Interpolating isogenies

and applications...

Benjamin Wesolowski, CNRS and ENS de Lyon

28 September 2023, Fundamental Algorithms and Algorithmic Complexity, IHP, Paris, France

Isogenies

Elliptic curves, isogenies, computational problems

Elliptic curves

Elliptic curve over $\mathbb{F}_q$: solutions (x,y) in $\mathbb{F}_q$ of

$$y^2 = x^3 + ax + b$$

$E(\mathbb{F}_q)$ is an additive group

Isogeny: a map

$$\varphi : E_1 \longrightarrow E_2$$

which preserves certain structures. In particular, it is a group homomorphism with a finite kernel

The **degree*** is $\deg(\varphi) = \#\ker(\varphi)$

* for separable isogenies

- $\deg(\varphi \circ \psi) = \deg(\varphi) \cdot \deg(\psi)$

The isogeny problem

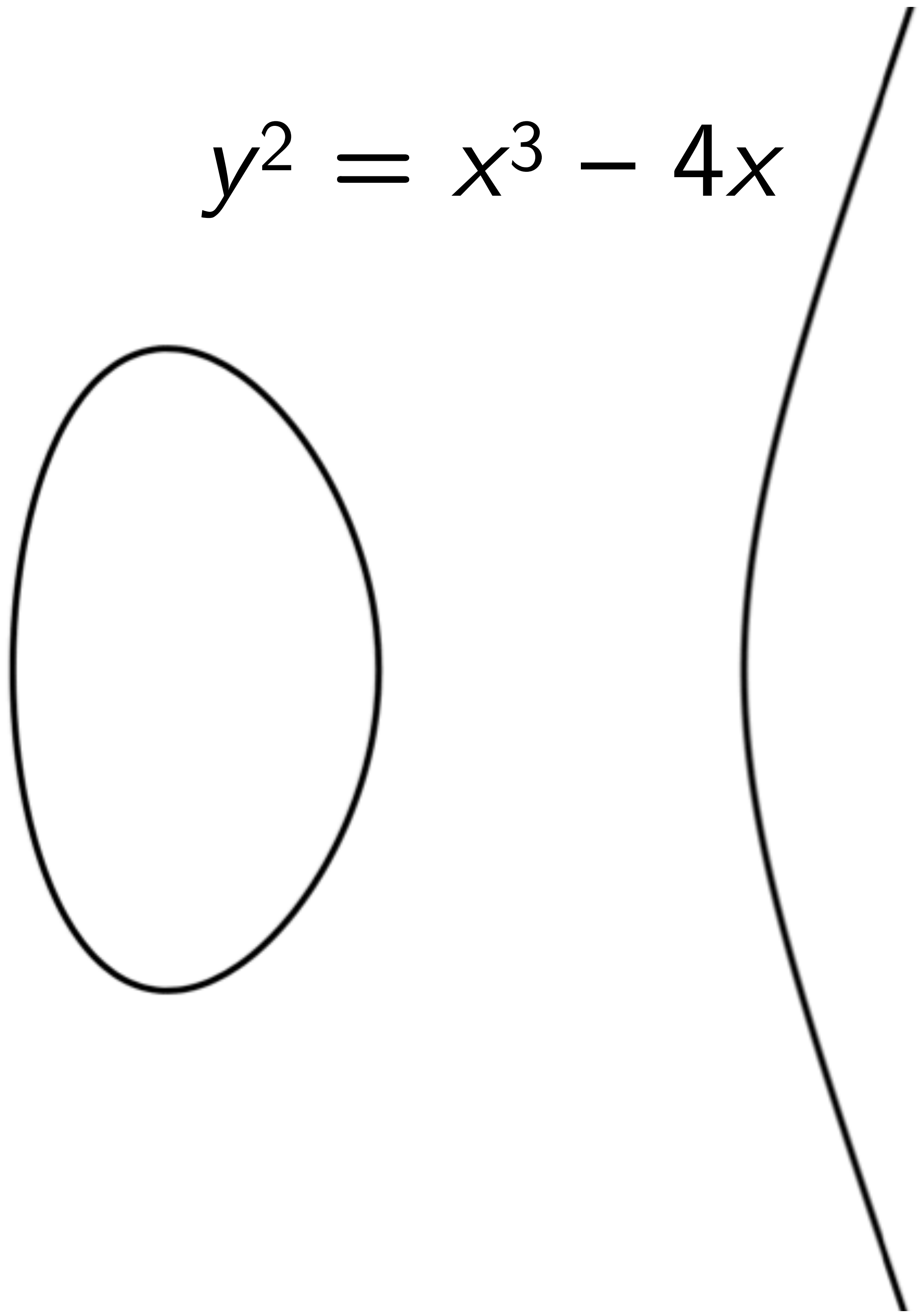
Isogeny problem: Given two elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$

$y^2 = x^3 + x$

elliptic curves:

