Computing Sparse Fourier Sum of Squares on Finite Abelian Groups

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Outline

Sparse Fourier SOS

Sparse FSOS of Integer Valued Functions

Lower Bounds of Functions on Finite Abelian Groups

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Sparse Fourier SOS

Sparse FSOS of Integer Valued Functions

Lower Bounds of Functions on Finite Abelian Groups

Group Theory

- $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ is the cyclic group of order n, with modular addition.
- If G is a finite abelian group, then

$$G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}.$$

for some integers $n_1, \ldots, n_k$.

Motivation and Problem Statement

- For $f : G \mapsto \mathbb{C}$

$$f \geq 0 \iff f \text{ has a Fourier SOS, i.e. } f = \sum_{i \in I} |g_i|^2.$$

- Applications: MAXSAT, MAXCUT.

MAXSAT Problem

- $C_2^n = \{-1,1\}^n \cong \mathbb{Z}_2^n$
- For $c = x_1 \vee \cdots \vee x_s \vee \neg x_{s+1} \vee \cdots \vee \neg x_n$, define its characteristic function:

$$f_c(y) = \frac{1}{2^{s+r}} \prod_{i=1}^{s} (1+y_i) \cdot \prod_{i=s+1}^{n} (1-y_i), \quad y \in C_2^n.$$

$$f_c(y) = \begin{cases} 1, & \text{if } y_1 = \cdots = y_s = 1(\text{false}), y_{s+1} = \cdots = y_n = -1(\text{true}) \\ 0, & \text{otherwise}. \end{cases}$$

- For the CNF formula $\phi = \bigwedge_{i=1}^{m} c_i$ in n variables, we define its characteristic function by

$$f_\phi(y) = \sum_{i=1}^{m} f_{c_i}(y), \quad y \in C_2^n.$$

- $f_\phi(y) = \#\{\text{falsified clauses in } \phi \text{ with assignment } y\}$.

Representation of Finite Abelian Group

Character

A nonzero complex function χ on finite abelian group G is called a character of G if it satisfies:

$$\chi(xy) = \chi(x)\chi(y), \quad \forall x, y \in G.$$

$\widehat{G}$: the set of all characters of G, also called Fourier basis, $\widehat{G} \simeq G$.

- Any $f : G \to \mathbb{C}$ has a unique form $f(x) = \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi(x)$.
- Support of f is $\mathrm{supp}(f) = \{\chi : \widehat{f}(\chi) \neq 0\}$.

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- Support of f is $\mathrm{supp}(f) = \{\chi : \widehat{f}(\chi) \neq 0\}$.

Example [Fawzi,Saunderson,Parrilo'16]

$f : \mathbb{Z}_6 \to \mathbb{C}, \; f(x) = 1 - \frac{1}{2}(\chi_1(x) + \chi_5(x)), \quad \chi_k(x) = e^{\frac{2ik\pi x}{6}}, \; k = 0, 1, 2, \cdots, 5.$

Gram Matrix and FSOS

- G : finite abelian group.
- $f : G \to \mathbb{C}$ map.
- $f \geq 0 \iff \exists Q = (Q_{\chi,\chi'})_{\chi,\chi' \in \widehat{G}} \in \mathbb{C}^{\widehat{G} \times \widehat{G}}$, s.t.

$$Q \succeq 0, \quad \sum_{\chi' \in \widehat{G}} Q_{\chi',\chi'\chi} = \widehat{f}(\chi), \quad \forall \chi \in \widehat{G}$$

$$\iff Q = M^* M, \; M = (M_{j,\chi})_{1 \leq j \leq r, \chi \in \widehat{G}} \in \mathbb{C}^{r \times \widehat{G}}$$

$$f = \sum_{j=1}^{r} |\sum_{\chi \in \widehat{G}} M_{j,\chi} \chi|^2 = \sum_{j=1}^{r} |g_j|^2.$$

- Q : Gram matrix of f, not unique.
- Sparsity of FSOS $f = \sum_{j=1}^{r} |g_j|^2$ is $\left|\bigcup_{j=1}^{r} \mathrm{supp}(g_j)\right|$.

