Superpolynomial lower bounds for circuits of constant depth

Nutan Limaye, Srikanth Srinivasan, Sébastien Tavenas

26 / 09 / 2023

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How many operations are needed for computing a polynomial?

Main result,

$\exists H$ which can not be written of the form:
$H(x_1, \ldots, x_N) = \sum_{i_1 \in [N]} \prod_{i_2 \in [N]} \cdots \sum_{i_{p-1} \in [N]} \prod_{i_p \in [N]} T_{i_1, \ldots, i_p}$
where

- T_i are constants or variables,
- the number of alternations between $\sum$ and $\prod$ is bounded by a constant.

Model(s) for Evaluating a Polynomial

Let $P(x_1, \ldots, x_N) \in \mathbb{F}[x_1, \ldots, x_N]$ (In this talk think $\mathbb{F} = \mathbb{Q}$)

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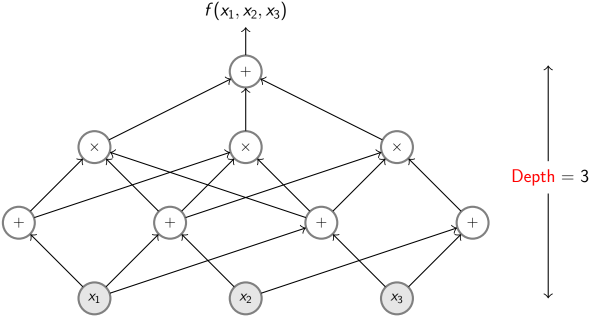

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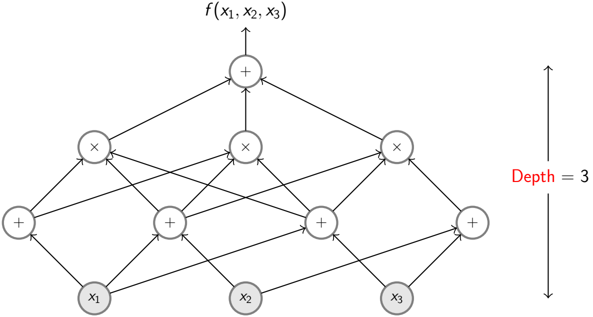

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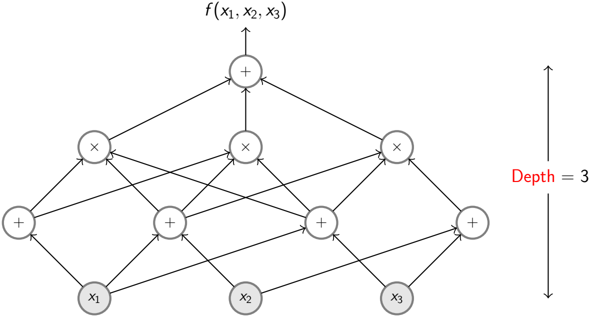

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We will always assume:
the top node is a $\sum$

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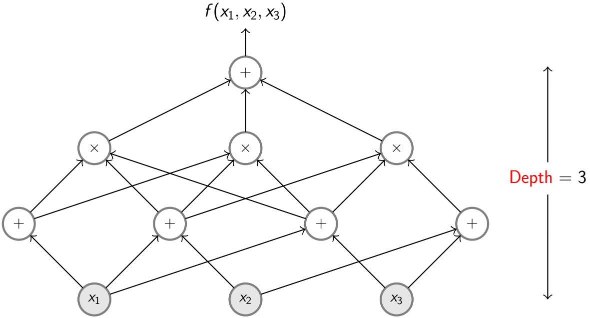

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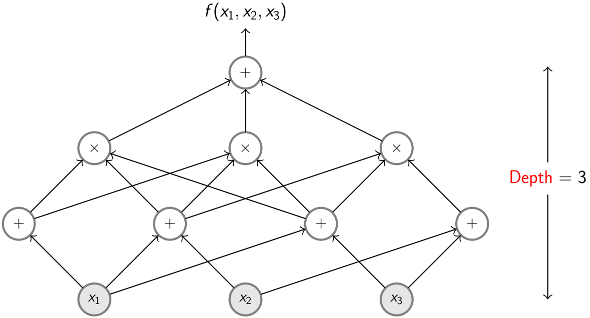

