

Integer multiplication in time $O(n \log n)$

David Harvey, Joris van der Hoeven



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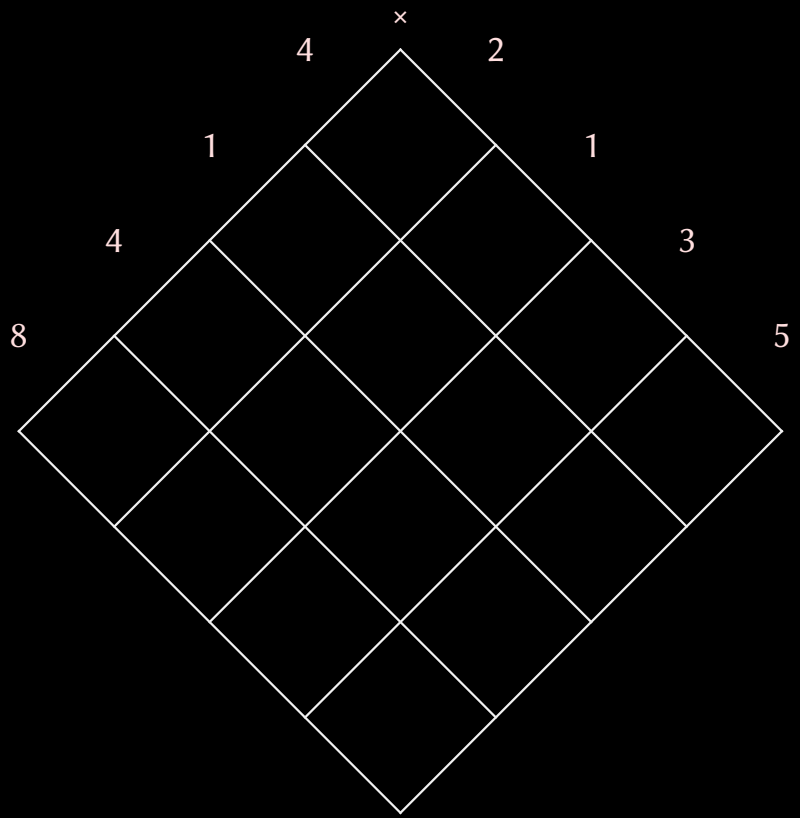
Computing e , π . $O(M(N) \log N)$

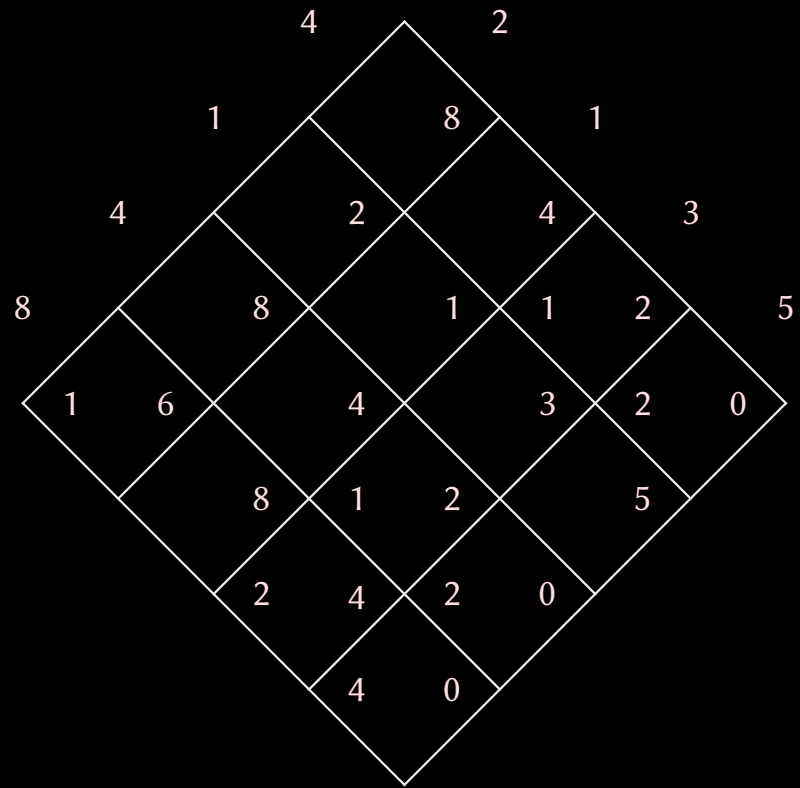
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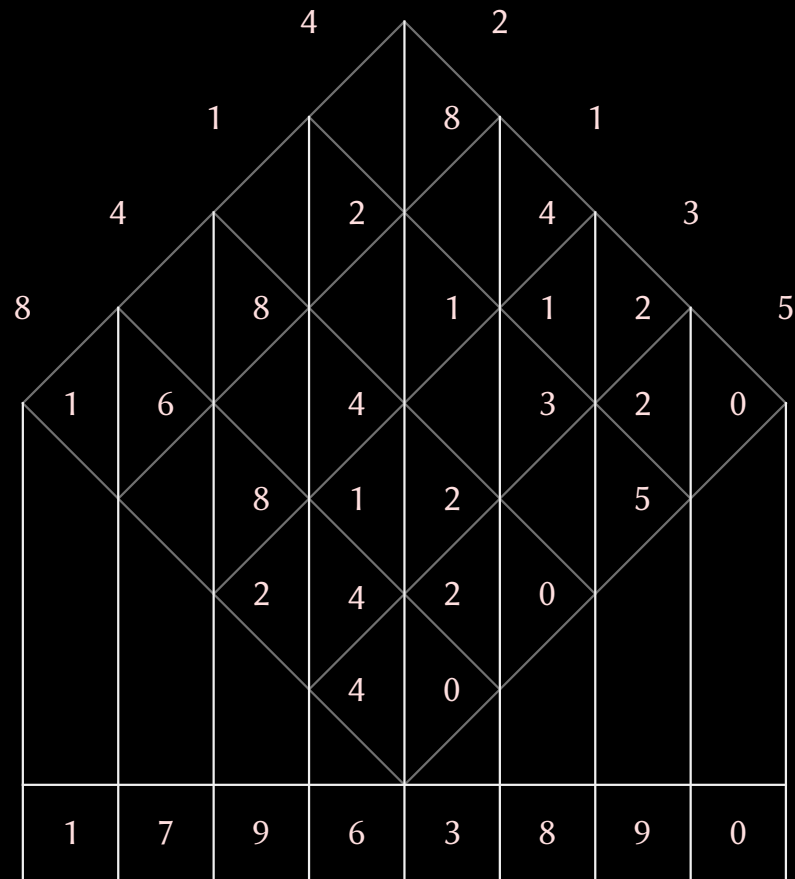
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$M(N)$ = speed of basic arithmetic

- Also**
- Better theoretical techniques $\xrightarrow{\text{often}}$ faster practical implementations
 - Asymptotic complexity abstracts from concrete machines
 - Mechanizing multiplication is a historically fascinating problem







Can we do better?