Example of Gram Matrix

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- Gram matrix of f:

$$\frac{1}{6} \cdot \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{array} \begin{array}{c} \begin{array}{cccccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 \end{array} \\ \begin{bmatrix} 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 1 \end{bmatrix} \end{array}.$$

FSOS Sparsity

Gram matrix $Q \Rightarrow$ FSOS (by Cholesky decomposition).

$$
\begin{aligned}
f = \quad & \frac{1}{6}\left|\frac{\chi_5}{2}+\frac{\chi_1}{2}-1\right|^2+\frac{1}{6}\left|\frac{\sqrt{3}\,\chi_5}{6}+\frac{\sqrt{3}\,\chi_2}{3}-\frac{\sqrt{3}\,\chi_1}{2}\right|^2 \\
& +\frac{1}{6}\left|\frac{\sqrt{6}\,\chi_5}{12}+\frac{\sqrt{6}\,\chi_3}{4}-\frac{\sqrt{6}\,\chi_2}{3}\right|^2+\frac{1}{6}\left|\frac{\sqrt{10}\,\chi_5}{20}+\frac{\sqrt{10}\,\chi_4}{5}-\frac{\sqrt{10}\,\chi_3}{4}\right|^2 \\
& +\frac{1}{6}\left|\frac{\sqrt{15}\,\chi_4}{5}-\frac{\sqrt{15}\,\chi_5}{5}\right|^2
\end{aligned}
$$

Sparsity of FSOS of f is 6.

Example of FSOS

- $f : \mathbb{Z}_6 \to \mathbb{C},\ f(x) = 1 - \frac{1}{2}(\chi_1(x) + \chi_5(x))$, $\chi_k(x) = e^{\frac{2ik\pi x}{6}}$.
- f admits a FSOS with sparsity 4. [Fawzi,Saunderson,Parrilo'16]

Example of FSOS

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- f admits a FSOS with sparsity 4. [Fawzi,Saunderson,Parrilo'16]
- f admits a FSOS with sparsity 2. [Yang,Ye,Zhi'22a]
  - Gram matrix of f:

$$
\begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 & \chi_4 & \chi_5 \end{array} \\
\begin{bmatrix}
1/2 & -1/2 & 0 & 0 & 0 & 0 \\
-1/2 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

  - $f(x) = \frac{1}{2}\left|1 - \chi_1(x)\right|^2$.

Sparse FSOS

Finding a sparse FSOS is equivalent to solving the minimization problems:

$$\min_{f=\sum_i g_i \overline{g_i}} \# \left(\bigcup_i \operatorname{supp}(g_i) \right)$$

$$\Updownarrow$$

$$\min_{Q\text{: Gram matrix}} \| \operatorname{diag}(Q) \|_{\ell^0}.$$

- $\operatorname{diag}(Q)$: diagonal elements of Q.

Convex Relaxation

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Convex Relaxation

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- Group case: $\|\operatorname{diag}(Q)\|_{\ell^1} = \text{constant}$:

$$\|\operatorname{diag}(Q)\|_{\ell^1} = \sum_{\chi \in \widehat{G}} Q(\chi, \chi) = \sum_{\chi \in \widehat{G}} Q(\chi, \chi\chi_0) = \widehat{f}(\chi_0).$$

$\chi_0(x) = 1, \forall x \in G$ is the character of the trivial representation of G.

Convex Relaxation

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- Group case: $\|\operatorname{diag}(Q)\|_{\ell^1} =$ constant:

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$\chi_0(x) = 1, \forall x \in G$ is the character of the trivial representation of G.

Theorem [Yang,Ye,Zhi'22a]

$$\min_{Q:\ \text{Gram matrix}} \|\operatorname{diag}(Q)\|_{\ell^0} = \min_{\substack{Q(\chi_0,\chi_0)\neq 0\\ Q:\ \text{Gram matrix}}} \#\{\chi \neq \chi_0 : Q(\chi,\chi) \neq 0\} + 1,$$