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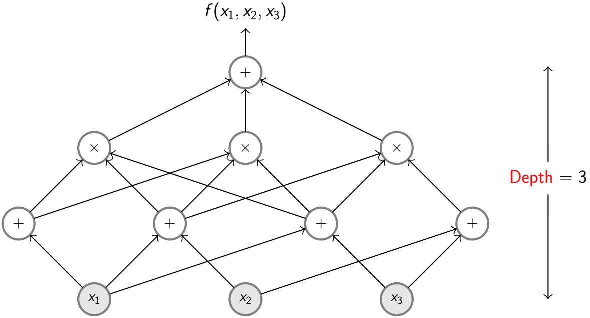

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A formula is a circuit with tree as the underlying undir. graph.

Algebraic analogue of P

Definition (VP – Valiant's P, or "efficiently computable")

Polynomials $f(x_1, \ldots, x_n)$ that can be computed by poly(n)-sized algebraic circuits?

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In particular they can be "efficiently" simulated by Boolean circuits (bits computation).

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Examples:

$$\text{[Ben-Or]} \quad \mathrm{ESym}_d(x_1, \cdots, x_n) = \sum_{S \subseteq [n], |S| = d} \prod_{i \in S} x_i$$

$$\text{[Berkowitz,Mahajan-Vinay]} \quad \mathrm{Det}_n = \begin{vmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{vmatrix}$$

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Fact: [Valiant] Det_n is complete* for VP.

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Examples:

$$\mathsf{HC}_n$$

$$\mathsf{Perm}_n = \mathrm{perm} \begin{bmatrix} x_{11} & \cdots & x_{n1} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \sum_{\pi \in S_n} \prod_{i=1}^{n} x_{i\pi(i)}$$

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An exponential sum of a VP polynomial $g(\mathbf{x}, \mathbf{y})$:

$$f(\mathbf{x}) \quad = \quad \sum_{\mathbf{y} \in \{0,1\}^m} g(\mathbf{x}, \mathbf{y})$$

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Under GRH, $\mathrm{VP} = \mathrm{VNP} \implies \mathrm{P/poly} = \mathrm{NP/poly} = \mathrm{PH/poly}$

How does one begin?

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"If you can't solve a problem, there is a simpler problem that you can't solve. Find it." – *George Pólya*

Lower bounds against constant depth?

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- Boolean lower bounds imply Algebraic lower bounds
- Exponentially lower bounds for Constant Depth circuits have been known for almost 50 years
- $\Longrightarrow$ We can combine them to get our goal!!!

Problem in the first point:

> - Small Algebraic Circuits simulated by Small Boolean ones
>
> But
>
> - Small Algebraic Circuits of constant depth are not simulated by Small Boolean Circuits of constant depth

See: a sum of variables

Are algebraic constant depth circuits a weak model?

- They can't be simulated by constant-depth Boolean circuits
- $\sum\prod\sum$ can compute $\mathsf{ESym}_{n,d}$ in a non-homogeneous way
- Can simulate general Algebraic Circuits with a subexponential cost!

IMM

Another example of problem in VP: (still almost VP-complete)

$$\underset{A}{\begin{pmatrix} \boxed{} & & \\ & & \\ & & \end{pmatrix}} = \underset{X_1}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}} \times \underset{X_2}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}} \times \cdots \times \underset{X_d}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}$$

$\mathrm{IMM}_{n,d}$ defined over variable sets $X_1, \ldots, X_d$, each of size n^2.

Each X_i thought of as an $n \times n$ matrix.

$\mathrm{IMM}_{n,d}$ is the $(1,1)$th entry of product $X_1 \cdot X_2 \cdot \ldots \cdot X_d$.
(polynomial with dn^2 variables and degree d)

Reduction to log-depth

Depth to compute $\mathrm{IMM}_{n,d}$?