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$$I(N) = \Theta(N^2)$$

!

?

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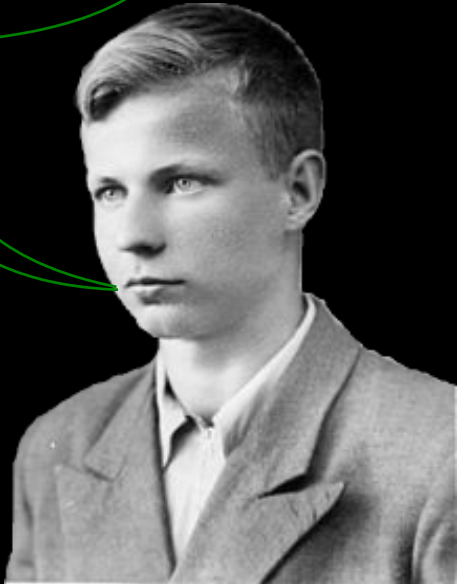


A black and white portrait of Donald Knuth, an older man with grey hair, wearing a suit and tie. A red speech bubble originates from his mouth.

$$I(N) = \Theta(N^2)$$

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A black and white portrait of Donald Knuth, a younger man with short hair, wearing a suit and tie. A green speech bubble originates from his mouth.

$$I(N) = O(N^{\log_2 3})$$

1962 Karatsuba	$O(N^{\log 3 / \log 2})$
1963 Toom	$O(N 2^{5\sqrt{\log N / \log 2}})$
1966 Schönhage	$O(N 2^{\sqrt{2\log N / \log 2}} (\log N)^{3/2})$
1969 Knuth	$O(N 2^{\sqrt{2\log N / \log 2}} \log N)$
1971 Pollard	$O(N \log N \log \log N \log \log \log N \dots)$
1971 Schönhage-Strassen	$O(N \log N \log \log N)$
2007 Fürer	$O(N \log N 2^{O(\log^* N)})$
2014 Harvey-vdH-Lecerf	$O(N \log N 8^{\log^* N})$
2017 Harvey	$O(N \log N 6^{\log^* N})$
2017 Harvey-vdH	$O(N \log N (4\sqrt{2})^{\log^* N})$
2018 Harvey-vdH	$O(N \log N 4^{\log^* N})$
2019 Harvey-vdH	$O(N \log N)$

Karatsuba multiplication

6/22

$$13022020 \times 31415926$$

Karatsuba multiplication

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$$1302 \ 2020 \times 3141 \ 5926$$

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$$\underbrace{1302}_a \underbrace{2020}_b \times \underbrace{3141}_c \underbrace{5926}_d$$

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$$(ax + b) \cdot (cx + d) = a \cdot c x^2 + (a \cdot d + b \cdot c) x + b \cdot d$$

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Complexity

$$M(n) \leq 3M(n/2) + Cn$$

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$$\begin{aligned} M(n) &\leq 3M(n/2) + Cn \\ &\leq 9M(n/4) + \frac{5}{2}Cn \end{aligned}$$

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Kronecker segmentation

$$4627579679788114 \times 4519170871966234$$

$$(4627x^3 + 5796x^2 + 7978x + 8114) \times (4519x^3 + 1708x^2 + 7196x + 6234)$$

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$$1004003 \times 2001005 = 2009015023015$$

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n : cycle length

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Summary so far

$$\mathbb{Z} \xrightarrow{\text{Kronecker}} \mathbb{K}[x] \xrightarrow{\text{Encode}} \mathbb{K}[x]/(x^n - 1)$$