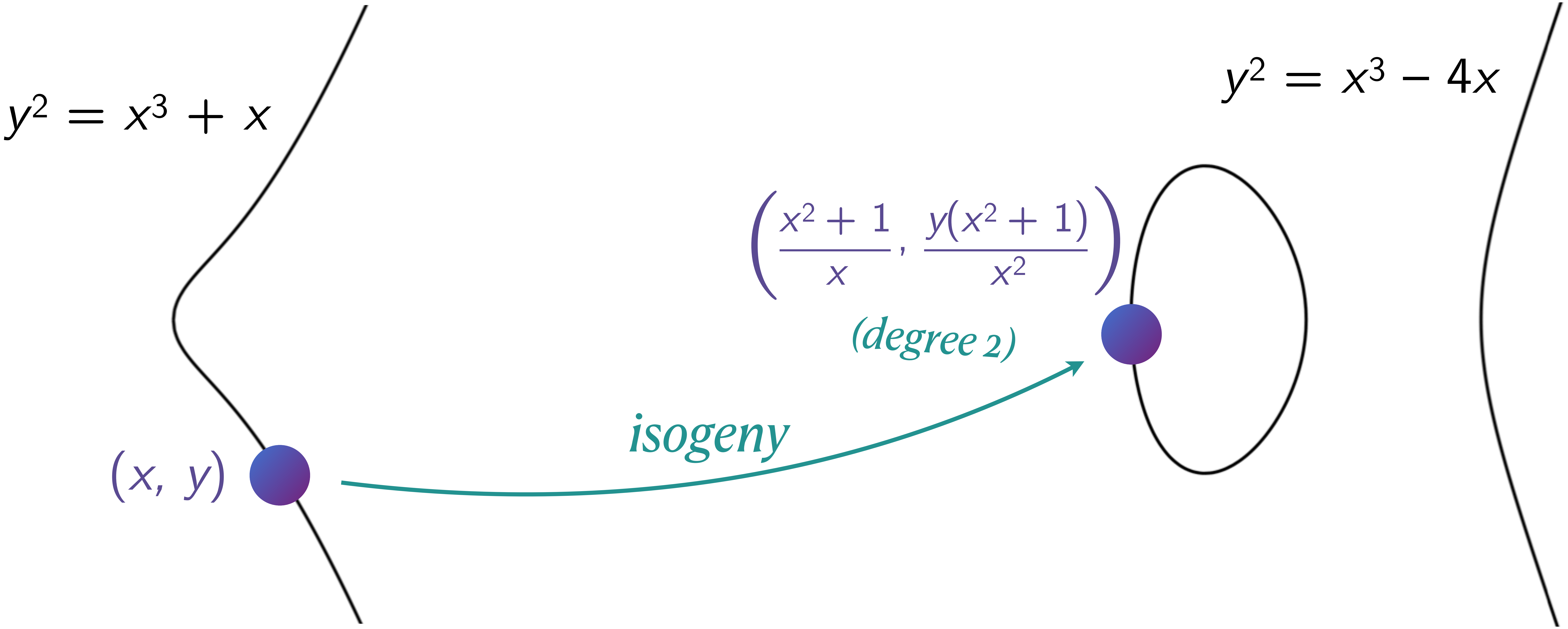
$$y^2 = x^3 + ax + b$$

$y^2 = x^3 - 4x$



The isogeny problem

Isogeny problem: Given two elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$



Applications of isogeny computation

- For the arithmetic of elliptic curves:
  - ➡ **counting points** over a finite field,
  - ➡ computing **endomorphism rings**,
  - ➡ computing **modular polynomials**...
- Classical cryptography: cryptanalysis of the discrete logarithm problem
- Post-quantum cryptography: cryptosystems "based on" hard versions of the isogeny problem
  - ➡ **digital signature** schemes,
  - ➡ **key exchange** protocols,
  - ➡ “Advanced” protocols...

The isogeny problem

Isogeny problem: Given two elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$

- Cryptosystems "based on" the isogeny problem?

Expectations: cryptosystems as secure as isogeny problem is hard

The isogeny problem = **Security of cryptosystems**

Hard even for quantum algorithms

Post-quantum cryptography

The isogeny problem

Isogeny problem: Given two elliptic curves E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$

- The solution φ is an isogeny…
- How to represent an isogeny?

Efficient isogenies

- Explicit polynomial formula, or Vélu's formulae... polynomial time in $\deg(\varphi)$

✓ Isogenies of *small* degree ℓ = 2, or 3... "ℓ-isogenies"

$$(x, y) \longmapsto \left(\frac{x^2+1}{x}, \frac{y(x^2+1)}{x^2}\right)$$

(degree 2)

Efficient isogenies

- Explicit polynomial formula, or Vélu's formulae… polynomial time in $\deg(\varphi)$

  ✓ Isogenies of *small* degree $\ell = 2$, or 3… "ℓ-isogenies"

- Given random E_1 and E_2, smallest $\varphi : E_1 \to E_2$ has degree poly(p)

  ✗ Typically in crypto, $p > 2^{256}$

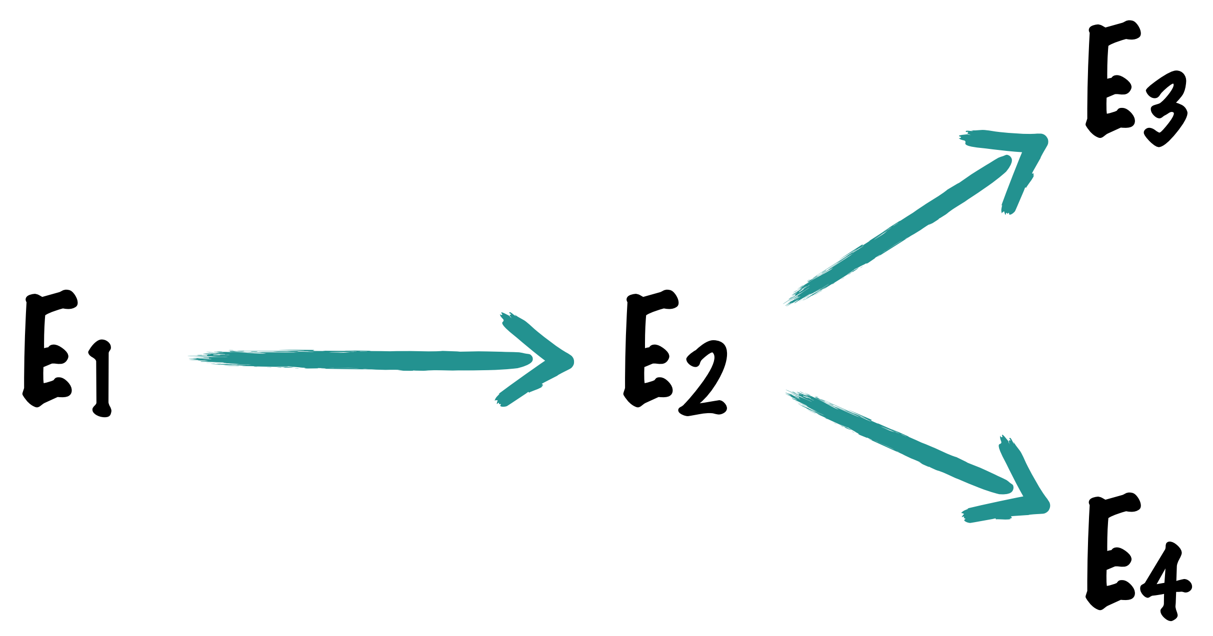
- Compose small isogenies to build bigger ones!