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"Divide and Conquer":

Compute (recursively) $\mathrm{IMM}_{n,d/2}$ on the left and on the right.

Recombine with one matrix multiplication.

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> $\mathrm{IMM}_{n,d}$ is computed by a circuit of size $\mathrm{poly}(n, d)$ and depth $O(\log d)$.

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[VSBR – AV/K/T – GKKS]

If P can be computed by a circuit of size s,
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Depth-4 circuits for $\mathrm{IMM}_{n,d}$?

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Split into $\sqrt{d}$ blocks of $\sqrt{d}$ matrices each.

Compute $\mathrm{IMM}_{n,\sqrt{d}}$ for each block.

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(Ex: $P(x_1, \ldots, x_N) = \sum_{S\subseteq[N]} \prod_{i\in S} x_i$ has 2^N monomials.)

Best known lower bound for $\sum\prod\sum$ circuits is $\Omega(N^3/\log^2 N)$
[Kayal,Saha,T.,2016].
Best known lower bound for $\sum\prod\sum\prod$ circuits is $\Omega(N^{2.5})$
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For depth $\Delta \geq 5$, lower bound in $N^{1+\Omega(1/\Delta)}$
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No superpolynomial lower bounds

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Superpolynomial Lower Bounds against Constant Depth Circuits

Main Theorem

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Let n, d be growing parameters with $d \leq \log n$.
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Any depth-Γ circuit for $\mathrm{IMM}_{n,d}$ must have size $n^{d^{\varepsilon_\Gamma}}$
where ε_Γ depends only on Γ.
Any depth-Γ circuit for Det_n must have size $n^{(\log n)^{\varepsilon_\Gamma}}$.

If $\Gamma = 3$, we have $\varepsilon_3 = 1/2$ (optimal for IMM).

If $\Gamma = 4$, we have $\varepsilon_4 = 1/4$.

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Subexponential time PIT

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Prior to this deterministic $n^{O(k)}$ time algorithm known for $\sum^{[k]} \prod \sum$ circuits. [Saxena,Seshadhri,2012]

Algebraic hardness vs. randomness (by [Chou,Kumar,Solomon,2018]) + our lower bound.

Builds on [Kabanets,Impagliazzo,2004], [Dvir,Shpilka,Yehudayoff,2009].

$$\begin{tikzcd}
\text{Lower bounds against general formulas} \\
\text{Lower bounds against weaker formulas} \arrow[u, "\text{Escalation}"']
\end{tikzcd}$$

Homogeneous/Set-multilinear restrictions

Set-multilinear polynomials

$P \in \mathbb{F}_{\mathsf{sm}}[X_1, \ldots, X_d]$, where $X_1, \ldots, X_d$ are sets of variables.

Each monomial uses exactly one variable per set.

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Homogeneous circuits/formulas

All gates compute homogeneous polynomials.

Set-multilinear circuits/formulas

All gates compute set-multilinear polynomials.

Homogeneization (Raz's approach)

Let $P(x_1, \ldots, x_N)$ be a set-multilinear polynomial of degree d.

[Raz 2009]

Formula of size s computing P

$$\Big\downarrow \text{Efficient conversion}$$

Set-multilinear formula computing P of size $\mathrm{poly}(s) \cdot (\log s)^{O(d)}$

Homogeneization (Raz's approach)

Let $P(x_1, \ldots, x_N)$ be a set-multilinear polynomial of degree $d = O(\log N / \log\log N)$.

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Caveat: Raz's transformation does not work for constant depth.

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Low degree regime - the blow-ups in size are all polynomial

Assumptions $d < \sqrt{\log n}$ and $\mathrm{char}(\mathbb{F}) \neq 0$.