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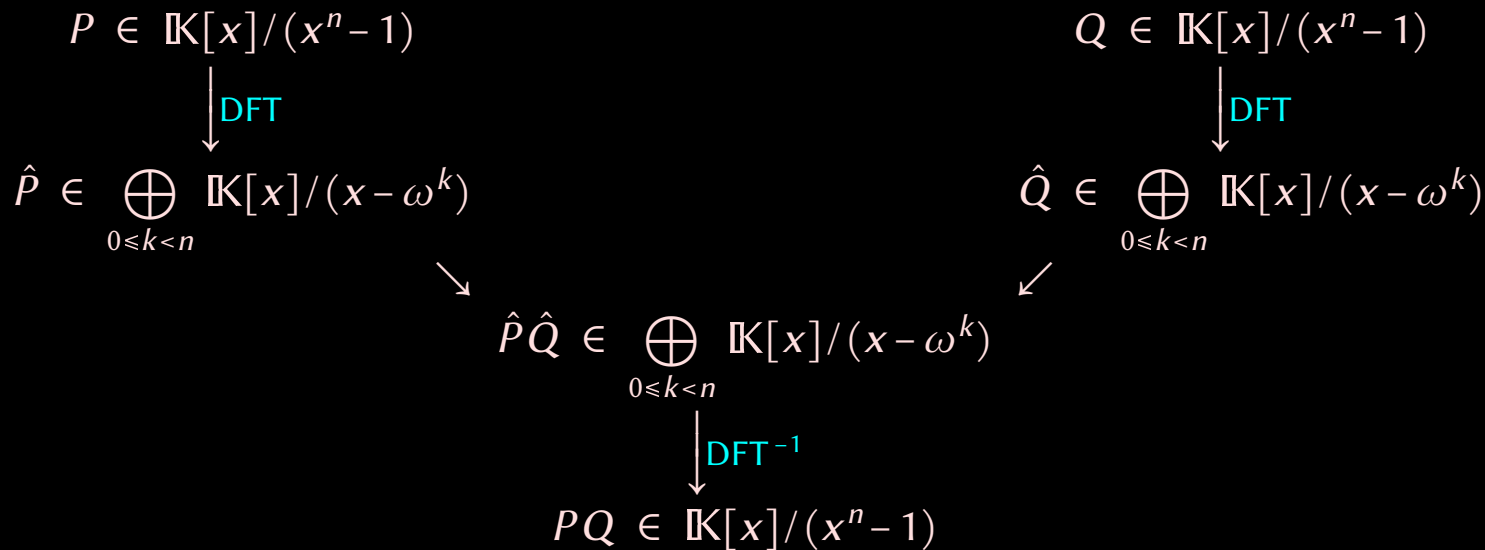
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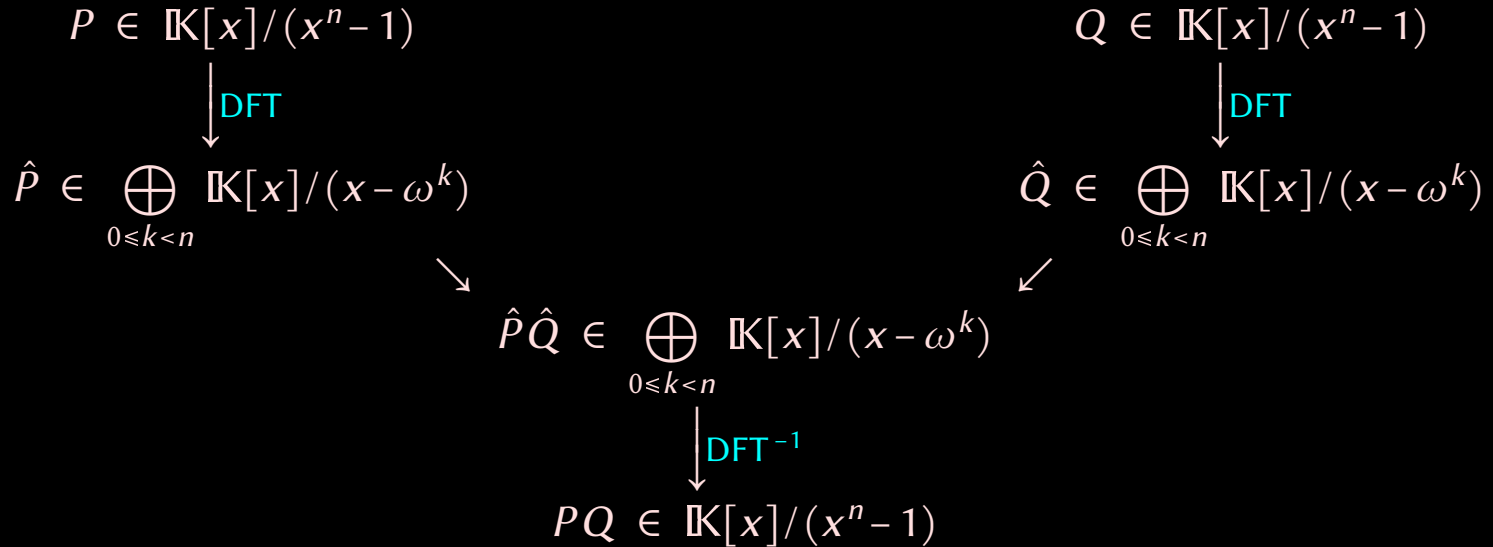
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$$\text{DFT}_\omega^{-1} \qquad \frac{1}{n} \text{DFT}_{\omega^{-1}}$$



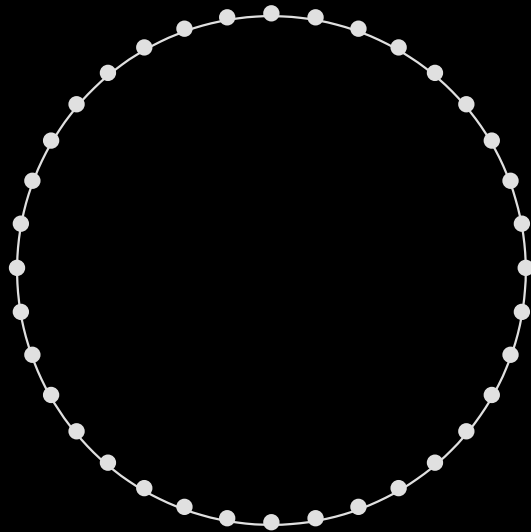


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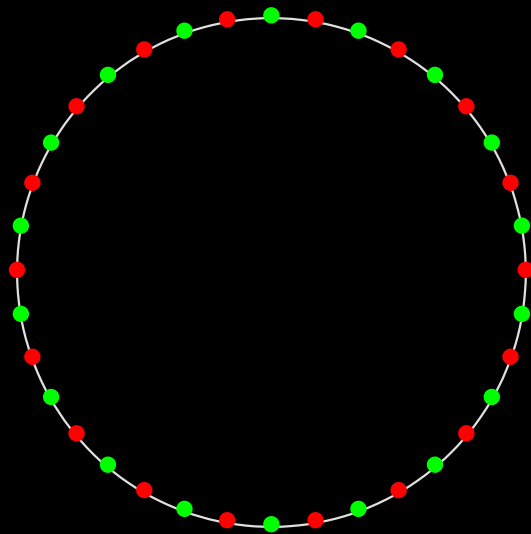
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Making the FFT fast

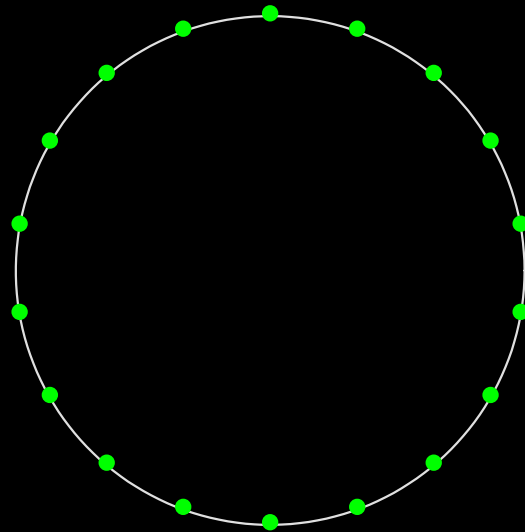
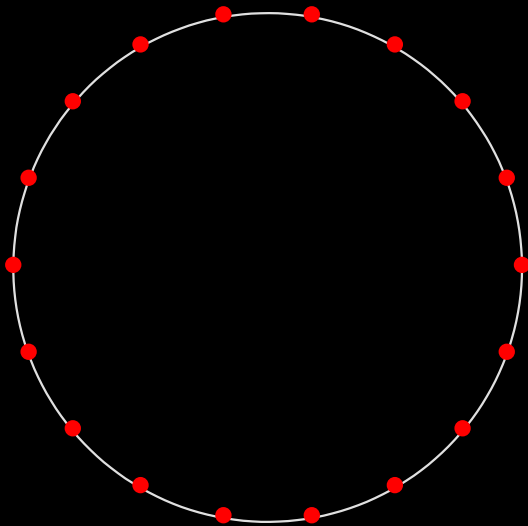
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Making the FFT fast