  ✓ Isogenies with **smooth degree** (small prime factors):

  $\varphi_n \circ \ldots \circ \varphi_2 \circ \varphi_1$ represented by ('compose', $\varphi_1, \varphi_2, \ldots, \varphi_n$), with $\deg(\varphi_i)$ small

Isogeny graph

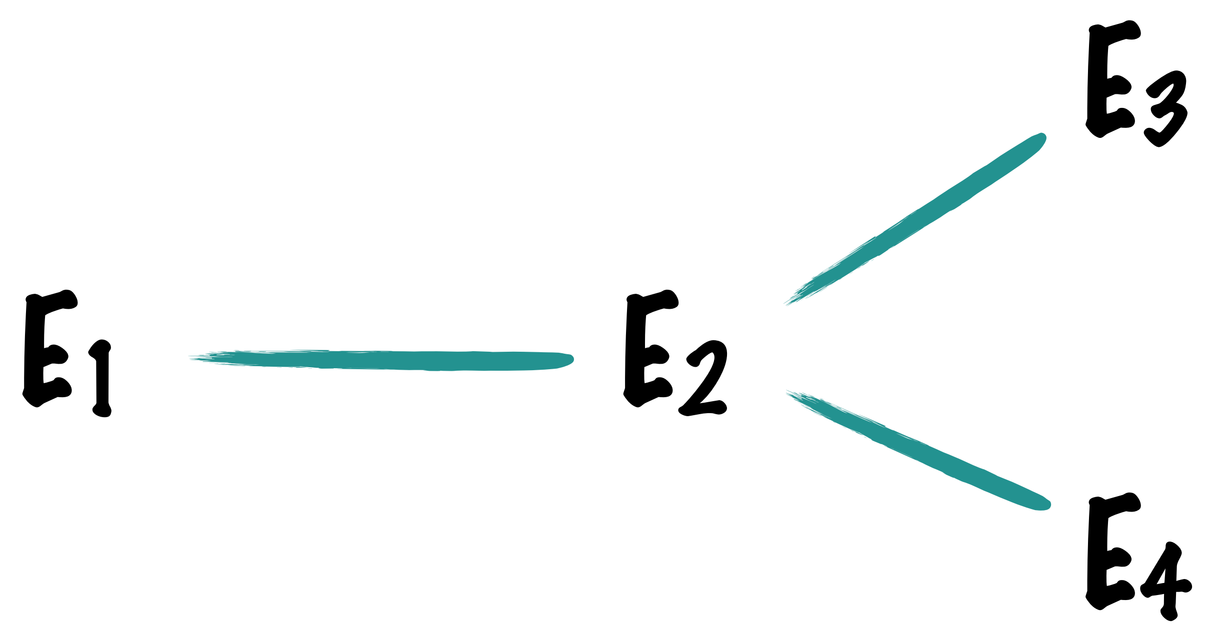
- Fix small ℓ (say, $\ell = 2$). Can easily compute ℓ-isogenies



an isogeny of degree ℓ = an edge in a graph

Isogeny graph

- Fix small ℓ (say, $\ell = 2$). Can easily compute ℓ-isogenies

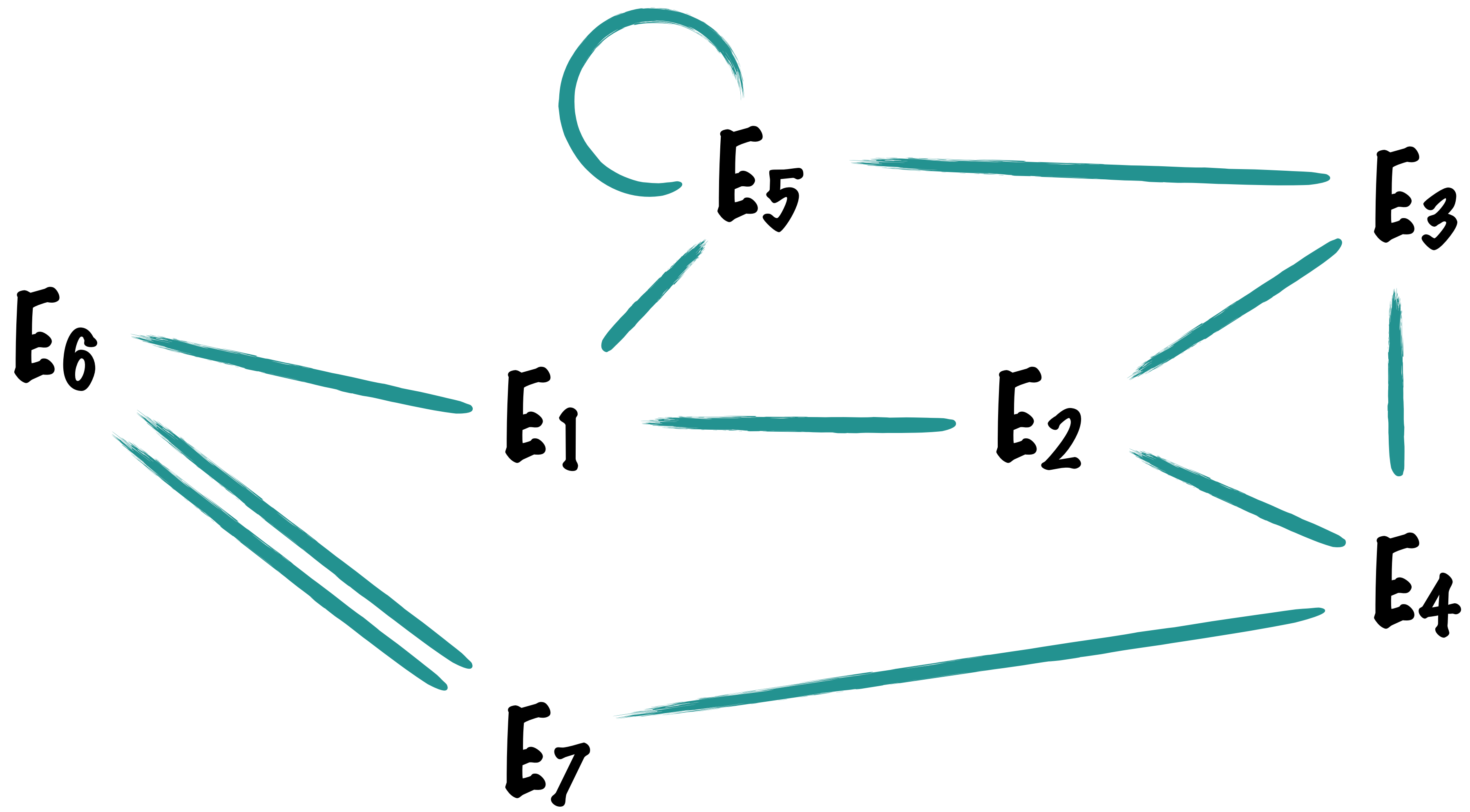


an isogeny of degree ℓ = an edge in a graph

$\exists\ \ell$-isogeny $E_1 \to E_2 \Rightarrow \exists\ \ell$-isogeny $E_2 \to E_1$

Isogeny graph

- Fix small ℓ (say, $\ell = 2$). Can easily compute ℓ-isogenies
- **The ℓ-isogeny graph** (supersingular...)



