Modular Tricks for Integer Sparse Polynomials

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Thank you!

Questions?

My Collaborators

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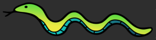
Plan

1. What are we doing?
2. How do we do it?
3. The modular tricks
4. Why did we do it?

Integer Sparse Polynomials

Setting

- Univariate polynomials in $\mathbb{Z}[X]$
- Multi-precision coefficients () AND exponents ()
- Necessarily "supersparse" / "lacunary"

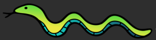
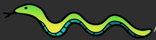
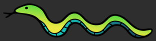
Example

$$f(X) = 123\,X^{987} + 346\,X^{765} + 567\,X^{543} + 789\,X^{321}$$

- $T = 4$ nonzero terms
- All coefficients and exponents at most $B = 1000$

Sparse Interpolation

Input

- Unknown $f(X) =$ $_1 X$$_1 + \cdots +$ $_T X$$_T$
- Way to evaluate at chosen points
- Bound T on sparsity (number of nonzero terms)
- Bound B on largest  or 

Output

- List of coefficient/exponent pairs $\in (\mathbb{Z} \times \mathbb{Z})^T$
- Total bit-length $O(T \log B)$

Optimal number of evaluations?

Quiz

Say f has $T \leq 4$ nonzero terms and coeffs, expons $\leq B = 1000$.
What is the fewest number of evaluations to recover f?

Optimal number of evaluations?

Quiz

Say f has $T \leq 4$ nonzero terms and coeffs, expons $\leq B = 1000$.
What is the fewest number of evaluations to recover f?

Answer

Just one evaluation over $\mathbb{Z}$ is enough!

Suppose $f(X) = 123\,X^{987} + 346\,X^{765} + 567\,X^{543} + 789\,X^{321}$

$f(1000) = 12300000 \cdots 0000034600000 \cdots 0000056700000 \cdots 0000078900000 \cdots 000$

But this has bit-length $O(TB)$, exponentially larger than output size $O(T \log B)$

Cost model

Evaluating f: "Modular Black Box"

Input: Modulus $m \in \mathbb{Z}$

Input: Point $\theta \in \mathbb{Z}/m\mathbb{Z}$

Output: $f(\theta) \bmod m$

Cost: evaluation bit-length $\log m$

Goal

- Total evaluation bit-length $O(T \log B)$
- Total computation bit-cost $\widetilde{O}(T \log B)$

where $\widetilde{O}(\blacksquare)$ is defined as $O(\blacksquare \cdot \text{polylog}(T + \log B))$

Plan

- What are we doing?
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Recovering tiny exponents from f_{tiny}

- Write $f_{\text{tiny}}(X) = \text{🐍}_1 X^{\text{🐘}_1}$
- Assume all $\text{🐍}_i \in \text{poly}(T + \log B)$ and all $\text{🐘}_i \in O(T \log B)$
- Let q a prime and $\omega \in \mathbb{F}_q$ where $q \in \text{poly}(T + \log B)$ and $\text{ord}_q(\omega) > \max \text{🐘}_i$

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Algorithm: Tiny Exponent Recovery

1. Evaluate $f_{\text{tiny}}(1), f_{\text{tiny}}(\omega), \cdots, f_{\text{tiny}}(\omega^{2T-1})$ modulo q
2. Fast Berlekamp-Massey to recover $\Lambda(Z) = \prod_i (Z - \omega^{\text{🐘}_i})$
3. Evaluate $\Lambda(1), \Lambda(\omega), \ldots, \Lambda(\omega^{\widetilde{O}(T \log B)})$
4. Roots of Λ reveal values of 🐘_i's

Recovering tiny exponents from f_{tiny}

- Write $f_{\text{tiny}}(X) =$  $_1 X$  $_1$
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COST: $O(T \log B))$ evaluation bits and $\widetilde{O}(T \log B)$ computation