Convex Relaxation

The convex relaxation problem for

$$\min_{\substack{Q(\chi_0,\chi_0)\neq 0\\ Q:\ \text{Gram matrix}}} \#\{\chi \neq \chi_0 : Q(\chi,\chi) \neq 0\}$$

can be formulated as an SDP problem:

$$\begin{aligned} \min_{Q\in\mathbb{C}^{\widehat{G}\times\widehat{G}}} \quad & \operatorname{trace}(Q) - Q(\chi_0,\chi_0) \\ \text{s.t.} \quad & Q \succeq 0, \quad \sum_{\chi'\in\widehat{G}} Q_{\chi',\chi'\chi} = \widehat{f}(\chi), \quad \forall \chi \in \widehat{G} \end{aligned}$$

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- Complexity of SDP: $\geq O(|G|^4)$.
- Closed form solution[Yang,Ye,Zhi'22a]: $O(|G|\log(|G|))$ (FFT).

Solution to the Relaxed Problem

Theorem [Yang,Ye,Zhi'22a]

Let $f : G \mapsto \mathbb{R}$, $f \geq 0$, h be its square root:

$$h(x) = \sqrt{f(x)} = \sum_{\chi \in \widehat{G}} a_\chi \chi(x), \quad x \in G,$$

and

$$Q_0(\chi, \chi') = \overline{a_\chi} a_{\chi'}.$$

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- Closed form solution.
- Complexity of FFT: $O(|G| \log(|G|))$

Example of FSOS (again)

- $f : \mathbb{Z}_6 \to \mathbb{C},\ f(x) = 1 - \frac{1}{2}(\chi_1(x) + \chi_5(x)) = 1 - \cos(\frac{2\pi x}{6}),\ \chi_k(x) = e^{\frac{2ik\pi x}{6}}$.
- By the fast Fourier transform, we have

$$\sqrt{f} = \left(\frac{\sqrt{2}}{3} + \frac{\sqrt{6}}{6}\right) - \left(\frac{\sqrt{2}}{12} + \frac{\sqrt{6}}{12}\right)(\chi + \chi^{-1}) + \left(\frac{\sqrt{2}}{12} - \frac{\sqrt{6}}{12}\right)(\chi^2 + \chi^{-2}) + \left(\frac{\sqrt{6}}{6} - \frac{\sqrt{2}}{3}\right)\chi^3.$$

- We obtain a rank one Gram matrix $Q_0 = u^* u$,

$$u = \left[\frac{\sqrt{2}}{3} + \frac{\sqrt{6}}{6} \quad -\frac{\sqrt{2}}{12} - \frac{\sqrt{6}}{12} \quad \frac{\sqrt{2}}{12} - \frac{\sqrt{6}}{12} \quad \frac{\sqrt{6}}{6} - \frac{\sqrt{2}}{3} \quad \frac{\sqrt{2}}{12} - \frac{\sqrt{6}}{12} \quad -\frac{\sqrt{2}}{12} - \frac{\sqrt{6}}{12}\right]$$

- Q_0 is not sparse, its large diagonal elements are useful for FSOS!

Error Bound by Square Root

Theorem [Yang,Ye,Zhi'22a]

Let $f: G \mapsto \mathbb{R}$, $f \geq 0$. Let S be a subset of $\widehat{G}$ s.t. $\mathrm{supp}(f) \subseteq S$, $S = S^{-1}$.

- Define h as the truncation of $\sqrt{f}$ at S, i.e.

$$\widehat{h}(\chi) = \begin{cases} \widehat{\sqrt{f}}(\chi), & \text{if } \chi \in S, \\ 0, & \text{if } \chi \notin S. \end{cases}$$

- The function $f + M$ has an FSOS with support S for

$$M := 2\|\widehat{\sqrt{f}} - \widehat{h}\|_{\ell^1} \cdot \|\widehat{h}\|_{\ell^1} + \|\widehat{\sqrt{f}} - \widehat{h}\|_{\ell^1}^2.$$

Main Steps of Our Algorithm

- Compute Q_0 by FFT, sort diagonal elements of Q_0 by $\sigma \in \mathfrak{S}_{|G|}$

$$Q_0(\chi_{\sigma(1)}, \chi_{\sigma(1)}) \geq Q_0(\chi_{\sigma(2)}, \chi_{\sigma(2)}) \geq \cdots \geq Q_0(\chi_{\sigma(|G|)}, \chi_{\sigma(|G|)}).$$

