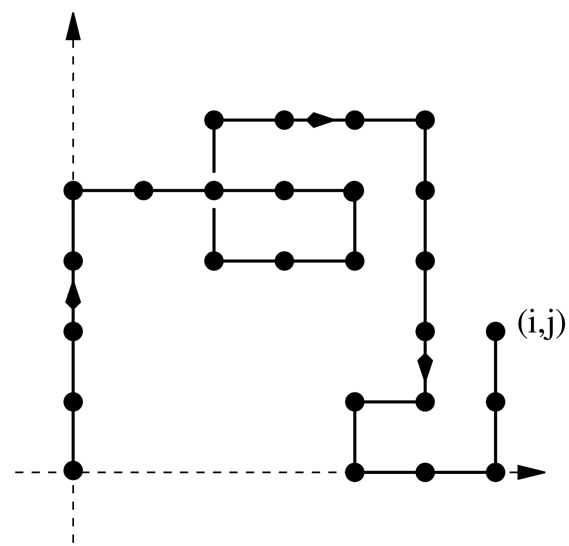
II. Guess

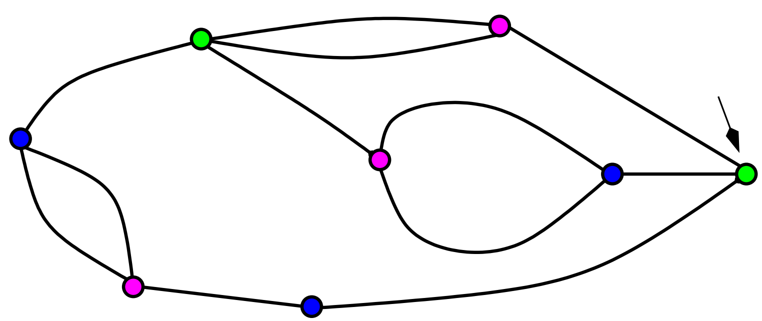
Setting

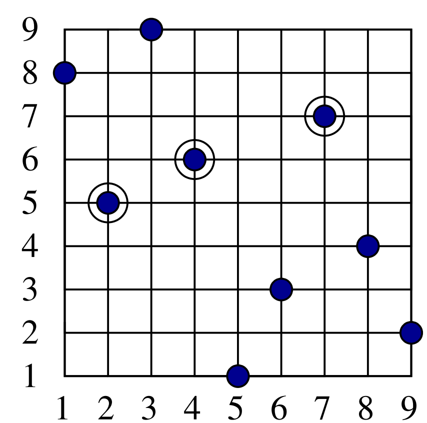
Let $a(n)$ be the number of objects of size n in the set $\mathcal{A}$.

Objective: guess a recurrence relation for $a(n)$ from the knowledge of $a(1), a(2), \ldots, a(N)$.









Hermite-Padé approximants for linear relations

Given the first coefficients $a_i(0)$, $a_i(1)$, ..., $a_i(n)$ of k series $A_i(t)$, i=1, ..., k, find polynomials $P_1(t)$, ..., $P_k(t)$ of small degree such that

$$P_1 A_1 + \cdots + P_k A_k = \mathcal{O}(t^{n+1})$$

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⇒ Needs about $n = kd$ coefficients in each series to guess an equation with $\deg(P_i) < d$.

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Example: a quadratic q-equation of order 2 corresponds to k=10 series

$$1, A(t), A(tq), A(tq^2),$$
$$A(t)^2, A(tq)^2, A(tq^2)^2, A(t)A(tq), A(t)A(t^2q), A(tq)A(t^2q).$$

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Same for an ODE of order e and degree δ.