- $(\ell + 1)$-regular, **connected** (for supersingular curves)

The ℓ-isogeny path problem

ℓ-isogeny path problem: Given E_1 and E_2, find an ℓ-isogeny path from E_1 to E_2

- Path finding in a graph
- Hard for supersingular curves! Best known algorithm = generic graph algorithm
- Typical meaning of ***"the isogeny problem"***

Isogeny-based cryptography

Expectations: cryptosystems as secure as isogeny problem is hard

The isogeny problem = **Security of cryptosystems**

Hard even for quantum algorithms

Post-quantum cryptography

Isogeny-based cryptography

Reality: a mess

Weird scheme-dependent variants of isogeny problems $\leq$ **Security of cryptosystems** $\leq$ **The isogeny problem**

The isogeny problem	=	CGL hash function (preimage)
One endomorphism	=	SQISign (soundness)
Vectorisation	=	CSIDH (key recovery)
SSI-T	=	SIDH (key recovery)

Isogeny-based cryptography

Reality: a mess

Weird scheme-dependent variants of isogeny problems	≤	Security of cryptosystems	≤	The isogeny problem

The isogeny problem with "torsion point information"...

[Jao, De Feo] PQCrypto 2011
Isogeny-based key exchange
NIST PQC alt-finalist

The isogeny problem	=	CGL hash function (preimage)
One endomorphism	=	SQISign (soundness)
Vectorisation	=	CSIDH (key recovery)
SSI-T	=	SIDH (key recovery)

SIDH

Jao–De Feo 2011

SIKE logo – Supersingular Isogeny Key Encapsulation

Isogeny from a kernel

- Let E be an elliptic curve
- Let G a finite subgroup of E
- **Quotienting by G:** there is a unique (separable) isogeny

$$\varphi : E \to E/G$$

  with $\ker(\varphi) = G$

- $\deg(\varphi) = \#G$
- **Computing an isogeny from its kernel:** Given generators of G, the isogeny φ can be computed in time poly(size of input, largest prime factor of $\#G$) [Vélu 1971]

➡ Given a smooth kernel, can efficiently compute the isogeny

SIDH

Fix reference elliptic curve E_0

Alice

Random subgroup G of E_0

Compute $\varphi_A : E_0 \to E_0/G$

Let $E_A = E_0/G$

Compute $\boldsymbol{E_{AB}} = E_B/G$

$$\xrightarrow{E_A}$$

$$\xleftarrow{E_B}$$

Bob

Random subgroup H of E_0

Compute $\varphi_B : E_0 \to E_0/H$

Let $E_B = E_0/H$

Compute $\boldsymbol{E_{BA}} = E_A/H$

$$\begin{tikzcd}
E_0 \arrow[r, "\varphi_B"] \arrow[d, "\varphi_A"'] & E_0/H = E_B \arrow[d] \\
E_A = E_0/G \arrow[r] & E_0/(G+H) = \boldsymbol{E_{AB}} = \boldsymbol{E_{BA}}
\end{tikzcd}$$

SIDH

Fix reference elliptic curve E_0

Alice

Random subgroup G of E_0

Compute $\varphi_A : E_0 \to E_0/G$

Let $E_A = E_0/G$

Compute $\boldsymbol{E_{AB}} = E_B/G$

$E_A \longrightarrow$

$\longleftarrow E_B$

Bob

Random subgroup H of E_0

Compute $\varphi_B : E_0 \to E_0/H$

Let $E_B = E_0/H$

Compute $\boldsymbol{E_{BA}} = E_A/H$

G is not a subgroup of E_B
$\varphi_B(G)$ is!

How to compute $\varphi_B(G)$?
Alice does not know φ_B...

Torsion

- The N-torsion of E is the subgroup

$$E[N] = \{P \in E \mid N \cdot P = P + P + \ldots + P = 0\}$$

- $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$

Idea:

- Alice picks a subgroup G of $E_0[2^n]$ ← Many choices, good entropy
- Bob gives φ_B on $E_0[2^n]$ ← φ_B remains secret everywhere else...
- Alice can compute $\varphi_B(G)$ ← Can compute shared secret $\boldsymbol{E_{AB}} = E_B/\varphi_B(G)$

SIDH

Fix: an elliptic curve E_0

Generators P_2, Q_2 of $E_0[2^n] \cong (\mathbb{Z}/2^n\mathbb{Z})^2$

Generators P_3, Q_3 of $E_0[3^m] \cong (\mathbb{Z}/3^m\mathbb{Z})^2$

Alice		Bob
Random subgroup G of $E_0[2^n]$		Random subgroup H of $E_0[3^m]$
Compute $\varphi_A : E_0 \to E_0/G$		Compute $\varphi_B : E_0 \to E_0/H$
Let $E_A = E_0/G$	$\xrightarrow{E_A,\ \varphi_A(P_3),\ \varphi_A(Q_3)}$	Let $E_B = E_0/H$
	$\xleftarrow{E_B,\ \varphi_B(P_2),\ \varphi_B(Q_2)}$	
Compute $\boldsymbol{E_{AB}} = E_B/\varphi_B(G)$		Compute $\boldsymbol{E_{BA}} = E_A/\varphi_A(H)$

The SSI-T problem

Context:

- two elliptic curves E_0 and E_1
- an isogeny $\varphi : E_0 \to E_1$ (say, of degree 3^m like Bob's isogeny)
- an integer N coprime to $\deg(\varphi)$ (say, $N = 2^n$...)
- generators P and Q of $E_0[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$

"torsion point information"

SSI-T: Given E_0, E_1, P, Q, $\varphi(P)$ and $\varphi(Q)$, find the isogeny $\varphi : E_0 \to E_1$