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- Parallelization of the circuits to depth $O(\log d)$. [VSBR83]
- Parallelization of the formulas to depth $O(\log s)$. [BKM73]
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Structural results

Homogeneization/Set-multilinearization of the circuits. [Str73,NW97]

Idem for formulas. [Raz13]

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where the depth is multiplied by at most 2.
[SW01,CKSV16,LST21]

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Sufficient to prove $n^{\omega(d)}$ lower bounds for set-multilinear formulas of depth $O(\log d)$!

Non-FPT Lower Bounds

Known lower bounds

Known set-multiliear formula lower bounds for constant depth. [NW 95, Raz 2009, RY 2009]

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For escalation to work, we need:

$$N^{\Omega(f(d))}$$

Our Lower Bound

A non-FPT lower bound for set-multilinear formulas.

Set-multilinear formula lower bound

Let $d \leq O(\log n)$.
For any $\Delta \geq 1$ any set-multilinear formula C computing $\mathrm{IMM}_{n,d}$ of depth Δ must have size $n^{d^{\varepsilon^{\Delta}}}$.

First case $\Delta = 5$: bound in $n^{\Omega(\sqrt{d})}$

Case $\Gamma = 3$

We just stated:

Set-multilinear formula lower bound

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Case $\Gamma = 3$

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Let $d \leq O(\log n)$. Any set-multilinear formula C computing $\mathrm{IMM}_{n,d}$ of depth 5 must have size $n^{\Omega(\sqrt{d})}$.

In particular,

General formula lower bound

Let n, d be growing parameters with $d = o(\log n)$.
Assume $\mathbb{F}$ is characteristic 0.
Any algebraic circuits of depth 3 computing $\mathrm{IMM}_{n,d}$ must have size $n^{\Omega(\sqrt{d})}$.

Techniques

A typical lower bound proof

The lower bound proof outline.

- Come up with a measure $\mu : \mathbb{F}_{\mathsf{sm}}[X_1, \dots, X_d] \to \mathbb{R}_{\geq 0}$.
- Show that $\mu(\mathsf{IMM}_{n,d})$ is large.
- Show that $\mu(\text{sm. } \sum\prod\sum\prod\sum)$ is small.

Partial Derivative Measure

Nisan and Wigderson [NW 95]

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Partition $[d]$ into $\mathcal{P}$ and $\mathcal{N}$.

$M^{\mathcal{P}}$ multilinear monomials over $(X_i : i \in \mathcal{P})$.

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For a polynomial f, define matrix M_f as follows.

$$\begin{array}{ccc} & M^{\mathcal{N}} \longrightarrow & \\ M^{\mathcal{P}} & m_2 & \\ \downarrow \quad m_1 & \begin{pmatrix} & & \\ & \boxed{} & \\ & & \end{pmatrix} & \end{array}$$

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Coeff of $m_1 \cdot m_2$ in f

The Partial Derivative Measure is the $\mathrm{rank}(M_f)$.

Properties of μ

$\mu : \mathbb{F}_{\mathsf{sm}}[X_1, \ldots, X_d] \to \mathbb{N}$

μ is sub-additive: $\mu(f+g) \leq \mu(f) + \mu(g)$

μ is multiplicative: $\mu(fg) = \mu(f)\mu(g)$

$\mu(f) \leq \min(M^{\mathcal{P}}, M^{\mathcal{N}})$

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The measure applied to $\mathrm{IMM}_{n,d}$

Recall that

$$\mathrm{IMM}_{n,d} = \sum_{i_1,\ldots,i_{d-1}\in[n]} X^{(1)}_{1,i_1}\cdot X^{(2)}_{i_1,i_2}\cdot X^{(3)}_{i_2,i_3}\cdots X^{(d)}_{i_{d-1},1}.$$

For $\mathcal{P} = \{i \mid i \text{ odd}\}$ and $\mathcal{N} = \{j \mid j \text{ even}\}$ (Assume d even)