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Same for an ODE of order e and degree δ.

numapprox[hermite_pade]

Special types of functional equations

- Guess **polynomial equations** (degree δ): linear relation between

$$1, A, \ \ldots, A^{\delta}$$

gfun[seriestoalgeq] [Salvy 94 →]

- Guess **linear differential equations** (order e): linear relation between

$$1, A, A', \ \ldots, A^{(e)}$$

gfun[seriestodiffeq]

- Guess **polynomial differential equations** (order e, degree δ):

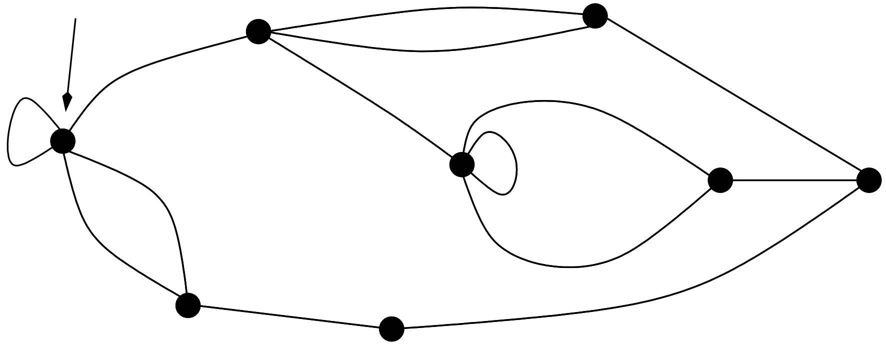
requires $\binom{\delta+e+1}{\delta}$ series.

FPS[delta2guess] [Teguia 23, Pantone 24+]

Example 1: in the 60's, Tutte and planar maps

Equation with a discrete derivative: planar maps by edges (t) and degree of the root vertex (x):

$$A(x) = 1 + tx^2 A(x)^2 + tx \frac{A(x) - A(1)}{x - 1}.$$



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$$A(1) = \bar{A}_1 := \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}.$$

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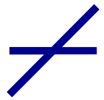
⇒ a guess for A(x) as an algebraic series of degree 4:

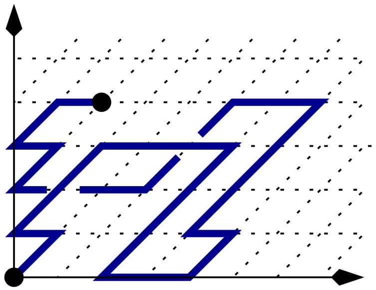
$$\bar{A}(x) = 1 + tx^2 \bar{A}(x)^2 + tx \frac{\bar{A}(x) - \bar{A}_1}{x - 1}.$$

Example 2: Gessel's quadrant walks

Equation with two discrete derivatives:

$$Q(x,y) = 1 + t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right) Q(x,y) - t\left(\frac{1}{x} + \frac{1}{xy}\right) Q(0,y) - \frac{t}{xy}\left(Q(x,0) - Q(0,0)\right)$$





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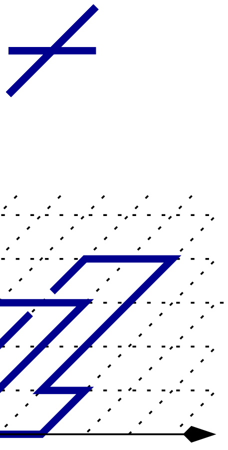
Gessel's ex-conjecture (∼2000)



$$Q(0,0) = \sum_{n\geq 0} 16^n \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n} t^{2n}$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.

Example 2: Gessel's quadrant walks

Equation with two discrete derivatives:

$$Q(x,y) = 1 + t\left(x + xy + \frac{1}{x} + \frac{1}{xy}\right) Q(x,y)$$
$$- t\left(\frac{1}{x} + \frac{1}{xy}\right) Q(0,y) - \frac{t}{xy}\left(Q(x,0) - Q(0,0)\right)$$

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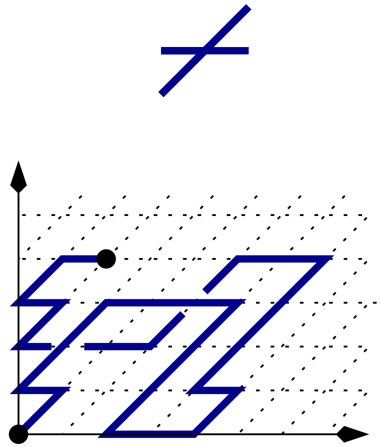

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with $(a)_n = a(a+1)\cdots(a+n-1)$.