SIDH key recovery $\Leftrightarrow$ SSI-T

An interpolation problem

SSI-T: given $\varphi(P)$ for a few $P \in E$, find φ

Polynomial interpolation: given $f(s)$ for a few $s \in K$, find f

An interpolation problem

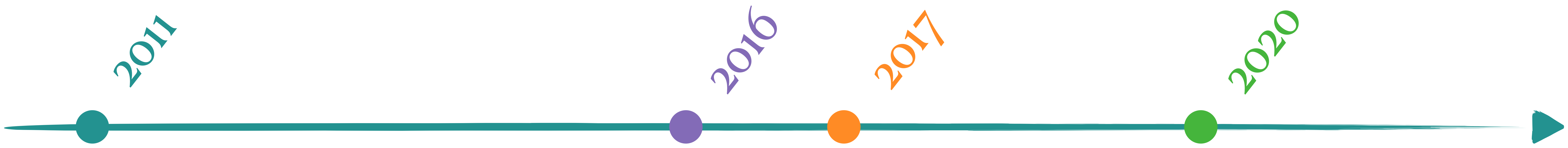
- Polynomial interpolation is not hard
- Isogenies are polynomials
- So isogeny interpolation (hence SSI-T) is easy??

$$\left(\frac{x^2+1}{x}, \frac{y(x^2+1)}{x^2}\right)$$

"Easy"? polynomial time in the length of the input…

- Polynomial interpolation: *length of the input* ≈ deg(f)
  - ➡ Need deg(f) + 1 values $f(s)$
- Isogeny interpolation: *length of the input* ≈ log(deg(φ))
  - ➡ A single $\varphi(P)$ also determines $\varphi(2P)$, $\varphi(3P)$, $\varphi(4P)$, …

Torsion point information: a weakness?



Birth of SIDH

[*Petit*] breaking “overstreched” SSI-T

[*Galbraith, Petit, Silva*] an active attack

[*de Quehen, Kutas, Leonardi, Martindale, Panny, Petit, Stange*] Improving Petit’s method

Standard SIDH parameters totally unaffected

The Snap

July 30 2022

July 30 2022

eprint 2022/975

An efficient key recovery attack on SIDH

Wouter Castryck, Thomas Decru

"Breaks SIKEp434 challenge in ten minutes"

Eurocrypt 2023 – “Isogeny 1” session

Efficient Key Recovery Attack on SIDH (Best Paper Award)

[Castryck, Decru]

A Direct Key Recovery Attack on SIDH (Honourable Mention)

[Maino, Martindale, Panny, Pope, W.]

Breaking SIDH in Polynomial Time (Honourable Mention)

[Robert]

Main result of the attacks

Interpolating isogenies [CD, MMPPW, R]**:**

- Let $\varphi : E_1 \to E_2$ of degree d
- Let P, Q generators of $E_1[2^n]$ such that $4 \deg(\varphi) \leq 2^{2n}$
- Given $(d, P, Q, \varphi(P), \varphi(Q))$, one can compute $\varphi(R)$ for any $R \in E_1$ in poly. time
- Interpolation: **Knowing φ on a few points $\Rightarrow$ Knowing φ everywhere**

Corollary: The few points leaked by SIDH leak the full secret.

Isogeny-based cryptography

Body count

Weird scheme-dependent variants of isogeny problems $\leq$ **Security of cryptosystems** $\leq$ **The isogeny problem**

The isogeny problem	=	CGL hash function (preimage)
One endomorphism	=	SQISign (soundness)
Vectorisation	=	CSIDH (key recovery)
~~SSI-T~~	~~=~~	~~SIDH (key recovery)~~

~~B-SIDH~~

~~k-SIDH~~

~~Séta~~

~~SHealS~~

Rundown of survivors

- **The isogeny path problem is unaffected**
- SQIsign [De Feo, Kohel, Leroux, Petit, W.] unaffected
  - ➡ Signature scheme, most compact $\mathtt{pk}$ + $\mathtt{sig}$ of all PQ schemes
  - ➡ Submitted to the NIST PQ signature call 2023
- CSIDH [Castryck, Lange, Martindale, Panny, Renes] unaffected
  - ➡ Key exchange very similar to Diffie–Hellman
- Wide variety of *CSIDH-inspired* constructions
  - ➡ "group action" cryptography
  - ➡ Signatures, PRFs, threshold stuff, oblivious stuff...

The algorithm

Isogenies in higher dimension

Dual

Let E an elliptic curve over $\mathbb{F}_q$ and N an integer

- Multiplication by N is an isogeny

$$[N] : E \to E : P \longmapsto [N]P = P + P + \ldots + P$$

- Let $\varphi : E_1 \to E_2$ be an isogeny
- **Dual of φ:** unique isogeny $\hat{\varphi} : E_2 \to E_1$ such that