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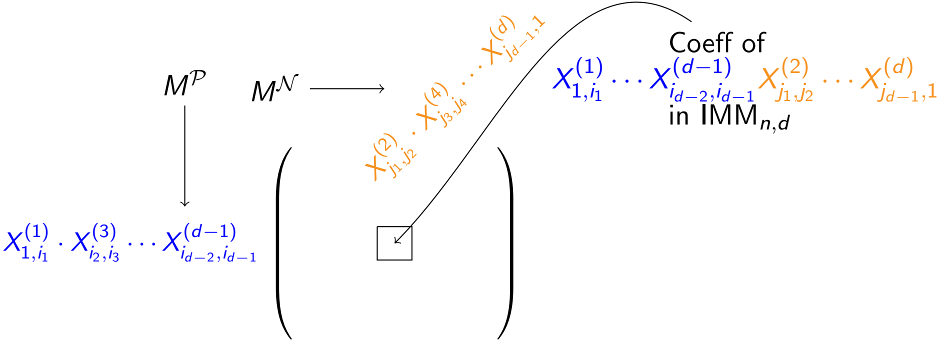

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$M^{\mathcal{P}}$ $\quad M^{\mathcal{N}} \longrightarrow$ $X^{(2)}_{j_1,j_2} \cdot X^{(4)}_{j_3,j_4} \cdots X^{(d)}_{j_{d-1},1}$

Coeff of $X^{(1)}_{1,i_1} \cdots X^{(d-1)}_{i_{d-2},i_{d-1}} X^{(2)}_{j_1,j_2} \cdots X^{(d)}_{j_{d-1},1}$ in $\mathrm{IMM}_{n,d}$

$$X^{(1)}_{1,i_1} \cdot X^{(3)}_{i_2,i_3} \cdots X^{(d-1)}_{i_{d-2},i_{d-1}} \quad \begin{pmatrix} & & \\ & \boxed{} & \\ & & \end{pmatrix} \quad \begin{cases} = 1 \text{ if } (i_1,..,i_{d-1}) = (j_1,..,j_{d-1} \\ = 0 \text{ otherwise.} \end{cases}$$

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Permutation matrix

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Permutation matrix

The matrix is full-rank! $\mathrm{rk}(\mathrm{IMM}_{n,d}) = n^{d-1}$.

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The lower bound proof outline.

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- Show that $\mu(\mathsf{IMM}_{n,d})$ is large. ✔
- Show that $\mu(\text{sm. } \sum\prod\sum\prod\sum)$ is small.

$\sum\prod\sum$ set-multilinear formulas

Let $(X_1, \ldots, X_d)$ be a partition of variables.

$$F(X) = \sum_{i=1}^{s} \prod_{j=1}^{d} \ell_{i,j}(X_j)$$

each $\ell_{i,j}$ homogeneous linear polynomial over X_j.

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Conclusion: $\sum\prod\sum$ s.m. form. for $\mathrm{IMM}_{n,d}$ has size $\geq n^{d-1}$.

$\sum\prod\sum\prod$ set-multilinear formulas

Product of Inner Products Polynomial.

Let $X_j = \{x_{j,1}, \ldots, x_{j,m}\}$ for $j \in [d]$.

$$\mathrm{PIP}(X_1, \ldots, X_d) = \prod_{j=1}^{d/2} \left(\sum_{k=1}^{m} x_{2j-1,k} \cdot x_{2j,k} \right)$$

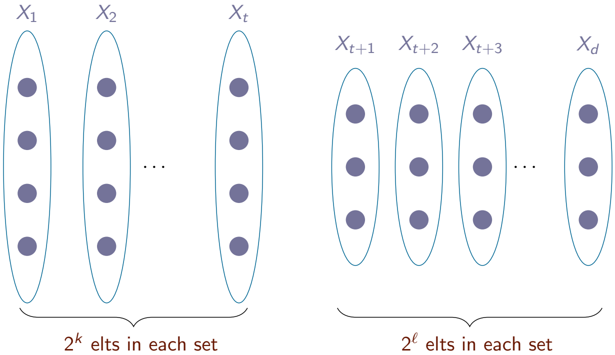
PIP has product-depth 2 set-multilinear formula of size $O(md)$.