Later... Q(0,0) satisfies an **polynomial equation** Pol(t,Q)= 0,

of bidegree (7,8) [Bostan & Kauers 10]

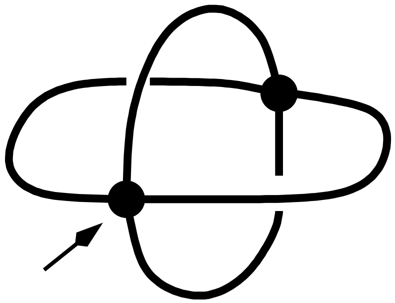
(+ Proof of the algebraicity of Q(x,y))

Example 3: bipartite quadrangulations, any genus

Substitutions in "catalytic" variables: [Louf 21]

$$2(1+2D)DA(x) = (A(x+1)+A(x-1)-2A(x)-2)(1+2D)A(x)$$

where D= t d/dt (plus value at x=1).



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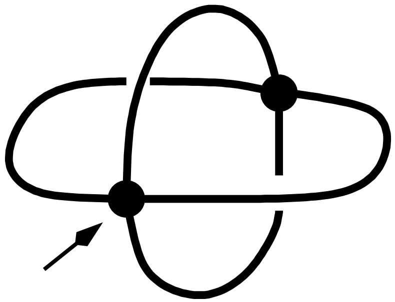
Guess: a quadratic, third order ODE in t

$$(1+D)A = t\,(3t+4x)\,A + t\,(11t+8x)\,DA + 12t^2D^{(2)}A + 4t^2D^{(3)}A$$
$$+3t^2A^2 + 12t^2A(DA) + 12t^2(DA)^2 + x^2$$

Proof [Carrell & Chapuy 15]



III. Prove

Setting

So far: a functional equation (E_1) for A(t,x,y...), possibly wild

Guessed: a simpler equation (E_2) for A(t,x,y...)

Two ingredients:

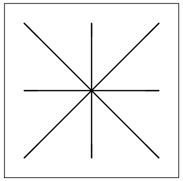
- **Uniqueness** of solution in (E_1)
- **Closure properties** of a class containing (E_2)

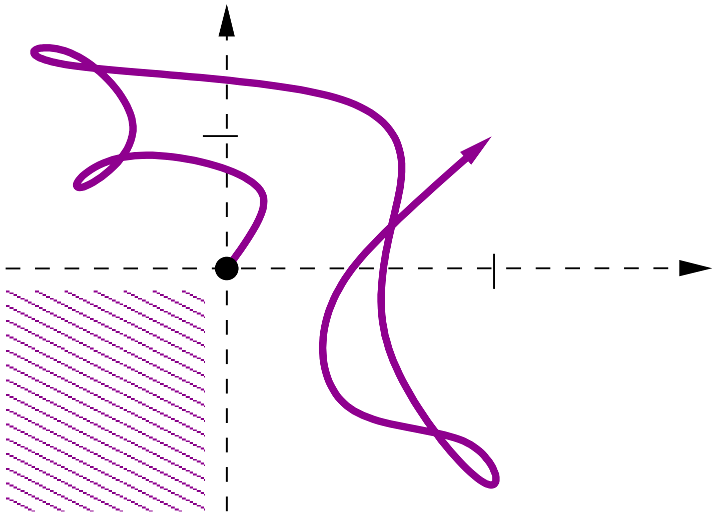
Example 1: a big algebraic system

King walks avoiding the negative quadrant [mbm & Wallner 23]

(E_1) A system of 4 polynomial equations in 4 series R_0, R_1, B_1, B_2

Degree in	R_0	R_1	B_1	B_2	t	Number of terms
Eq. 1	5	3	1	1	7	72
Eq. 2	6	4	2	2	7	132
Eq. 3	5	5	2	2	9	192
Eq. 4	6	6	3	3	10	276





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(E_2) **Guessed minimal polynomials** for all four series, and **rational expressions** in terms of two "simple" series T and U (deg. 12, 24).

Generating function	Degree in GF	Degree in t	Number of terms
R_0	24	36	323
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B_1	12	24	229
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thanks to Mark van Hoeij !

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Plug in (E_1) and check by reduction mod minimal polynomials of T and U.