$$\hat{\varphi} \circ \varphi = [\deg(\varphi)]$$

Abelian varieties

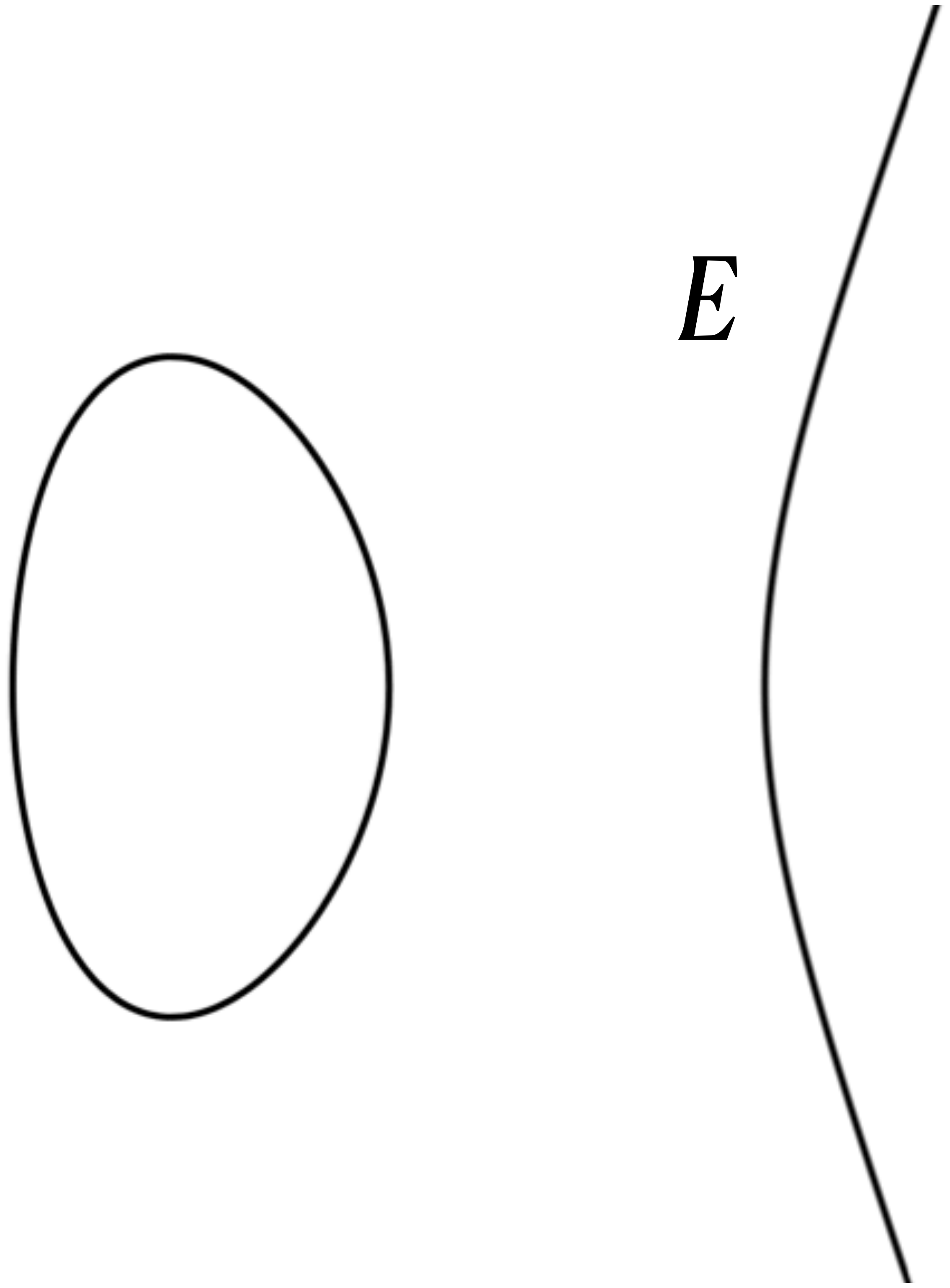
Elliptic curve: a curve that is also a group

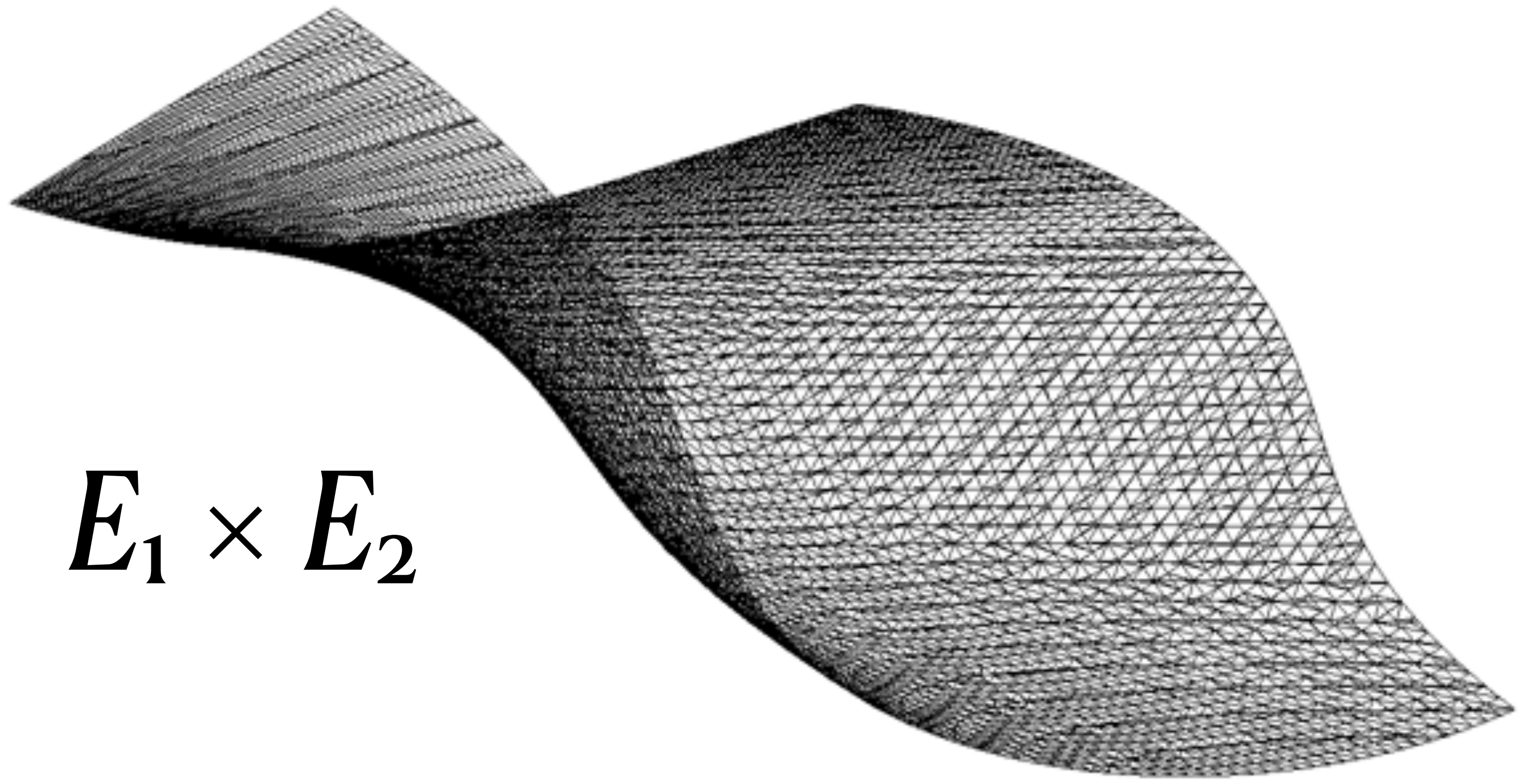
Abelian surface: surface that is also a group