Recovering big coefficients from f_{wide}

- Write $f_{\text{wide}}(X) =$ $_1 X^{}$$^{1} + \cdots +$ $_T X^{}$^{T}
- Assume all $_i \leq B$ and all $_i \in O(T \log B)$

Recovering big coefficients from f_{wide}

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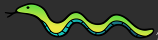
- Assume all $_i \leq B$ and all $_i \in O(T \log B)$

Algorithm: Big Coefficient Recovery

1. Choose small prime $q \in \text{poly}(T + \log B)$ and larger modulus $m \geq B$
2. Use Tiny Exponent Recovery mod q to obtain all $_i$'s
3. Evaluate $f_{\text{wide}}(1), f_{\text{wide}}(\omega), \ldots, f_{\text{wide}}(\omega^{T-1})$ mod m
4. Solve Transposed Vandermonde system (**next slide) to recover $_i$'s

Recovering big coefficients from f_{wide}

- Write $f_{\text{wide}}(X) =$  $_1 X$  $_1$ $+ \cdots +$  $_T X$  $_T$
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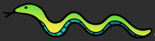
4. Solve Transposed Vandermonde system (**next slide) to recover  $_i$'s

COST: $O(T \log B))$ evaluation bits and $\widetilde{O}(T \log B)$ computation

Aside: Transposed Vandermonde

Recall:

- $f_{\text{wide}}(X) = \text{🐍}_1 X^{\text{🐘}_1} + \cdots + \text{🐍}_T X^{\text{🐘}_T}$
- We know 🐘_i's and have $f_{\text{wide}}(\omega^i)$'s

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega^{\text{🐘}_1} & \omega^{\text{🐘}_2} & \cdots & \omega^{\text{🐘}_T} \\ \omega^{2\text{🐘}_1} & \omega^{2\text{🐘}_2} & \cdots & \omega^{2\text{🐘}_T} \\ \vdots & \vdots & \vdots & \vdots \\ \omega^{(T-1)\text{🐘}_1} & \omega^{(T-1)\text{🐘}_2} & \cdots & \omega^{(T-1)\text{🐘}_T} \end{bmatrix} \begin{bmatrix} \text{🐍}_1 \\ \text{🐍}_2 \\ \vdots \\ \text{🐍}_T \end{bmatrix} = \begin{bmatrix} f_{\text{wide}}(1) \\ f_{\text{wide}}(\omega) \\ f_{\text{wide}}(\omega^2) \\ \vdots \\ f_{\text{wide}}(\omega^{T-1}) \end{bmatrix}$$

Can solve using $\widetilde{O}(T)$ field operations using fast polynomial arithmetic

Recovering big exponents from f

- Write $f(X) =$  $_1 X$  $^{1} + \cdots +$  $_T X$  ^{T}
- Assume all  $_i,$  $_i \le B$

Recovering big exponents from f

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Algorithm: Big Exponent Recovery

1. Create implicit polynomial g with tiny exponents and full exponents embedded in the coefficients
2. Use Big Coefficient Recovery to obtain coefficients of g
3. Extract actual 's and 's from coefficients of g

Plan

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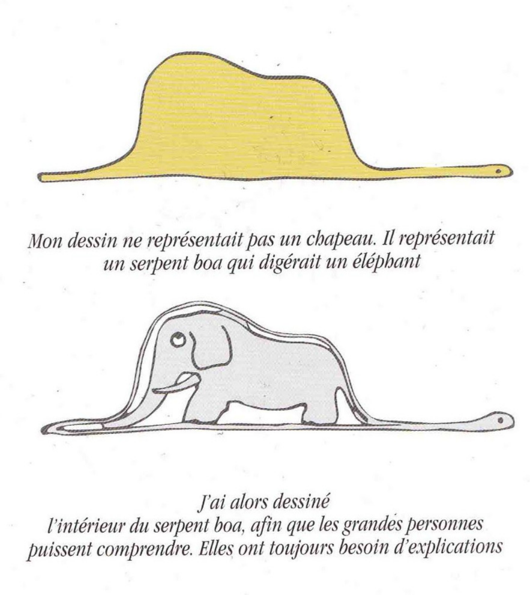