Example 2: in the 60's, Tutte and planar maps

Planar maps by edges (t) and degree of the root vertex (x):

$$A(x) = 1 + tx^2 A(x)^2 + tx \frac{A(x) - A(1)}{x - 1}.$$

Uniqueness: there exists a unique solution A(x) that is a formal power series in t. Its coefficients are polynomials in x.

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Guessing for A(1):

$$A(1) = \bar{A}_1 := \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}.$$

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To do: prove that $\bar{A}(x)$ has **polynomial coeffs**. in x, so that $\bar{A}_1 = \bar{A}(1)$.

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or

$$(x-1)\left(\bar{A}(x) - 1 - tx^2 \bar{A}(x)^2\right) = tx\left(\bar{A}(x) - \bar{A}_1\right).$$

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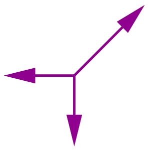
Example 3: Kreweras' walks in the quadrant



Two discrete derivatives:

$$\left(xy - t\left(x + y + x^2y^2\right)\right) Q(x,y) = xy - A(x) - A(y)$$

where $A(x)=txQ(x,0)$.

[Bostan & Kauers 10]

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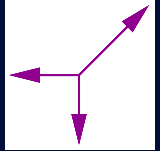

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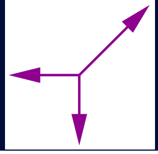

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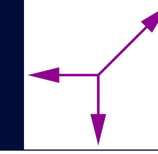

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To do:

- Prove that $\bar{A}(x)$ has **polynomial coefficients** in x.
- Prove that (E_1) holds for $\bar{A}(x)$ by computing a polynomial annihilating the rhs of (E_1), and checking first coefficients.

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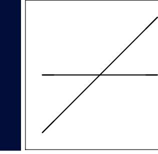
Example 4: more walks in the quadrant



Two discrete derivatives:

$$K(x,y)Q(x,y) = xy - A(x) - B(y)$$

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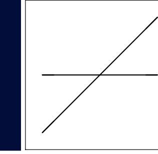

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$$\begin{aligned} 0 &= X(y)y - A(X(y)) - B(y), \\ 0 &= xY(x) - A(x) - B(Y(x)), \end{aligned} \qquad (E_1)$$

where $X(y)$ (resp. $Y(x)$) is the only root of K that is a formal series in t.

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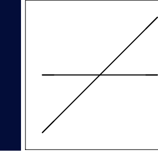

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To do:

- Prove that the guessed solutions have **polynomial coefficients**
- Prove that (E_1) holds for the guessed series by **polynomial elimination** and checking first coefficients.

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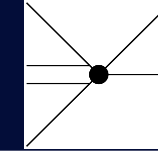

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$$0 = X(y)y - A(X(y)) - B(y),$$
$$0 = xY(x) - A(x) - B(Y(x)),$$
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where $X(y)$ (resp. $Y(x)$) is the only root of K that is a formal series in t.

Guess (E_2): differential ideals (in ∂t and ∂x, resp. ∂y) for $A(x)$ and $B(y)$

To do:

- Prove that the guessed solutions have **polynomial coefficients**
- Prove that (E_1) holds for the guessed series by **differential elimination** and checking first coefficients.

[Bostan, mbm, Kauers & Melczer 16]

IV. Simplify

Setting

Given a series A(t,x,y...) and a defining functional equation (algebraic, D-finite, D-algebraic), get a **better understanding** of A.

- Find a simple description of A
- Understand the properties of A
- Determine singularities, asymptotics
- ...

Simplifying in the algebraic world

Classical tools: polynomial factorization, resultants, Gröbner bases...

Given a minimal polynomial P(t,A)=0:

- genus, **rational parametrization** (if genus 0), Weierstrass form for (hyper)elliptic solutions (algcurves)

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Question: Given an algebraic series A(t;x,y...) given by its minimal polynomial over K=$\mathbb{Q}$(t,x,y...), find a "simple" series generating K(A).