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- Find the minimal k s.t $S_k \neq \emptyset$ via an SDP solver, where

$$S_k = \{Q \in \mathbb{C}^{\widehat{G} \times \widehat{G}} : \text{Gram matrix of } f, \text{ with supports } \chi_{\sigma(1)}, \ldots, \chi_{\sigma(k)}\}.$$

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Theorem [Yang,Ye,Zhi'22a]

The total complexity of our algorithm with input $f : G \mapsto \mathbb{R}$ is at most

$$\mathrm{O}(|G| \log |G| + \mathrm{poly}(t)),$$

which is quasi-linear in $|G|$, and polynomial in the FSOS sparsity t.

Numerical Experiment: Bounded Degree Case

group	FSOS sparsity	time(s)	bound
$\mathbb{Z}_{10000}$	16.7	1.49	648
$\mathbb{Z}_{20000}$	18.6	2.42	720
$\mathbb{Z}_{30000}$	19	2.82	792
$\mathbb{Z}_{40000}$	17.8	3.03	792
$\mathbb{Z}_{50000}$	18.8	3.38	864
$\mathbb{Z}_{60000}$	19	3.89	864

Table: Bounded degree (not greater than 25)

Theoretical bounds [Fawzi,Saunderson,Parrilo'16]: $3d\log_2(N/d)$.

Numerical Experiment: Product Groups

group	FSOS sparsity	time(s)
$\mathbb{Z}_{500} \times \mathbb{Z}_{500}$	51.4	24.3
$\mathbb{Z}_{1000} \times \mathbb{Z}_{1000}$	50.8	63.3
$\mathbb{Z}_{1500} \times \mathbb{Z}_{1500}$	49.2	123.3
$\mathbb{Z}_{2000} \times \mathbb{Z}_{2000}$	49.6	208.4
$\mathbb{Z}_{2500} \times \mathbb{Z}_{2500}$	50.6	318.6
$\mathbb{Z}_{3000} \times \mathbb{Z}_{3000}$	50.2	457.6
$\mathbb{Z}_{3500} \times \mathbb{Z}_{3500}$	49.2	632.8
$\mathbb{Z}_{4000} \times \mathbb{Z}_{4000}$	49.8	831.7
$\mathbb{Z}_{4500} \times \mathbb{Z}_{4500}$	48.2	1066.0
$\mathbb{Z}_{5000} \times \mathbb{Z}_{5000}$	50.6	1325.1

Table: Bounded FSOS support (at least 10).

URL: github.com/jty-AMSS/FSOS

Outline

Sparse Fourier SOS

Sparse FSOS of Integer Valued Functions

Lower Bounds of Functions on Finite Abelian Groups

Integer-valued Functions over Finite Abelian Groups

- Integer-valued function: $f : G \mapsto \mathbb{Z}$. $\max_{x \in G} |f(x)| = \mathrm{O}(\mathrm{polylog}(|G|))$.

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Integer-valued function $f \geq L,\ L \in \mathbb{Z}$.

$$\Updownarrow$$

$$\max_{x \in G} \left| (f(x) - L) - \sum_{j \in J} |g_j(x)|^2 \right| < 1, \text{ for some } \{g_j\}_{j \in J}.$$

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- We call $\{g_j\}_{j \in J}$ a certificate of $f \geq L$.

FSOS with Error

Theorem [Blekherman,Gouveia,Pfeiffer'16]

$$f(x_1,\ldots,x_n) = \left(\sum_{j=1}^{n} x_j - \left\lfloor \frac{n}{2} \right\rfloor\right)\left(\sum_{j=1}^{n} x_j - \left\lfloor \frac{n}{2} \right\rfloor - 1\right)$$

is non-negative on $\{0,1\}^n$, f has no polynomial or rational FSOS of degree less than $\frac{n-1}{2}$.

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- FSOS certificate of degree 1 and sparsity $n+1$ [Yang,Ye,Zhi'22b]:

$$f \text{ integer-valued, } f(x_1,\ldots,x_n) = \left(\sum_{j=1}^{n} x_j - \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

$$\Longrightarrow \left| f(x) - \left(\sum_{j=1}^{n} x_j - \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{2}\right)^2 \right| = \frac{1}{4} < 1,$$

$$\Longrightarrow f \geq 0.$$

Integer-valued Functions over Finite Abelian Groups

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Integer-valued Functions over Finite Abelian Groups

- If $G = \mathbb{Z}_2^n$, complexity of FFT: $O(n \cdot 2^n)$.
- Question: can we compute the FSOS faster (polynomial in n)?