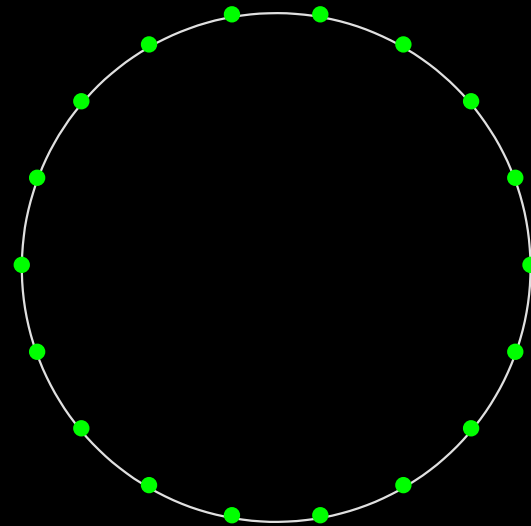
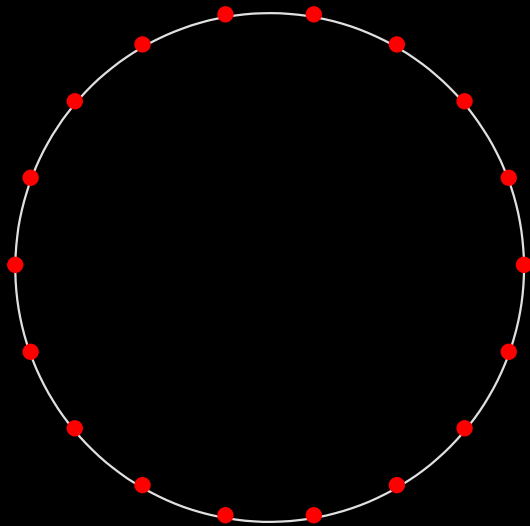


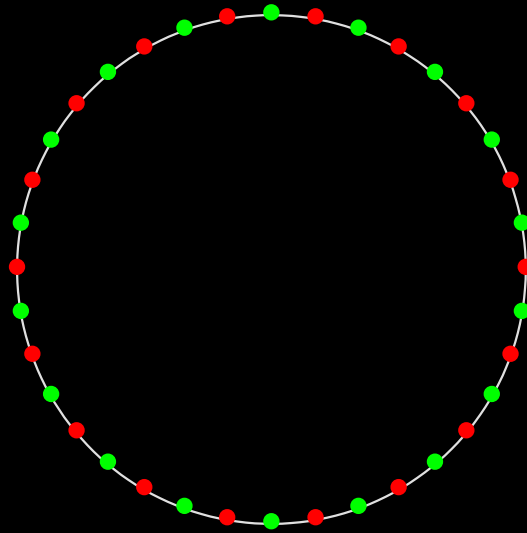
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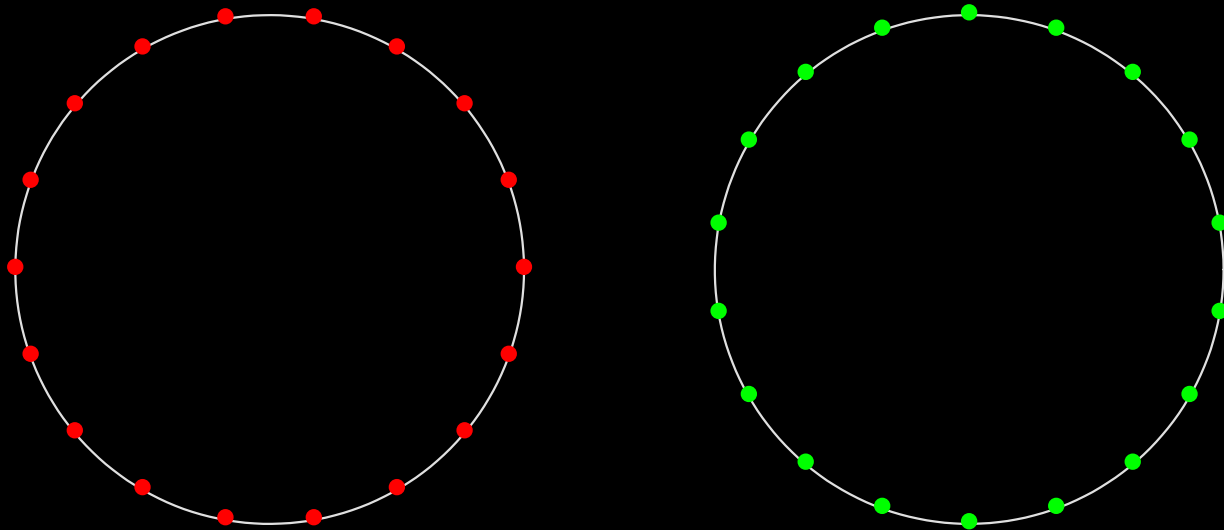
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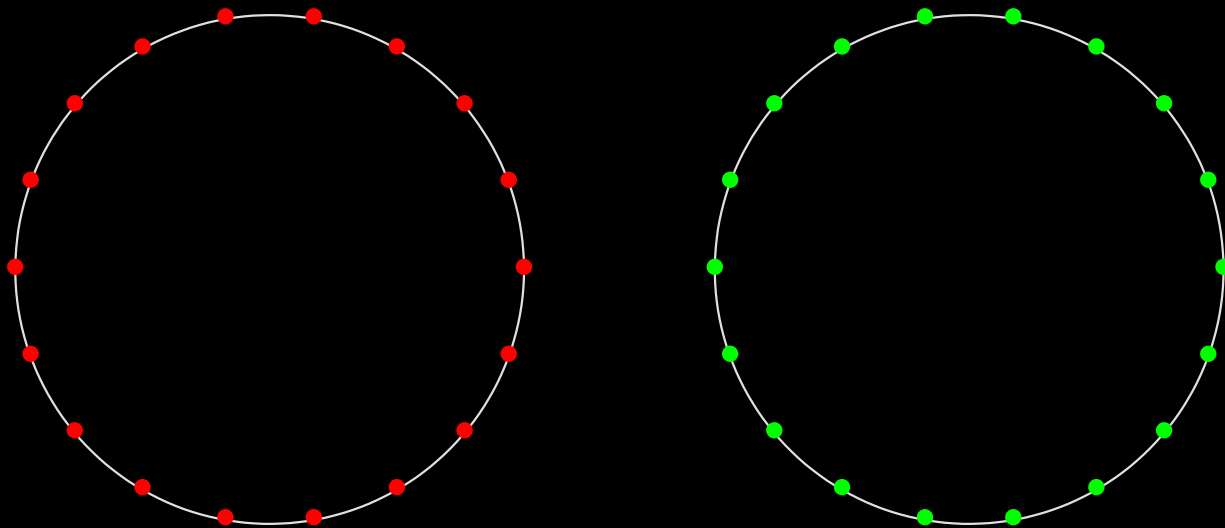