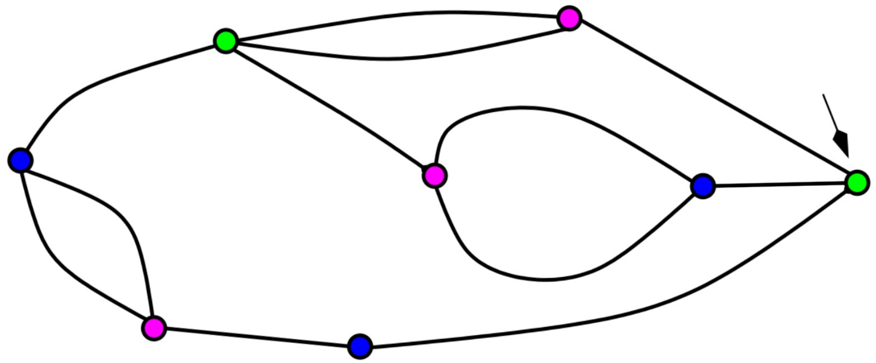
Same question for the subfields between K and K(A).

A small example: properly 3-coloured planar maps

How does one go from this **polynomial of bidegree** (6, 4) in (t,A):

$$-12500A^4t^6+24t^4(1000t-71)A^3-2t^2\left(3600t^3+7216t^2-1020t+39\right)A^2$$
$$-\left(864t^5-9040t^4-1712t^3+536t^2-42t+1\right)A-40t+540t^2-2720t^3+432t^4+1=0$$

to...



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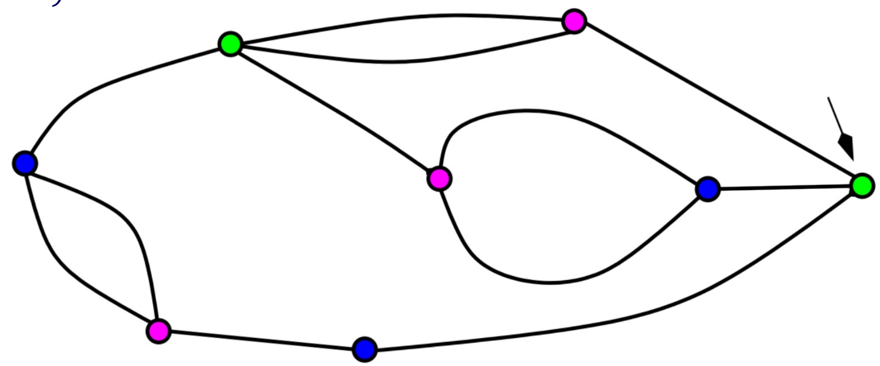
to...

$$A = 2T - \frac{T^2(1+2T)(1+2T^2+2T^4)}{(1-2T^3)^3}$$

with $T = t\dfrac{(1+2T)^3}{1-2T^3}$?



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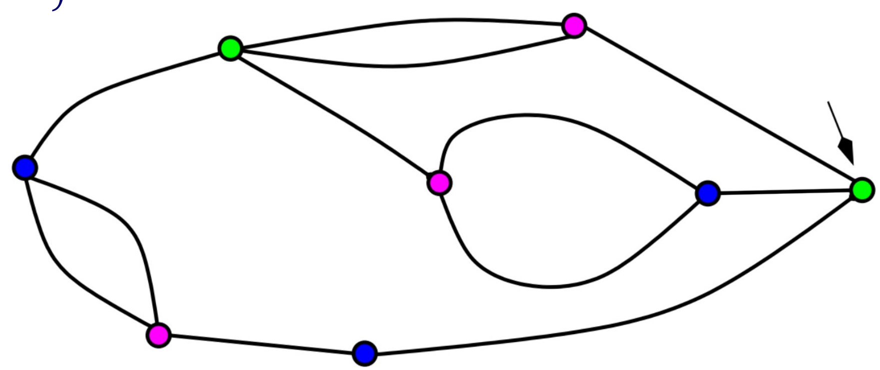

algcurves[parametrization] gives *some* parametrization

$$t=\frac{S^3-6S^2+12S-10}{S^3(S-2)}$$

(genus 0)

[Bernardi & mbm 09]