What is needed

1. Make exponents tiny: turn  $\mapsto$ 
2. Embed exponents in coefficients:  $+$  $\mapsto$ 
3. Make small q and $\omega \in \mathbb{F}_q$ "compatible with" large m and $\omega \in \mathbb{Z}/m\mathbb{Z}$

Trick 1: Linnick

 $\mapsto$ 

1. Choose tiny prime $p \in O(T \log B)$
2. Try to find a prime $q = pr + 1$

How large should we try for r?

How long will this take?

The dirty work

Sedunova 2018, Corollary 1.5

Let $\pi(x)$ denote the number of prime numbers $\leq x$, $\pi(x; m, a)$ the number of prime numbers $\leq x$ that are congruent to a modulo m, and $\ell(x)$ the smaller prime divisor of x. Then for any $\gamma \geq 4$ and $\lambda_1 \leq \lambda_2 \leq \gamma^{1/2}$,

$$\sum_{\substack{m \leq \lambda_2 \\ \ell(m) > \lambda_1}} \max_{2 \leq y \leq \gamma} \max_{a : \gcd(a,m)=1} \left| \pi(y; m, a) - \frac{\pi(y)}{\phi(m)} \right|$$
$$\leq 122.77 \left(14 \frac{\gamma}{\lambda_1} + 4\gamma^{1/2}\lambda_2 + 15\gamma^{2/3}\lambda_2^{1/2} + 4\gamma^{5/6} \ln(\frac{\lambda_2}{\lambda_1}) \right) (\ln \gamma)^{7/2}.$$

Our version

Giorgi, Grenet, Perret du Cray, R 2022

Given a bit-size $b \geq 60$, in worst-case poly(b) time, we can find a triple (p, q, ω) where w.h.p.

- p is a b-bit prime
- q is a prime with at most $6b$ bits
- $p \mid (q-1)$
- ω is a p-PRU in $\mathbb{F}_q$

Trick 2: Paillier

How to implicitly embed exponents in coefficients?

- Assume we can evaluate $\frac{d}{dX} f(X)$

  OR

- Use the fact that $(1+m)^e \bmod m^2 = 1 + em$

One-step coefficients and exponents embedding

Recall $f(X) =$  $_1 X$  $^{1} + \cdots +$  $_T X$  ^{T}

Fact

Let $g(X) = X \cdot f(X + m) + (1 - X) \cdot f(X) \mod m^2$.

Then the coefficient of X^i in g is  $_i \cdot (1 +$  $_i \cdot m) \mod m^2$

One-step coefficients and exponents embedding

Recall $f(X) =$  $_1 X$  $^{1} + \cdots +$  $_T X$  ^{T}

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Then the coefficient of X^i in g is  $_i \cdot (1 +$  $_i \cdot m) \mod m^2$

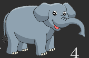
We can recover both  $_i$ and  $_i$ if m is large enough

Trick 3: Newton

Find large m, ω_m "consistent with" small q and ω

- Use $m = q^k$ for $k \geq \log_q B$
- Each p-PRU in $\mathbb{Z}/q^k\mathbb{Z}$ is 1-1 with the p-PRUs in $\mathbb{Z}/q^i\mathbb{Z}$ for $1 \leq i \leq k$
- We construct a Newton iteration to lift $\omega_m \mod q^k$ from $\omega \mod q$ in $O(\log k)$ steps.