- Example: product $E_1 \times E_2$

Abelian variety: same but any dimension

- Example: product $E_1 \times E_2 \times \ldots \times E_n$





Isogenies between products

$$\Psi : \; E_1 \times E_2 \longrightarrow F_1 \times F_2$$

Isogenies between products

$$
\begin{tikzcd}
\Psi : E_1 \times E_2 \arrow[r, bend left=80, "\varphi_{12}"] \arrow[r, bend left=40, "\varphi_{11}"] \arrow[r] \arrow[r, bend right=40, "\varphi_{21}"'] \arrow[r, bend right=80, "\varphi_{22}"'] & F_1 \times F_2
\end{tikzcd}
$$

$$(P_1, P_2) \longmapsto (\varphi_{11}(P_1) + \varphi_{21}(P_2), \varphi_{12}(P_1) + \varphi_{22}(P_2))$$

$$= \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \cdot \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

Isogenies between products

Every isogeny $\Psi : E_1 \times E_2 \to F_1 \times F_2$ is of the form

$$\begin{aligned} \Psi : \quad E_1 \times E_2 &\longrightarrow F_1 \times F_2 \\ (P_1, P_2) &\longmapsto \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \cdot \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \end{aligned}$$

where $\varphi_{ij} : E_i \to F_j$

- It is an ***N*-isogeny** if

$$\begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \cdot \begin{pmatrix} \hat{\varphi}_{11} & \hat{\varphi}_{12} \\ \hat{\varphi}_{21} & \hat{\varphi}_{22} \end{pmatrix} = \begin{pmatrix} [N] & 0 \\ 0 & [N] \end{pmatrix}$$

- Given the kernel of a 2^n-isogeny, can evaluate it in polynomial time

HD embedding of an isogeny

- Let $\varphi : E_1 \to E_2$ of degree $\deg(\varphi) = d$ (Bob's secret)
- Suppose $2^n - \deg(\varphi) = a^2$ is a square
- Define $\Psi : E_1 \times E_2 \to E_1 \times E_2$ as

$$\Psi = \begin{pmatrix} [a] & -\hat{\varphi} \\ \varphi & [a] \end{pmatrix}$$

- If we can evaluate Ψ, we can evaluate φ:

$$\begin{array}{ccccccc} E_1 & \xrightarrow{\text{inclusion}} & E_1 \times E_2 & \xrightarrow{\Psi} & E_1 \times E_2 & \xrightarrow{\text{projection}} & E_2 \\ P_1 & & (P_1, O) & & (aP_1, \varphi(P_1)) & & \varphi(P_1) \end{array}$$

HD embedding of an isogeny

- Let $\varphi : E_1 \longrightarrow E_2$ of degree $\deg(\varphi) = d$ (Bob's secret) $\hat{\varphi} \circ \varphi = [d]$
- Suppose $2^n - \deg(\varphi) = a^2$ is a square
- Define $\Psi : E_1 \times E_2 \longrightarrow E_1 \times E_2$ as

$$\Psi = \begin{pmatrix} [a] & -\hat{\varphi} \\ \varphi & [a] \end{pmatrix}$$

- Is it a 2^n-isogeny?

$$\begin{pmatrix} [a] & -\hat{\varphi} \\ \varphi & [a] \end{pmatrix} \cdot \begin{pmatrix} [a] & \hat{\varphi} \\ -\varphi & [a] \end{pmatrix} = \begin{pmatrix} [a^2] + [d] & 0 \\ 0 & [a^2] + [d] \end{pmatrix} = \begin{pmatrix} [2^n] & 0 \\ 0 & [2^n] \end{pmatrix}$$

- $\ker(\Psi) = \{ ([d]P, [a]\varphi(P)) \mid P \in E_1[2^n] \}$
- Given φ on $E_1[2^n]$ (torsion information) $\Rightarrow$ can compute $\ker(\Psi)$ $\Rightarrow$ can compute φ

4D embedding of an isogeny

- **2^n – deg(φ) not a square**? [Robert] has a solution
- Suppose $2^n - \deg(\varphi) = a^2 + b^2$ is a **sum of 2 squares**...
- Define $\Psi : E_1 \times E_1 \times E_2 \times E_2 \longrightarrow E_1 \times E_1 \times E_2 \times E_2$ as

$$\begin{pmatrix} a & b & -\hat{\varphi} & 0 \\ -b & a & 0 & -\hat{\varphi} \\ \varphi & 0 & a & -b \\ 0 & \varphi & b & a \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$$

- It is a 2^n-isogeny
- Isogeny in dimension 4
- Many integers are sum of 2 squares... but not all

8D embedding of an isogeny

- Every integer is a **sum of 4 squares**: $2^n - \deg(\varphi) = a^2 + b^2 + c^2 + d^2$
- [Robert] has another trick for that case: Zarhin's trick

$$
\begin{pmatrix}
a & -b & -c & -d & -\hat{\varphi} & & & \\
b & a & d & -c & & -\hat{\varphi} & 0 & \\
c & -d & a & b & & & -\hat{\varphi} & \\
d & c & -b & a & 0 & & & -\hat{\varphi} \\
\varphi & & & & a & b & c & d \\
 & \varphi & 0 & & -b & a & -d & c \\
 & & \varphi & & -c & d & a & -b \\
0 & & & \varphi & -d & -c & b & a
\end{pmatrix}
$$

Generalisation

Interpolating isogenies: ∃ an algorithm that for any isogeny $\varphi : E_1 \to E_2$, given:

- the curves E_1 and E_2, and the degree $\deg(\varphi)$
- points $P, Q \in E_1$ generating a subgroup G with $4\deg(\varphi) \leq \#G$
- the points $\varphi(P), \varphi(Q)$
- a point $S \in E_1$

returns $\varphi(S)$ in poly. time in: length of the input, largest prime factor of $\#G$, and degree of the field of definition of $E_i[\ell^e]$ for each prime-power factor ℓ^e of $\#G$.

Open question: what about $\#G$ not smooth?

Representing isogenies

Back to the foundations

The isogeny problem

"Idealised" isogeny problem: Given E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$

ℓ-isogeny path problem: Given E_1 and E_2, find an ℓ-isogeny path from E_1 to E_2

- The **ℓ-isogeny path problem** is the standard version of "**the isogeny problem**" because... no other way to represent solution $\varphi : E_1 \to E_2$ than as a path?

    ➡ Strong restriction on φ because of technical obstacle

- **How to represent an isogeny?**

Efficient representation of isogenies

How to represent an isogeny?