Polynomial Approximation of $\sqrt{f(x)}$

Suppose

- $0 \le f(x) \le m$, $\|f\|_{\ell^\infty} = \max_{x \in G} |f(x)|$.
- polynomial $q : \mathbb{R} \mapsto \mathbb{R}$ satisfying $|q(x) - \sqrt{x}| < \varepsilon, \forall x \in [0, m]$.

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Then we have

- $\|q(f(x)) - \sqrt{f(x)}\|_{\ell^\infty} < \varepsilon$, $\left\|q(f(x))^2 - f(x)\right\|_{\ell^\infty} \le 2\sqrt{m}\varepsilon + \varepsilon^2$.

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- error of Fourier coefficients satisfies:

$$\left|\widehat{q(f)}(\chi) - \widehat{\sqrt{f}}(\chi)\right| < \varepsilon,\ \forall \chi \in \widehat{G}.$$

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Remark: sparsity of f is $s \Rightarrow$ sparsity of $q(f) \le s^{\deg q}$.

Low Degree Polynomial Approximation

Theorem [Stoer,Bulirsch'02]

Let $a < b$ be two real numbers. For each $p \in C^{d+1}([a,b])$, we have

$$\max_{t\in[a,b]} |p(t) - q(t)| \leq \left(\frac{b-a}{2}\right)^{d+1} \frac{\max_{y\in[a,b]} |p^{(d+1)}(t)|}{2^d (d+1)!},$$

q is the degree d Chebyshev interpolation polynomial for p on $[a,b]$.

Low Degree Polynomial FSOS Certificate

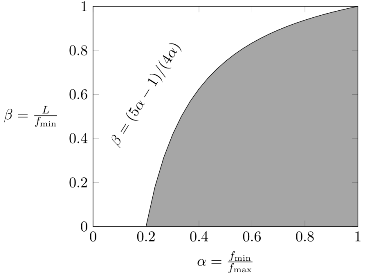
Applying the above result to $p(t) := \sqrt{t}$ on $[\alpha, 1]$ for some $\alpha > 0$, we have

Theorem [Yang,Ye,Zhi'22b]

Let $f : G \mapsto \mathbb{Z}$, if $f_{\max} < (5-4\beta) f_{\min}$, $\beta \in [0,1)$, there exists a rank-one FSOS certificate g, $\| f - L - |g|^2 \|_{\ell^\infty} < 1$ for $f \geq L := \lceil \beta f_{\min} \rceil$ s.t.

$$\deg(g) \leq \deg(f) \left\lfloor \frac{3 + \log(f_{\max} - L)}{2 + \log(f_{\min} - L) - \log(f_{\max} - f_{\min})} \right\rfloor$$

- $\deg(g)$ is bounded by $O(\log(f_{\max}))$.



Impossibility Theorem

- It is impossible to approximate the square root function exponentially and uniformly [Varga'87].

Theorem [Yang,Ye,Zhi'22b]

Let $0 < \varepsilon < 1/6$ and p be a polynomial such that $\left|p(i) - \sqrt{i}\right| \leq \varepsilon$ for each integer $0 \leq i \leq m$, then

$$\deg p \geq \sqrt{\frac{(1-2\varepsilon)m}{1+\sqrt{m}}} = O(m^{\frac{1}{4}}).$$

Low Degree Rational Function Approximation

Let E_d be the approximation error of rational functions $r(t)$ to $\sqrt{t}$ on $[0,1]$:

$$E_d := \inf_{r \in R_d} \left\{ \max_{t\in[0,1]} |\sqrt{t} - r(t)| \right\}.$$

[Vyacheslavov'75,Stahl'03]

There exists some constant $C > 0$ s.t. for each $d \in \mathbb{N}$, it holds that

$$\frac{1}{3} e^{-2\pi\sqrt{\frac{d}{2}}} \leq E_d \leq C e^{-2\pi\sqrt{\frac{d}{2}}}.$$