$$\mathbb{K}[x]/(x^{2n} - 1)$$



$$\mathbb{K}[x]/(x^{2n} - 1) \cong \mathbb{K}[x]/(x^n - 1) \oplus \mathbb{K}[x]/(x^n + 1)$$



$$\begin{aligned}
 \mathbb{K}[x]/(x^{2n}-1) &\cong \mathbb{K}[x]/(x^n-1) \oplus \mathbb{K}[x]/(x^n+1) \\
 &\cong \mathbb{K}[x]/(x^n-1) \oplus \mathbb{K}[x]/(\tilde{x}^n-1) \\
 &\quad \tilde{x} = \omega x \\
 &\quad \omega^n = -1
 \end{aligned}$$

$$F_{\mathbb{K}}(2n) \leq 2F_{\mathbb{K}}(n) + n \text{add}_{\mathbb{K}} + n \text{sub}_{\mathbb{K}} + n \text{mul}_{\omega^{\mathbb{N}}}$$

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$$n = 2^{\lg n} \implies F_{\mathbb{K}}(n) \leq n \lg n \left(\text{add}_{\mathbb{K}} + \frac{1}{2} \text{mul}_{\omega^{\mathbb{N}}} \right)$$

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| II. $M(N) = O(NM(\log N))$ | $M(N) = O(N \log N \log \log N \cdots)$ |
| III. $M^\circ(N) \leq 2\sqrt{N}M^\circ(\sqrt{N}) + O(N \log N)$ | $M(N) = O(N \log N \log \log N)$ |

$M^\circ(N)$: cost of multiplication in $\mathbb{Z}/(2^N + 1)\mathbb{Z}$

A careful construction yields

$$M^{\Theta}(n) \leq Cn \log n + 2n^{1/2} M^{\Theta}(n^{1/2})$$

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$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 2n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + Cn \log n + 4n^{3/4} M^\ominus(n^{1/4})\end{aligned}$$

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$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 2n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + Cn \log n + 4n^{3/4} M^\ominus(n^{1/4}) \\ &\leq Cn \log n + Cn \log n + Cn \log n + 8n^{7/8} M^\ominus(n^{1/8})\end{aligned}$$

A careful construction yields

$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 2n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + Cn \log n + 4n^{3/4} M^\ominus(n^{1/4}) \\ &\leq Cn \log n + Cn \log n + Cn \log n + 8n^{7/8} M^\ominus(n^{1/8}) \\ &\vdots \\ &\leq Cn \log n + \overset{\log \log n \times}{\dots} + Cn \log n + O(n \log n)\end{aligned}$$

What if...

$$M^{\Theta}(n) \leq C n \log n + 1.98 n^{1/2} M^{\Theta}(n^{1/2})$$

What if...

$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 1.98 n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + 0.99 Cn \log n + 1.98^2 n^{3/4} M^\ominus(n^{1/4})\end{aligned}$$

What if...

$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 1.98 n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + 0.99 Cn \log n + 1.98^2 n^{3/4} M^\ominus(n^{1/4}) \\ &\leq Cn \log n + 0.99 Cn \log n + 0.99^2 Cn \log n + 1.98^3 n^{7/8} M^\ominus(n^{1/8})\end{aligned}$$

What if...

$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 1.98 n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + 0.99 Cn \log n + 1.98^2 n^{3/4} M^\ominus(n^{1/4}) \\ &\leq Cn \log n + 0.99 Cn \log n + 0.99^2 Cn \log n + 1.98^3 n^{7/8} M^\ominus(n^{1/8}) \\ &\vdots \\ &\leq O(n \log n)\end{aligned}$$

What if...

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Next aim

$$M(n) \leq Cn \log n + (d - \epsilon) n^{1-1/d} M(n^{1/d})$$

What if...

$$\begin{aligned}M^\ominus(n) &\leq Cn \log n + 1.98 n^{1/2} M^\ominus(n^{1/2}) \\ &\leq Cn \log n + 0.99 Cn \log n + 1.98^2 n^{3/4} M^\ominus(n^{1/4}) \\ &\leq Cn \log n + 0.99 Cn \log n + 0.99^2 Cn \log n + 1.98^3 n^{7/8} M^\ominus(n^{1/8}) \\ &\vdots \\ &\leq O(n \log n)\end{aligned}$$