- an **efficient representation** of φ: can evaluate $\varphi(P)$ in poly. time for any P

Examples:

- Small degree isogenies
- Compositions of small degree isogenies
- Linear combinations of compositions of small degree isogenies…

Main result of the attacks

Interpolating isogenies [CD23, MMPPW23, Rob23]**:**

- Let $\varphi : E_1 \to E_2$ of degree d
- Let P, Q in E_1 such that $4 \deg(\varphi) \leq \#\langle P, Q \rangle$
- Given $(d, P, Q, \varphi(P), \varphi(Q))$, one can compute $\varphi(R)$ for any $R \in E_1$ in poly. time
- Interpolation: **Knowing φ on a few points ⇒ Knowing φ everywhere**

Corollary: $(d, P, Q, \varphi(P), \varphi(Q))$ is an efficient representation of φ.

- **“Interpolation representation”** of φ, or “HD representation”
- Universal! Given any efficient repr. of φ, can compute its interpolation repr.

The universal isogeny problem

The universal isogeny problem: Given E_1 and E_2, find an isogeny $\varphi : E_1 \to E_2$ represented by interpolation.

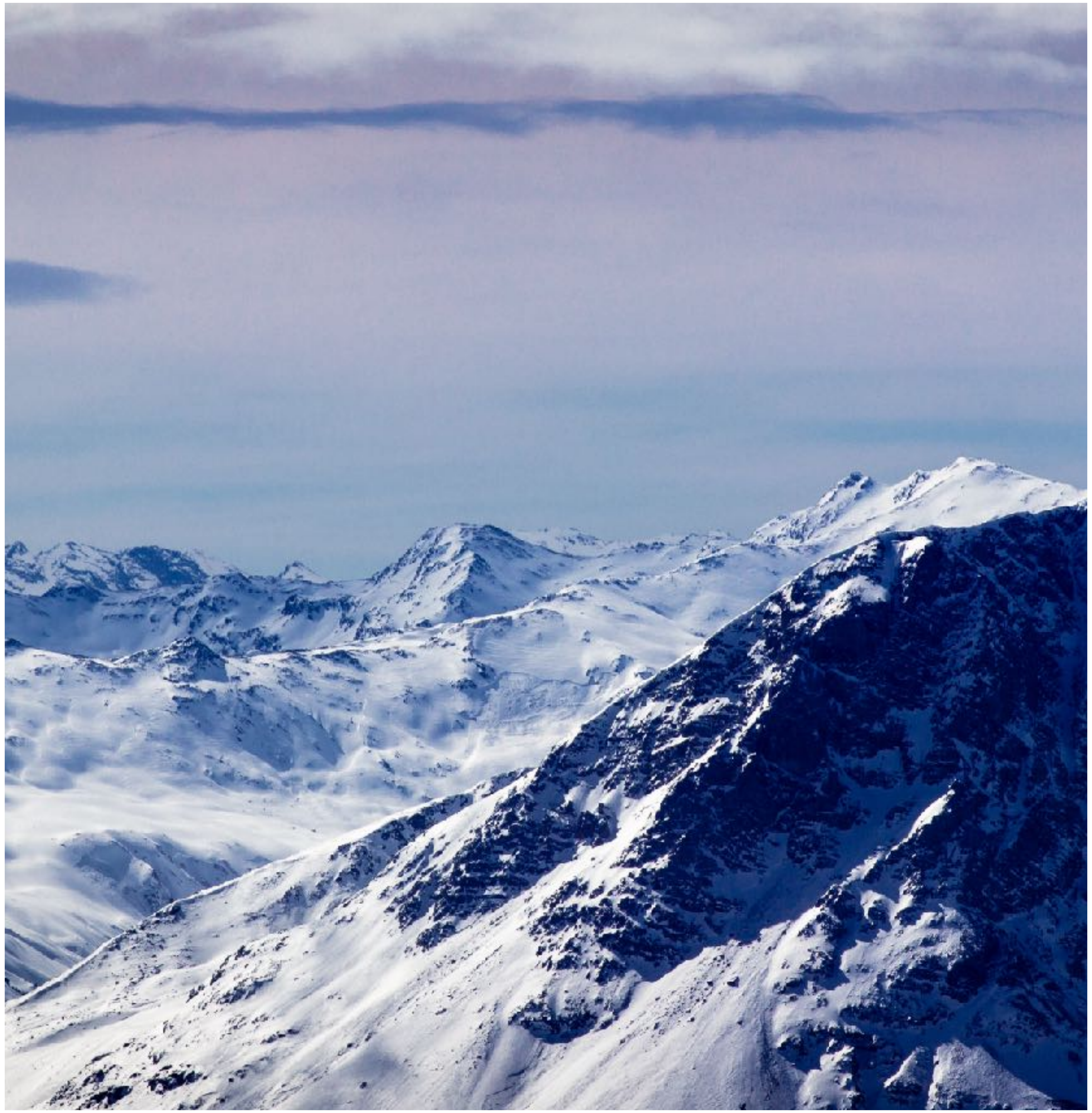
- No restriction on φ like in ℓ-isogeny path: any φ can be a valid response

Universal isogeny $\Leftrightarrow$ ℓ-isogeny path

[Page, W.] preprint 2023

Applications

In cryptography
and number theory



New cryptosystems

- **FESTA** [Basso, Maino, Pope]: Fast Encryption from Supersingular Torsion Attacks
  - ➡ **2D isogenies** for decryption
  - ➡ Well-studied, “Richelot isogenies”, **efficient**
  - ➡ Good implementations available
- **SQIsign HD** [Dartois, Leroux, Robert, W.]: signature scheme inspired by SQIsign
  - ➡ **4D isogenies** for verification
  - ➡ Not well studied
  - ➡ Very promising ongoing work by Dartois

New computational equivalences

[Page, W.] **The supersingular Endomorphism Ring and One Endomorphism problems are equivalent.** 2023

- Finding an ℓ-isogeny path is equivalent to finding any isogeny
- Finding one endomorphism is equivalent to finding them all

[Arpin, Clements, Dartois, Eriksen, Kutas, W.] **Finding orientations of supersingular elliptic curves and quaternion orders.** 2023

- Deciding if an elliptic curve has a certain endomorphism is equivalent to finding said endomorphism (subexponential equivalence)

New algorithms

[Robert] **Some applications of higher dimensional isogenies to elliptic curves.** 2022

- Computing ordinary endomorphism rings, canonical lifts, Siegel modular polynomials…

[Herlédan Le Merdy, W.] **The supersingular endomorphism ring problem given one endomorphism.** 2023

- Given a supersingular elliptic curve E and some $\alpha \in \mathrm{End}(E)$, compute $\mathrm{End}(E)$ in subexponential time (assuming GRH)