For $\mathcal{P} = \{i \mid i \text{ odd}\}$ and $\mathcal{N} = \{i \mid i \text{ even}\}$,

M_{PIP} is a permutation matrix.

rk(PIP) is full.

Idea: Different set sizes



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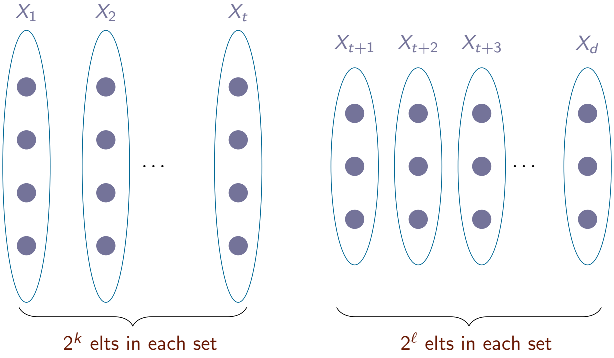

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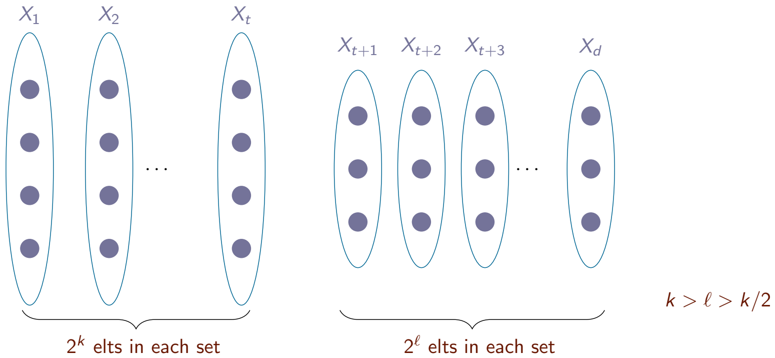

We want to ensure $|M^{\mathcal{P}}| = |M^{\mathcal{N}}|$.

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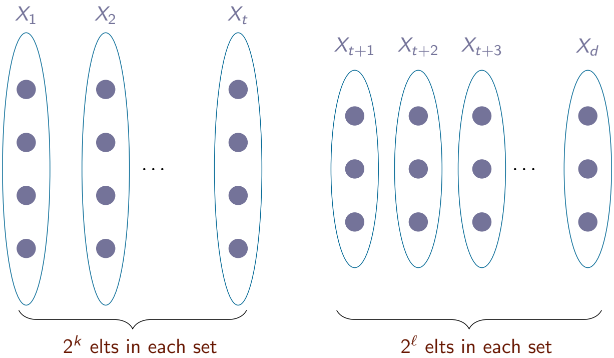

$k > \ell > k/2$
$kt = (d-t)\ell$

We want to ensure $|M^{\mathcal{P}}| = |M^{\mathcal{N}}|$.

$\sum\prod\sum\prod\sum$ set-multilinear formulas

- Sets of size 2^k, 2^ℓ
- $k > \ell > k/2$
- Full rank $= 2^{kt}$

$$\sum \quad \overset{F}{\boxed{\prod_j \underbrace{\left(\sum\prod\sum\right)}_{F_j}}}$$

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Focus on one term F, which is $F_1 \times F_2 \times \ldots \times F_r$.

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Focus on one term F, which is $F_1 \times F_2 \times \ldots \times F_r$.

Sufficient to show $\mu(F) \leq \frac{2^{kt}}{n^{\sqrt{d}/100}} = \frac{\sqrt{2^{kt}2^{\ell(d-t)}}}{2^{k\sqrt{d}/100}}$.

Each F_j is a $(\sum\prod\sum)$ set-multilinear formula.

It covers p_j $\mathcal{P}$-variables-sets and q_j from $\mathcal{N}$.