A bigger example: king walks avoiding a quadrant

How does one go from this **polynomial of bidegree (24, 12)** in (t,A):

$$
\begin{aligned}
&\left(1544682349732742644432896t^6 + 2859956429703196777316352t^5 + 1371747210064046280769536t^4\right.\\
&\left.+2618686066483670565551936t^3 + 2068591227551829350645 76t^2 + 98613397010845517414 4t + 655923393268641792\right) A^{12}\\
&+\left(119088381814379102884331 52t^8 + 27491842869484512619266048t^7 + 22066168998404344966742016t^6\right.\\
&+9456378844969952000409600t^5 + 3577317106243476992311296t^4 + 725362067373633286668288t^3\\
&\left.+123324842335532119326720t^2 + 4261627989408261242 88t + 2498755783880540 16\right) A^{11}\\
&+ [\cdots]\\
&-2\left(1099511627776t^{16} + 4947802324992t^{15} + 8908835913728t^{14} + 8010919313408t^{13} + 3551066587136t^{12}\right.\\
&+601824952320t^{11} + 128619544576t^{10} + 260050427904t^9 + 187250317568t^8 + 66799107968t^7 + 13529493584t^6\\
&\left.+1545216528t^5 + 86381746t^4 + 1570596t^3 + 920t^2 + 38t - 1\right)(4t+1)^4(8t-1)^4 A\\
&+ 3t^2(t+1)^2(4t+1)^6(8t-1)^{10} = 0
\end{aligned}
$$

to…

[mbm & Wallner 23]

A bigger example: king walks avoiding a quadrant

How does one go from this **polynomial of bidegree (24, 12)** in (t,A):

$$
\begin{aligned}
&\left(1544682349732742644432896t^6 + 2859956429703196777316352t^5 + 1371747210064046280769536t^4\right.\\
&\left.+261868606648367056551936t^3 + 20685912275518293506457 6t^2 + 9861339701084551741 44t + 655923393268641792\right) A^{12}\\
&+\left(119088381814379102884331 52t^8 + 2749184286948451261926604 8t^7 + 2206616899840434496674201 6t^6\right.\\
&+9456378844969952000409600t^5 + 3577317106243476992311296t^4 + 725362067373633286668288t^3\\
&\left.+123324842335532119326720t^2 + 4261627989408261242 88t + 249875578388054016\right) A^{11}\\
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$$A = 3(1-8t)\,\frac{T^2(1+4T+T^2)(T^2-1)(1+2T)}{2(1-3T^2-4T^3)^3(1+4T-2T^3)},$$

with

$$\frac{T\left(T^2+T+1\right)\left(1+3T-T^3\right)^3}{\left(T^2+4T+1\right)\left(1-3T^2-4T^3\right)^3} = \frac{t(1+t)}{1-8t}.$$

(genus 4)

[mbm & Wallner 23]

A recurrent question: dependence on parameters

Subfields. If P(t,A)=0, what are the subfields of $\mathbb{Q}$(t,A) ?

Example. Starting from P(t,a) of bidegree (24, 12), the command

evala(Subfields(subs(t=10^k, P(t,a)),4)

yields a subfield of degree 4 over $\mathbb{Q}$(t) for each value of t.

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E.g, for t=10,

$$RootOf\left(59059089842541\_Z^4 + 40291825844958\_Z^3 - 1436343349704265470\_Z^2 + 3848807433734406268482\_Z - 29043956303983559748 5204\right)$$

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⇒ Reconstruction?

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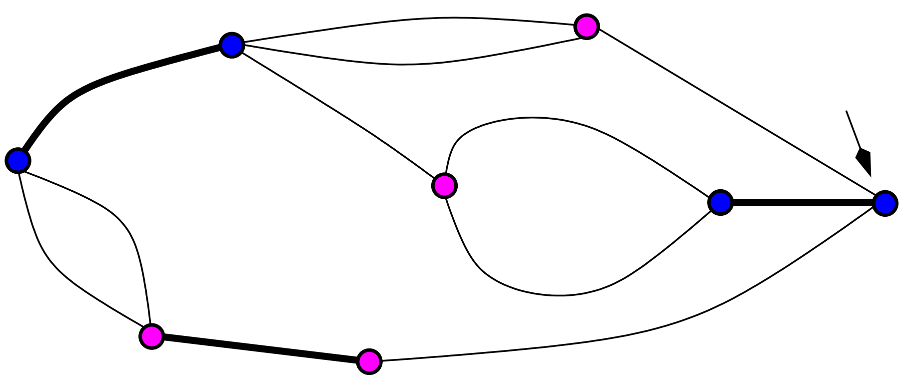