Lower Degree Rational FSOS Certificate

Theorem [Yang,Ye,Zhi'22b]

Let $f : G \mapsto \mathbb{Z}$, $L \leq f_{\min}$, there exists a rational FSOS certificate (g,h) for $f \geq L$,

$$\|f - L - \frac{|g|^2}{|h|^2}\|_{\ell^\infty} < 1,$$

with

$$\deg g = \deg h \leq \deg(f) \left\lceil \frac{(\log(f_{\max} - L) + c)^2}{2\pi^2} \right\rceil.$$

for some constant $c > 0$.

Lower Degree Rational FSOS Certificate

Theorem [Yang,Ye,Zhi'22b]

Let $f : G \mapsto \mathbb{Z}$, $L \leq f_{\min}$, there exists a rational FSOS certificate (g,h) for $f \geq L$,

$$\|f - L - \frac{|g|^2}{|h|^2}\|_{\ell^\infty} < 1,$$

with

$$\deg g = \deg h \leq \deg(f) \left\lceil \frac{(\log(f_{\max} - L) + c)^2}{2\pi^2} \right\rceil.$$

for some constant $c > 0$.

- Rational SOS: $O(\log^2(f_{\max}))$
- Polynomial SOS : $O((f_{\max})^{\frac{1}{4}})$

Validation by ℓ^1-norm

- $\| f - L - \frac{\sum_{j\in J} |g_j|^2}{\sum_{i\in I} |h_i|^2} \|_{\ell^\infty} < 1$.

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Theorem [Yang,Ye,Zhi'22b]

Suppose $h_i, g_j : G \mapsto \mathbb{R}$ satisfy

- Gram matrix V of $\sum_{i\in I}|h_i|^2$, $V \succeq \mathrm{Id}$.
- error function $e(y) = \sum_{j\in J}|g_j(y)|^2 - (\sum_{i\in I}|h_i(y)|^2)\cdot(f(y) - L)$.
- $\|\widehat{e}\|_{\ell^1} \leq \frac{1}{2}$, sum of absolute value of Fourier coefficients is less than $\frac{1}{2}$.

Then $\|f - L - \frac{\sum_{j\in J}|g_j|^2}{\sum_{i\in I}|h_i|^2}\|_{\ell^\infty} \leq \frac{1}{2}$.

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Then $\|f - L - \frac{\sum_{j\in J}|g_j|^2}{\sum_{i\in I}|h_i|^2}\|_{\ell^\infty} \le \frac{1}{2}$.

The condition $\|\widehat{e}\|_{\ell^1} \le \frac{1}{2}$ can be verified by solving an SDP problem.

Validation by Sampling on $G = C_2^n$

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Theorem [Yang,Ye,Zhi'22b]

Let $f:\Gamma_2^n \mapsto \mathbb{Z}$, $(\{g_j\}_{j\in J}, \{h_i\}_{i\in I})$ is a pair of families of functions on Γ_2^n s.t.

(i) $\sum_{i\in I} |h_i|^2 \geq 1$;

(ii) $d := 2\max_{j\in J, i\in I}\{\deg g_j, \deg h_i\}$,$\deg(f) + d \leq n$ ($d = O(\log^2 n)$).

(iii) $\left|\sum_{j\in J} |g_j(y)|^2 - \left(\sum_{i\in I} |h_i(y)|^2\right)(f(y) - L)\right| \leq n^{-2(\deg(f)+d)}$ for $y \in \Gamma_2^n$ s.t.

$$\sum_{i=1}^{n} y_i \leq 2\deg(f) - n + 2d.$$

Then $(\{g_j\}_{j\in J}, \{h_i\}_{i\in I})$ is a rational FSOS certificate for $f \geq L$.

- It is sufficient to check $\mathbf{n}^{\mathbf{O}(\log^2(\mathbf{n}))}$ many inequalities.