Example: Setup

$f(X) =$  $_1 X$  $^1 +$  $_2 X$  $^2 + \cdots +$  $_4 X$  ^{4}

Bounds: $T = 4$, $B = 1000$

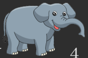
Example: Recover tiny exponents

$f(X) =$ 🐍$_1 X^{🐘_1} +$ 🐍$_2 X^{🐘_2} + \cdots +$ 🐍$_4 X^{🐘_4}$

Bounds: $T = 4,\ B = 1000$

1. Choose tiny $p = 11$, small $q = 23$, p-PRU $\omega = 6$
2. $2T$ evals $f(\omega_i) \mod q = 8, 22, 3, 5, 17, 11, 11, 8$
3. Berlekamp-Massey to find $\Lambda(Z) = x^4 + 18x^3 + 7x^2 + 3x + 16$
4. Multi-point evals to find roots: $\Lambda(\omega^2) = \Lambda(\omega^4) = \Lambda(\omega^6) = \Lambda(\omega^8) = 0$
5. Therefore 🐘's are $[2, 4, 6, 8]$

Example: Recover big coefficients

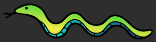
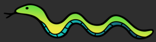
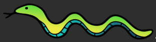
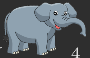
$f(X) =$  $_1 X$  $^1 +$  $_2 X$  $^2 + \cdots +$  $_4 X$  ^{4}

Bounds: $T = 4,\ B = 1000$

Recall: $p = 11$, small $q = 23$, p-PRU $\omega = 6$, and 's are $[2, 4, 6, 8]$

6. Set $m = q^3 = 12167$ and lift ω to p-PRU $\omega_m = 90603883 \bmod m^2$
7. Define $g(X) = X \cdot f(X + m) + (1 - X) \cdot f(X) \mod m^2$
8. T evals $g(\omega_m^i) \bmod m^2 = 126319619, 41994848, 22280517, 104183726$
9. Solve system to get s $= 120806932, 45091469, 111729907, 144763089$

Example: Recover f

$f(X) =$  $_1 X$  $^1 + $  $_2 X$  $^2 + \cdots +$  $_4 X$  ^{4}

Bounds: $T = 4$, $B = 1000$

Recall: $p = 11$, small $q = 23$, $m = 12167$,
and  s are $120806932, 45091469, 111729907, 144763089$

10 Unpack each  as in:

- $120806932 = 9929m + 789$
-  $_1 = 789$
-  $_1 = (9929/789) \bmod m = 321$

11 $f = 789X^{321} + 567X^{543} + 346X^{765} + 123X^{987}$

Plan

- What are we doing?
- How do we do it?
- The modular tricks
- Why did we do it?

What we have

Given a modular black box for an unknown polynomial $f \in \mathbb{Z}[x]$,
and bounds on f's sparsity and (uniform) term bit-length,
we can recover f in time proportional to the worst-case output size.

Easily extends to multivariate polynomials and rational coefficients

What we don't (yet) have

- Fast sparse interpolation using only low-precision evaluations
- Sensitivity to average term bit-length
- (Super)sparse rational function recovery
- Softly-optimal sparse interpolation over finite fields
- Numerical stability with (soft)-optimal complexity

Applications and Connections

- Sparse polynomial multiplication
- More generally: Avoid intermediate expression swell
- Related to Reed-Solomon decoding, exponential analysis, Hermite-Pade, ...

A brief history

- Ben-Or & Tiwari 1988
- Zippel 1990
- Kaltofen & Lakshman 1988
- Kaltofen, Lakshman, Wiley 1990
- Grigoriev, Karpinski, & Singer 1990
- Mansour 1995
- Huang & Rao 1996
- Murao & Fujise 1996
- Kaltofen & Lee 2003
- Avendaño, Krick, & Pacetti 2006
- Garg & Schost 2009
- Kaltofen 2010
- Javadi & Monagan 2010
- Giesbrecht & R 2011
- Cuyt & Lee 2011
- Arnold & R 2014
- van der Hoeven & Lecerf 2015
- Arnold, Giesbrecht & R 2016
- Huang & Gao 2017
- van der Hoven & Lecerf 2019
- Huang 2020
- Giorgi, Grenet, Perret du Cray, & R 2022

Merci encore !