Next aim

$$\begin{aligned}M(n) &\leq Cn \log n + (d - \epsilon) n^{1-1/d} M(n^{1/d}) \quad \text{or} \\ M(n^d) &\leq Cdn^d \log n + (d - \epsilon) n^{d-1} M(n)\end{aligned}$$

$$\mathbb{L} := \mathbb{K}[u]/(u^n - 1)$$

$$\mathbb{L} := \mathbb{K}[u]/(u^n - 1)$$

Schönhage–Strassen

$$\mathbb{L}[x]/(x^n - 1) \xrightleftharpoons{\text{DFT}} \mathbb{L}^n$$

$$\text{mul}_{\mathbb{L}[x]/(x^n - 1)} \leq n \text{ mul}_{\mathbb{L}} + O(n^2 \log n)$$

$$\mathbb{L} := \mathbb{K}[u]/(u^n - 1)$$

Schönhage–Strassen

$$\begin{aligned} \mathbb{L}[x]/(x^n - 1) &\stackrel{\text{DFT}}{\iff} \mathbb{L}^n \\ \text{mul}_{\mathbb{L}[x]/(x^n - 1)} &\leq n \text{mul}_{\mathbb{L}} + O(n^2 \log n) \end{aligned}$$

Nussbaumer

$$\begin{aligned} \mathbb{L}[u_2, \dots, u_d]/(u_2^n - 1, \dots, u_d^n - 1) &\stackrel{\text{DFT}}{\iff} \mathbb{L}^{n^{d-1}} \\ \text{mul}_{\mathbb{L}[u_2, \dots, u_d]/(u_2^n - 1, \dots, u_d^n - 1)} &\leq n^{d-1} \text{mul}_{\mathbb{L}} + O(d n^d \log n) \end{aligned}$$

$$\mathbb{L} := \mathbb{K}[u]/(u^n - 1)$$

Schönhage–Strassen

$$\begin{aligned} \mathbb{L}[x]/(x^n - 1) &\xrightarrow{\text{DFT}} \mathbb{L}^n \\ \text{mul}_{\mathbb{L}[x]/(x^n - 1)} &\leq n \text{mul}_{\mathbb{L}} + O(n^2 \log n) \end{aligned}$$

Nussbaumer

$$\begin{aligned} \mathbb{L}[u_2, \dots, u_d]/(u_2^n - 1, \dots, u_d^n - 1) &\xrightarrow{\text{DFT}} \mathbb{L}^{n^{d-1}} \\ \text{mul}_{\mathbb{L}[u_2, \dots, u_d]/(u_2^n - 1, \dots, u_d^n - 1)} &\leq n^{d-1} \text{mul}_{\mathbb{L}} + O(dn^d \log n) \end{aligned}$$

What if...

$$\mathbb{K}[x]/(x^{n^d} - 1) \xrightarrow{?} \mathbb{K}[u_1, \dots, u_d]/(u_1^n - 1, \dots, u_d^n - 1)$$

Lifting the Chinese remainder theorem

s_1, \dots, s_d pairwise coprime

s_1, \dots, s_d pairwise coprime

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong \mathbb{Z}/s_1 \mathbb{Z} + \cdots + \mathbb{Z}/s_d \mathbb{Z}$$

s_1, \dots, s_d pairwise coprime

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong \mathbb{Z}/s_1 \mathbb{Z} + \cdots + \mathbb{Z}/s_d \mathbb{Z}$$

$$\mathcal{X}^{\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z})} \cong \mathcal{U}_1^{\mathbb{Z}/s_1 \mathbb{Z}} \times \cdots \times \mathcal{U}_d^{\mathbb{Z}/s_d \mathbb{Z}}$$

s_1, \dots, s_d pairwise coprime

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong \mathbb{Z}/s_1 \mathbb{Z} + \cdots + \mathbb{Z}/s_d \mathbb{Z}$$

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong u_1^{\mathbb{Z}/s_1 \mathbb{Z}} \times \cdots \times u_d^{\mathbb{Z}/s_d \mathbb{Z}}$$

$$\begin{aligned} \mathbb{K}[x]/(x^{s_1 \cdots s_d} - 1) &\cong \mathbb{K}[u_1]/(u_1^{s_1} - 1) \otimes \cdots \otimes \mathbb{K}[u_d]/(u_d^{s_d} - 1) \\ &\cong \mathbb{K}[u_1, \dots, u_d]/(u_1^{s_1} - 1, \dots, u_d^{s_d} - 1) \end{aligned}$$

s_1, \dots, s_d pairwise coprime

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong \mathbb{Z}/s_1 \mathbb{Z} + \cdots + \mathbb{Z}/s_d \mathbb{Z}$$

$$x^{\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z})} \cong u_1^{\mathbb{Z}/s_1 \mathbb{Z}} \times \cdots \times u_d^{\mathbb{Z}/s_d \mathbb{Z}}$$

$$\begin{aligned} \mathbb{K}[x]/(x^{s_1 \cdots s_d} - 1) &\cong \mathbb{K}[u_1]/(u_1^{s_1} - 1) \otimes \cdots \otimes \mathbb{K}[u_d]/(u_d^{s_d} - 1) \\ &\cong \mathbb{K}[u_1, \dots, u_d]/(u_1^{s_1} - 1, \dots, u_d^{s_d} - 1) \end{aligned}$$

Conclusion

$$\mathbb{K}[x]/(x^{s_1 \cdots s_d} - 1) \longrightarrow \mathbb{K}[u_1, \dots, u_d]/(u_1^{s_1} - 1, \dots, u_d^{s_d} - 1)$$

Achieved: s_1, \dots, s_d pairwise coprime

Required: $s_1 = \cdots = s_d = n$

s_1, \dots, s_d pairwise coprime

$$\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z}) \cong \mathbb{Z}/s_1 \mathbb{Z} + \cdots + \mathbb{Z}/s_d \mathbb{Z}$$

$$x^{\mathbb{Z}/(s_1 \cdots s_d \mathbb{Z})} \cong u_1^{\mathbb{Z}/s_1 \mathbb{Z}} \times \cdots \times u_d^{\mathbb{Z}/s_d \mathbb{Z}}$$

$$\begin{aligned} \mathbb{K}[x]/(x^{s_1 \cdots s_d} - 1) &\cong \mathbb{K}[u_1]/(u_1^{s_1} - 1) \otimes \cdots \otimes \mathbb{K}[u_d]/(u_d^{s_d} - 1) \\ &\cong \mathbb{K}[u_1, \dots, u_d]/(u_1^{s_1} - 1, \dots, u_d^{s_d} - 1) \end{aligned}$$

Conclusion

$$\mathbb{K}[x]/(x^{s_1 \cdots s_d} - 1) \longrightarrow \mathbb{K}[u_1, \dots, u_d]/(u_1^{s_1} - 1, \dots, u_d^{s_d} - 1)$$

Achieved: s_1, \dots, s_d pairwise coprime

Required: $s_1 = \cdots = s_d = n$

What if... possible to slightly change s_1, \dots, s_d ?