Outline

Sparse Fourier SOS

Sparse FSOS of Integer Valued Functions

Lower Bounds of Functions on Finite Abelian Groups

FSOS with Error

Theorem [Yang,Ye,Zhi'23]

- Let $f : G \mapsto \mathbb{R}$, G is a finite abelian group, $S \subseteq \widehat{G}$.
- Let λ be the minimal eigenvalue of Hermitian matrix $Q \in \mathbb{C}^{S\times S}$,

$$e := f - v_S^* Q v_S, \ v_S := (\chi)_{\chi \in S}.$$

We have

- f is bounded below by

$$\min_{x\in G} f(x) \geq -\|\widehat{e}\|_{\ell^1} + \lambda |S|.$$

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$$\min_{x\in G} f(x) \geq -\|\widehat{e}\|_{\ell^1} + \lambda |S|.$$

- $Q - \lambda \, \mathrm{Id} \succeq 0$ is a Gram matrix of the function $f - e - \lambda |S|$.

Difference between finite abelian group G and $\mathbb{R}$

Let $f(z_1, z_2) = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2z_1z_2$.

- on $C_2^2 = \{-1, 1\}^2$, we have

$$f = |z_1 + z_2|^2 - (|z_1|^2 + |z_2|^2) \geq -(|z_1|^2 + |z_2|^2) \geq -2.$$

- on $\mathbb{R}^2$:

$$f = (z_1 + z_2)^2 - (z_1^2 + z_2^2) \geq \min_{(z_1, z_2) \in \mathbb{R}^2} -(z_1^2 + z_2^2) = -\infty$$

Lower Bound by FSOS

Given $S \subseteq \widehat{G}$, compute a lower bound of $f : G \to \mathbb{R}$ by FSOS

$$\max_{\mathrm{supp}(h_i) \subseteq S} \alpha, \quad \text{s.t. } f - \alpha = \sum_{i \in I} |h_i|^2.$$

It is equivalent to the SDP problem:

$$\begin{aligned} \max\nolimits_{Q \in \mathbb{C}^{S \times S}} \quad & \widehat{f}(\chi_0) - \mathrm{trace}(Q), \\ \text{s.t.} \quad & \sum_{\chi' \in \widehat{G}} Q_{\chi', \chi'\chi} = \widehat{f}(\chi), \ \chi \neq \chi_0 \in \widehat{G} \\ & Q \succeq 0 \end{aligned}$$

Computation of Lower Bounds

- Any Hermitian matrix $Q \in \mathbb{C}^{S\times S}$ gives a lower bound of f,
- We solve the unconstrained optimization problem

$$\max_{Q=Q^*\in\mathbb{C}^{S\times S}} \widehat{f}(\chi_0) - F(Q)$$

where

$$F : \mathbb{C}^{S\times S} \mapsto \mathbb{R},\ F(Q) = \mathrm{trace}(Q) + \|E(Q)\|_{\ell^1} - \lambda_{\mathsf{min}}(Q)|S|,$$

and $E(Q) = \widehat{e - e_0}$, $e := f - v_S^* Q v_S$.

Solving Unconstrained Optimization Problem

Advantages of minimizing F without constraints:

- The subgradient of F is given explicitly by

$$\partial F = \mathrm{Id} + (\partial E)^* \, \mathrm{sign}(E(Q)) - |S| u u^*,$$

  $\mathrm{sign}(x)$: sign function, u: unit eigenvector of Q w.r.t. $\lambda_{\min}(Q)$.
- Early termination;
- Adaptive to more SDP solvers;
- Size reduction.

Random examples

For group $G = C_2^{25}$, C_3^{15}, C_5^{10} randomly generate $f : G \to \mathbb{R}$ with sparsity at least 450 or 200, $f_{min} = 1$.

No	group	sp	Our Algorithm		TSSOS		CS-TSSOS	
			bound	time	bound	time	bound	time
1	C_2^{25}	451	1.00	1058.00	1.00	1027.03	1.00	1451.53
2	C_2^{25}	451	0.67	867.38	-8.72	853.47	-7.23	1483.55
3	C_2^{25}	451	0.75	773.06	1.00	1442.03	1.00	1846.23
4	C_2^{25}	451	0.99	906.15	1.00	1519.08	1.00	1831.33
5	C_2^{25}	451	0.02	718.21	1.00	1364.66	1.00	1710.47
6	C_3^{15}	203	0.97	327.58	–	–	1.00	7336.23
7	C_3^{15}	203	1.00	223.73	–	–	1.00	2876.34
8	C_3^{15}	203	1.00	212.43	–	–	1.00	1353.14
9	C_5^{10}	201	0.97	191.54	–	–	-	-
10	C_5^{10}	201	0.94	236.33	–	–	1.00	559.68
11	C_5^{10}	213	0.90	233.82	–	–	-	-

TSSOS, CS-TSSOS: [Wang,Victor,Lasserre'22].