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[mbm & Bernardi 09]

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$$T = t\,\frac{\left(1+3xT-3xT^2-x^2T^3\right)^2}{1-2T+2x^2T^3-x^2T^4}.$$

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[mbm & Bernardi 09]

Simplifying in the D-finite world

Classical tools for linear ODEs

- Closure properties [Gfun]
- Factorisation of differential operators
- ODE of minimal order satisfied by a D-finite series
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Steve Melczer's talk

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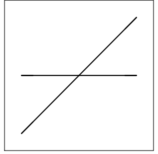
Gessel's quadrant walks ending on the y-axis



- Start from the polynomial equation for A=Q(0,1):

$$1090491731185059590307 84A^8t^6 + 1211657479094510655897 6t^4\left(16t+1\right)A^6$$

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- The solution (LREToo ls[hypergeomsols])

$$a(n) = \frac{4\sqrt{3}\,\Gamma\left(\frac{5}{6}\right) 16^n \Gamma\left(n+\frac{1}{2}\right) \Gamma\left(n+\frac{7}{6}\right)}{9\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right) \Gamma(n+2) \Gamma\left(n+\frac{4}{3}\right)} + \frac{2\Gamma\left(\frac{2}{3}\right) 16^n \Gamma\left(n+\frac{5}{6}\right) \Gamma\left(n+\frac{1}{2}\right)}{9\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right) \Gamma(n+2) \Gamma\left(n+\frac{5}{3}\right)}$$

Simplifying in the D-finite world

Question: decide whether a given D-finite series is algebraic
[Bostan 17, Bostan, Caruso & Roques 23(a), Singer 80]

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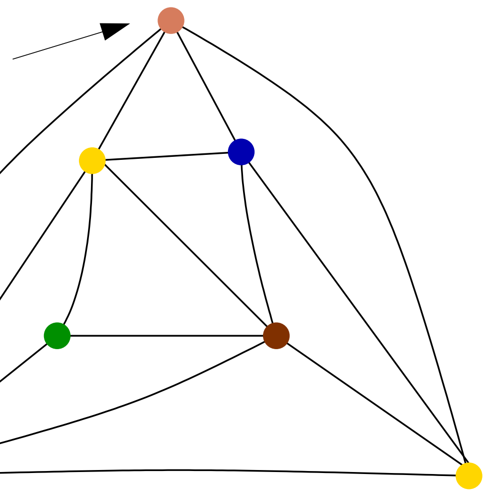

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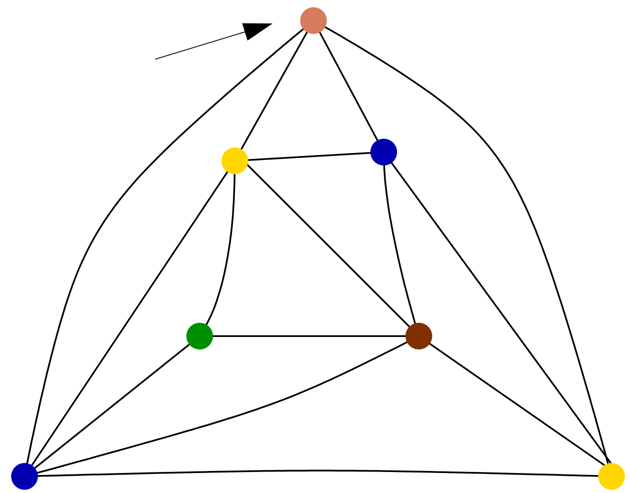

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[Tutte 73-84]
[Bettinelli]

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α = 4. Properly 5-coloured triangulations, probably not D-finite

[Tutte 73-84]
[Bettinelli]

My favourite tool...

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Ask people !

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The A≠B team...

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Thanks for your attention