DFT of length $p=5$

$$\begin{pmatrix} A(1) \\ A(\omega^1) \\ A(\omega^2) \\ A(\omega^3) \\ A(\omega^4) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \in \mathbb{K}[x]/(x^5 - 1)$$

DFT of length $p=5$

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$$\begin{pmatrix} A(1) \\ A(\omega^{2^0}) \\ A(\omega^{2^1}) \\ A(\omega^{2^3}) \\ A(\omega^{2^2}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{2^0} & \omega^{2^1} & \omega^{2^3} & \omega^{2^2} \\ 1 & \omega^{2^1} & \omega^{2^2} & \omega^{2^0} & \omega^{2^3} \\ 1 & \omega^{2^3} & \omega^{2^0} & \omega^{2^2} & \omega^{2^1} \\ 1 & \omega^{2^2} & \omega^{2^3} & \omega^{2^1} & \omega^{2^0} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$1 = 2^0, \quad 2 = 2^1, \quad 3 = 2^3, \quad 4 = 2^2 \pmod{5}$$

DFT of length $p=5$

$$\begin{pmatrix} A(1) \\ A(\omega^{2^0}) \\ A(\omega^{2^1}) \\ \rightarrow A(\omega^{2^2}) \\ \rightarrow A(\omega^{2^3}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^{2^0} & \omega^{2^1} & \omega^{2^3} & \omega^{2^2} \\ 1 & \omega^{2^1} & \omega^{2^2} & \omega^{2^0} & \omega^{2^3} \\ \rightarrow 1 & \omega^{2^2} & \omega^{2^3} & \omega^{2^1} & \omega^{2^0} \\ \rightarrow 1 & \omega^{2^3} & \omega^{2^0} & \omega^{2^2} & \omega^{2^1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

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DFT of length $p=5$

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \omega^1 & \omega^2 & \omega^4 & \omega^3 \\ \omega^2 & \omega^4 & \omega^3 & \omega^1 \\ \omega^4 & \omega^3 & \omega^1 & \omega^2 \\ \omega^3 & \omega^1 & \omega^2 & \omega^4 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\Updownarrow$$

$$v_0 + v_1 x + v_2 x^2 + v_3 x^3 = (\omega^1 + \omega^2 x + \omega^4 x^2 + \omega^3 x^3) (u_0 + u_1 x + u_2 x^2 + u_3 x^3)$$

modulo $x^4 - 1$

DFT of length $p=5$

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\Updownarrow

$$v_0 + v_1 x + v_2 x^2 + v_3 x^3 = (\omega^1 + \omega^2 x + \omega^4 x^2 + \omega^3 x^3) (u_0 + u_1 x + u_2 x^2 + u_3 x^3) \text{ modulo } x^4 - 1$$

$$F(p) \leq M_{\mathbb{K}, \text{fixed}}^{\circ}(p-1) + 2p \cdot \text{add}_{\mathbb{K}}$$

$M_{\mathbb{K}}^{\circ}(n)$: cost of one multiplication in $\mathbb{K}[x]/(x^n - 1)$

$M_{\mathbb{K}, \text{fixed}}^{\circ}(n)$: when one argument is fixed

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$$v_0 + v_1 x + v_2 x^2 + v_3 x^3 = (\omega^1 + \omega^2 x + \omega^4 x^2 + \omega^3 x^3) (u_0 + u_1 x + u_2 x^2 + u_3 x^3) \text{ modulo } x^4 - 1$$

$$\begin{aligned} F_{\mathbb{K}}(p) &\leq M_{\mathbb{K},\text{fixed}}^{\circ}(p-1) + 2p \cdot \text{add}_{\mathbb{K}} \\ &\leq 2F_{\mathbb{K}}(p-1) + 2p \cdot \text{add}_{\mathbb{K}} \end{aligned}$$

$M_{\mathbb{K}}^{\circ}(n)$: cost of one multiplication in $\mathbb{K}[x]/(x^n - 1)$

$M_{\mathbb{K},\text{fixed}}^{\circ}(n)$: when one argument is fixed

Univariate reduction

FFT in $\mathbb{K}[x]/(x^p - 1)$ \longrightarrow multiplication in $\mathbb{K} \oplus \mathbb{K}[x]/(x^{p-1} - 1)$

Multivariate reduction

FFT in $\bigotimes_{1 \leq i \leq d} \mathbb{K}[x_i]/(x_i^{p_i} - 1)$ \longrightarrow multiplication in $\bigotimes_{1 \leq i \leq d} (\mathbb{K} \oplus \mathbb{K}[x_i]/(x_i^{p_i-1} - 1))$

Essentially

$\mathbb{K}[x]/(x^{p_1 \cdots p_d} - 1)$ \longrightarrow $\mathbb{K}[u_1, \dots, u_d]/(u_1^{p_1-1} - 1, \dots, u_d^{p_d-1} - 1)$

How to choose our primes

$$p_i = q_i 2^l + 1, \quad q_i \ll 2^l, \quad q_i \text{ odd prime}, \quad i = 1, \dots, d$$

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OK with “probability” l^{-1} for “random” prime with $q_i \ll 2^l$

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Note: we assumed q_i prime for convenience, but this is not really essential

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$l = 8$	$q = 3, 13, 31, 37, 157, 163, 181, 193, \dots$
$l = 16$	$q = 37, 103, 307, 313, 397, 421, 487, 541, \dots$
$l = 32$	$q = 43, 73, 157, 181, 211, 433, 571, 601, \dots$
$l = 64$	$q = 163, 337, 487, 907, 1051, 1297, 1453, 1567, \dots$
$l = 128$	$q = 1171, 2551, 3607, 3907, 4021, 4483, 4567, 4603, \dots$
$l = 256$	$q = 607, 1567, 1783, 2683, 2797, 4993, 6577, 6871, \dots$
$l = 512$	$q = 223, 2083, 2803, 3853, 4783, 9403, 9781, 10303, \dots$
$l = 1024$	$q = 1987, 4447, 15031, 22807, 26713, 46153, 46507, 47653, \dots$