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So $\mu(F) = \quad \mu(F_1) \quad \times \cdots \times \quad \mu(F_r)$

$$= \frac{\sqrt{2^{kp_1}2^{\ell q_1}}}{\mathrm{Loss}(F_1)} \times \cdots \times \frac{\sqrt{2^{kp_r}2^{\ell q_r}}}{\mathrm{Loss}(F_r)}$$

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$\sum\prod\sum\prod\sum$ set-multilinear formulas

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We saw $\mu(\sum\prod\sum) \leq \mathrm{size}(\sum\prod\sum)$.

If the size is $\geq 2^{k\sqrt{d}/50}$ 

Otherwise

$$2^{k\sqrt{d}/50} \geq \mu(F_j) = \frac{\sqrt{2^{kp_j}2^{\ell q_j}}}{\mathrm{Loss}(F_j)} \geq \frac{2^{k\sqrt{d}/8}}{\mathrm{Loss}(F_j)}.$$

$\sum\prod\sum\prod\sum$ set-multilinear formulas

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Conclusion: $\mathrm{Loss}(F_j) \geq 2^{k\sqrt{d}/100}$.

$\sum\prod\sum\prod\sum$ set-multilinear formulas

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Case 2 All F_j have degree $< \sqrt{d}/2$.

$\sum\prod\sum\prod\sum$ set-multilinear formulas

$$F$$

$$\underbrace{\boxed{\prod_j \left(\sum\prod\sum\right)}}_{F_j}$$

- Sets of size 2^k, 2^ℓ
- $k > \ell > k/2$
- Full rank $= 2^{kt}$
- $\mu(F_j) = \frac{\sqrt{2^{kp_j} 2^{\ell q_j}}}{\mathrm{Loss}(F_j)}$
- We want:

$$2^{k\sqrt{d}/100} \leq \prod \mathrm{Loss}(F_j)$$

Case 2 All F_j have degree $< \sqrt{d}/2$.

Let us choose $\ell = \lfloor k - k/(10\sqrt{d}) \rfloor$.

Focus on the ratio between the # of rows and of columns:

$$|kp_j - \ell q_j| > \frac{q_j k}{10\sqrt{d}}.$$

So $\mathrm{Loss}(F_j) \geq 2^{q_j k/(20\sqrt{d})}$.

Conclusion: $\prod \mathrm{Loss}(F_j) \geq \prod 2^{q_j k/(20\sqrt{d})} \geq 2^{k\sqrt{d}/40}$.

A typical lower bound proof

The lower bound proof outline.

- Come up with a measure $\mu : \mathbb{F}_{\mathsf{sm}}[X_1, \ldots, X_d] \to \mathbb{R}_{\geq 0}$. ✔
- Show that $\mu(\mathsf{IMM}_{n,d})$ is large. ✔
- Show that $\mu(\text{sm. } \sum\prod\sum\prod\sum)$ is small. ✔

We just showed:

Set-multilinear formula lower bound

Let $d \leq O(\log n)$. Any set-multilinear formula C computing $\mathrm{IMM}_{n,d}$ of depth 5 must have size $n^{\Omega(\sqrt{d})}$.

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In particular,

General formula lower bound

Let n, d be growing parameters with $d = o(\log n)$.
Assume $\mathbb{F}$ is characteristic 0.
Any algebraic circuits of depth 3 computing $\mathrm{IMM}_{n,d}$ must have size $n^{\Omega(\sqrt{d})}$.

General case

Set-multilinear formula lower bound

Let $d \leq O(\log n)$. Any set-multilinear formula C computing $\mathrm{IMM}_{n,d}$ of depth Δ must have size $n^{d^{\exp(-O(\Delta))}}$.

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In particular,

General formula lower bound

Let n, d be growing parameters with $d = o(\log n)$.
Assume $\mathbb{F}$ is characteristic 0.
Any algebraic circuits of depth Γ computing $\mathrm{IMM}_{n,d}$ must have size $n^{d^{\exp(-O(\Gamma))}}$.

Open Questions

Can the lower bound be improved? What about $n^{\Omega(d^{1/\Delta})}$?

Can we remove the characteristic 0 condition?

Can we get better lower bounds if we consider non-commutative computations?

Can combining known measures give better lower bounds?