MAX-2SAT benchmark with 120 variables

No	clause	min	Our Algorithm		TSSOS		CS-TSSOS	
			bound	time	bound	time	bound	time
1	1200	161	159.5	370	146.7	45	146.7	52
2	1200	159	156.7	327	143.1	49	143.1	55
3	1200	160	159.0	362	146.8	46	146.8	64
4	1300	180	177.5	450	162.4	52	162.4	73
5	1300	172	170.6	417	156.2	47	156.2	65
6	1300	173	171.6	432	158.8	44	158.8	58
7	1400	197	194.8	506	179.8	46	179.8	75
8	1400	191	189.3	499	174.3	51	174.3	87
9	1400	189	187.2	504	172.1	58	172.1	78

Table: Unweighted MAX-2SAT problems

TSSOS, CS-TSSOS run out of memory when the relaxation order ≥ 2.

Computation of Feasible Solutions

Rounding by Gram Matrix:

- Compute $v \in \mathbb{R}^S$, $(Q - \lambda_{\min}(Q)\,\mathrm{Id})v = 0,\ v(\chi_0) = 1$.
- Recover $g \in C_2^n$ by $\mathrm{sign}(v)$.

$f(z_1,z_2,z_3) = 4 + z_1 + z_2 + z_3 + z_1z_2z_3 \geq 0$ on C_2^3

- we obtain a Hermitian matrix by SDPNAL+:

$$Q = \begin{array}{c} \\ 1 \\ z_1 \\ z_2 \\ z_3 \\ z_1z_2z_3 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & z_1 & z_2 & z_3 & z_1z_2z_3 \end{array} \\ \begin{bmatrix} 1.9546 & 0.5000 & 0.5000 & 0.5000 & 0.5000 \\ 0.5000 & 0.4536 & 0.0000 & 0.0000 & 0.0000 \\ 0.5000 & 0.0000 & 0.4536 & 0.0000 & 0.0000 \\ 0.5000 & 0.0000 & 0.0000 & 0.4536 & 0.0000 \\ 0.5000 & 0.0000 & 0.0000 & 0.0000 & 0.4536 \end{bmatrix} \end{array},$$

whose eigenvalues are $-0.0462, 0.4536, 0.4536, 0.4536, 2.4544.$

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  whose eigenvalues are $-0.0462, 0.4536, 0.4536, 0.4536, 2.4544$.

- The normalized null vector of $Q + 0.0462\,\mathrm{Id}$ is

$$v = \begin{array}{c} \begin{array}{ccccc} 1 & z_1 & z_2 & z_3 & z_1z_2z_3 \end{array} \\ \begin{bmatrix} 1 & -1.000447 & -1.000447 & -1.000447 & -1.000447 \end{bmatrix} \end{array}.$$

$f(z_1, z_2, z_3) = 4 + z_1 + z_2 + z_3 + z_1 z_2 z_3 \geq 0$ on C_2^3

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- $f_{min} = 0$ is achieved at $z_1 = z_2 = z_3 = -1$ (rounding elements of v).

MAX-2SAT benchmark with 120 variables.

No	clause	min	Gram	ρ_i^N	$2^{-(i-1)}$
1	1200	161	162	225	227
2	1200	159	159	215	194
3	1200	160	160	162	160
4	1300	180	180	226	243
5	1300	172	173	225	230
6	1300	173	173	245	253
7	1400	197	198	234	270
8	1400	191	192	255	246
9	1400	189	189	227	231

Table: rounding on MAX-2SAT benchmarks

- "Gram" : our rounding methods
- ρ_i^N and $2^{-(i-1)}$: rounding methods in [Maaren,Norden,Heule'08].

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Thank you for your attention!