How to choose our primes

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$$\begin{aligned} & \mathbb{K}[x_1, \dots, x_d] / (x_1^{p_1-1} - 1, \dots, x_d^{p_d-1} - 1) \\ & \cong \mathbb{K}[u_1, \dots, u_d, v_1, \dots, v_d] / (u_1^{q_1} - 1, \dots, u_d^{q_d} - 1, v_1^{2^l} - 1, \dots, v_d^{2^l} - 1) \\ & \cong \mathbb{K}[y, v_2, \dots, v_d] / (y^{q_1 \cdots q_d 2^l} - 1, v_2^{2^l} - 1, \dots, v_d^{2^l} - 1) \end{aligned}$$

$$p_i = q_i 2^l + 1, \quad q_i \ll 2^l, \quad q_i \text{ odd prime}, \quad i = 1, \dots, d$$

OK with “probability” l^{-1} for “random” prime with $q_i \ll 2^l$

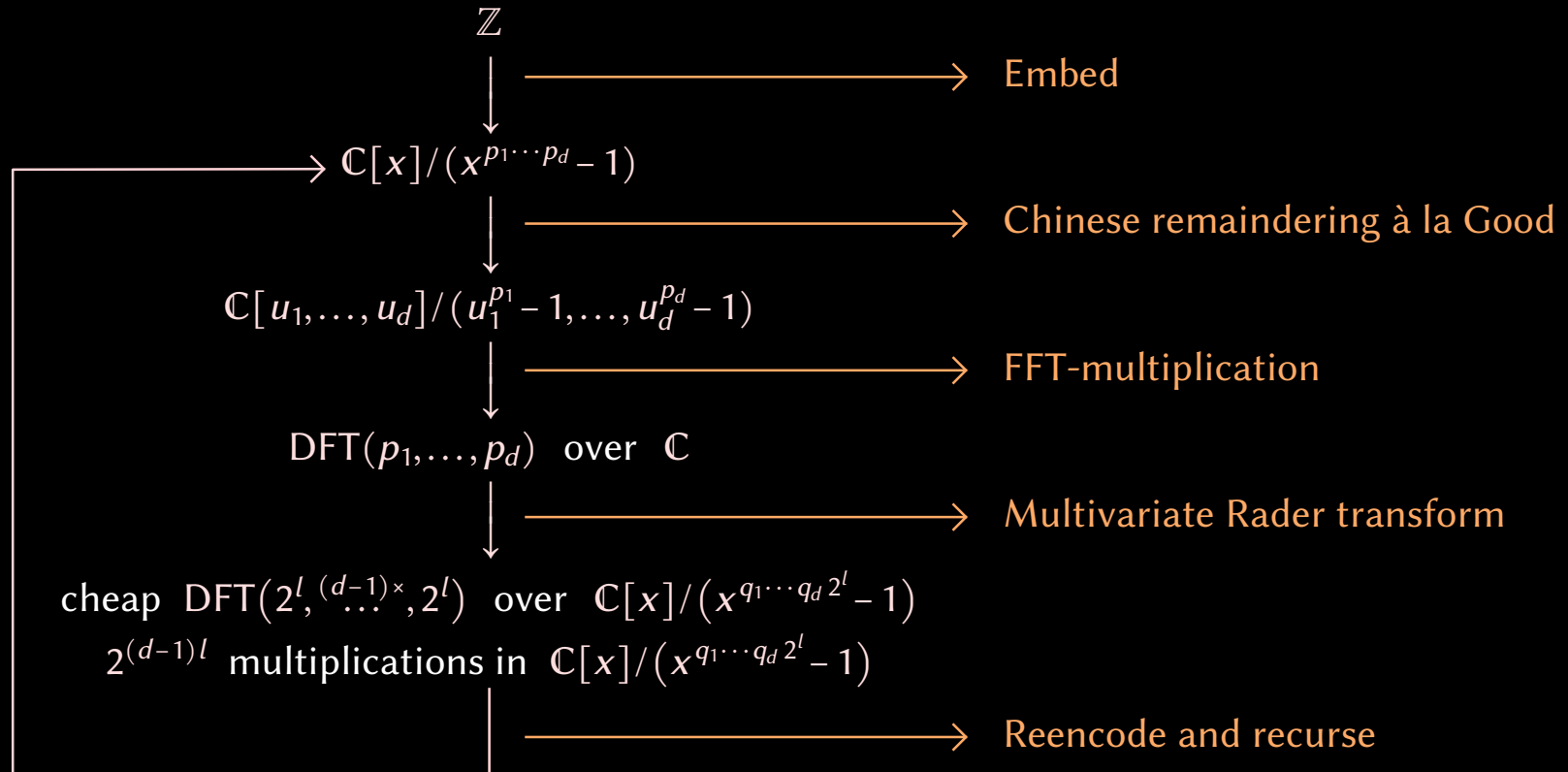
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Conclusion

$$\mathbb{K}[x] / (x^{p_1 \cdots p_d} - 1) \longrightarrow \mathbb{K}[y, v_2, \dots, v_d] / (y^{q_1 \cdots q_d 2^l} - 1, v_2^{2^l} - 1, \dots, v_d^{2^l} - 1)$$

$$M_{\mathbb{K}}^{\circ}(\underbrace{p_1 \cdots p_d}_{\geq 2^{(d+\epsilon)l}}) \leq 2^{(d-1)l} M_{\mathbb{K}}^{\circ}(\underbrace{q_1 \cdots q_d 2^l}_{2^{(1+\epsilon)l}}) + O(d 2^l \log 2^l \text{ add}_{\mathbb{K}})$$



Linnik constants

$$P(a, k) := \min \{ck + a : c \in \mathbb{N}, ck + a \text{ is prime}\}$$

$$P(k) := \max \{P(a, k) : 0 < a < k, a \wedge k = 1\}$$

$$L \text{ is a Linnik constant} : \Leftrightarrow P(k) = O(k^L)$$

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Theorem

If there exists a Linnik constant $L < 1 + \frac{1}{303}$, then

$$I(N) = O(N \log N).$$

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Theorem

If there exists a Linnik constant $L < 1 + \frac{1}{303}$, then

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Theorem

If there exists a Linnik constant $L < 1 + 2^{-1162}$, then

$$M_{\mathbb{F}_q}(n) = O(n \log q \log(n \log q)),$$

uniformly in q .

Thank you !



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