Border rank, homogeneity and de-bordering paradigms in GCT

Based on joint works with – Prateek Dwivedi (IITK), Gorav Jindal (MPI), Fulvio Gesmundo (U. Saarland), Christian Ikenmeyer (U. Warwick), Vladimir Lysikov (U. Bochum), Nitin Saxena (IITK).

Pranjal Dutta
National University of Singapore

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- We study different measure on $S^d\mathbb{C}^n := \mathbb{C}[x_1, \cdots, x_n]_d$ = set of d-degree homogeneous polynomials.

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- Very few techniques are known that could potentially break this barrier.

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  - GCT captures 'algebraic approximations'.

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VNP = "hard to compute?" [Valiant 1979]

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(2) Over $\mathbb{R}$, $m \geq (n-1)^2 + 1$ [Yabe'15].

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  - Assuming GRH (Generalized Riemann hypothesis), the results hold over $\mathbb{C}$ as well.
- $\mathsf{P/poly} = \mathsf{NP/poly} \implies \mathsf{PH} = \Sigma_2$ (i.e. Polynomial Hierarchy collapses) [Karp-Lipton 1980].

Waring and border Waring rank

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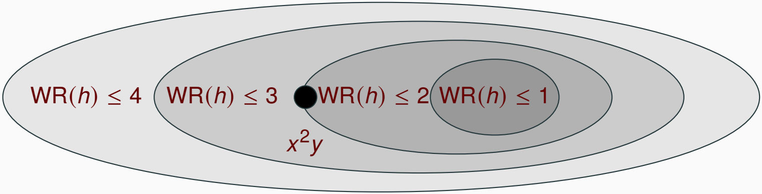

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GCT, border complexity and its advantages

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- Padding has issues pointed out in: [Kadish-Landsberg 2012], [Ikenmeyer-Panova 2015], [Bürgisser-Ikenmeyer-Panova 2016], via their *no-go* theorem, to study the *inconsistency* between representations of $\overline{\mathsf{GL}_{r^2} \circ \mathsf{det}_r}$ and $\overline{\mathsf{GL}_{r^2} \circ \ell^{r-n}\mathsf{perm}_n}$.

GCT's disadvantages: Homogeneous vs. non-homogeneous

- Valiant's determinant computation via entries with *affine linear polynomial* ($\mathsf{det}(\ell_1, \ldots, \ell_{n^2})$) introduces *non-homogeneity*.

- For using algebraic geometric tools, we require *padding*:

$$\overline{\mathsf{dc}}(f) \leq r \iff \ell^{r-\mathsf{deg}(f)} \cdot f \in \overline{\mathsf{GL}_{r^2} \circ \mathsf{det}_r}\ .$$

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- It is possible to frame these questions without *padding*! [DGIJL'23].

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  - Some more to come in the next slides...
- Upper bounds and lower bounds are *dual* to each other.
- Further potential applications in identity testing and understanding its 'robustness'.

A few more complexity measures

Kumar's complexity

- [Kumar'20, DGJIL'23] The *Kumar's complexity* of f, denoted $\mathsf{Kc}(f)$ is:

$$\mathsf{Kc}(f) \;:=\; \min\left\{ r \;:\; f \;=\; \alpha\left(\prod_{i=1}^{r}(1+\ell_j) - 1\right), \ell_j \text{ linear forms, } \alpha \in \mathbb{C} \right\}.$$

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- $\mathsf{Kc}(\ell^d) = d$, since for $\omega := \exp(2\pi\iota/d)$,

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Kumar's result and its converse

Universality [Kumar 2020]

For any $f \in S^d\mathbb{C}^n$, $\overline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \mathsf{WR}(f) < \infty$.

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Converse of Kumar [DGIJL'23]

For any $f \in S^d\mathbb{C}^n$, either $f = \prod_{i\in[d]} \ell_i$, or $\overline{\mathsf{WR}}(f) \leq \overline{\mathsf{Kc}}(f) \leq \deg(f) \cdot \overline{\mathsf{WR}}(f)$.

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Let $f \in S^d\mathbb{C}^n$. Chow rank of h, $\mathrm{CR}(f)$, is the smallest r such that h can be written as a sum of d-product of linear forms ℓ_i, i.e. $f = \sum_{i=1}^{r} \prod_{j=1}^{d} \ell_{i,j}$.

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- Exponential-gap between $\mathsf{WR}(f)$ and $\mathsf{CR}(f)$ (same in border):
  $\mathsf{WR}(x_1 \cdots x_n + y_1 \cdots y_n) = 2^n$, while $\mathsf{CR}(x_1 \cdots x_n + y_1 \cdots y_n) = 2$!

Chow rank vs. Kumar's complexity

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- [Kumar'20] For any $f \in \mathbb{C}[x_1, \ldots, x_n]$, we have $\overline{\mathsf{CR}^{\mathsf{aff}}}(f) \leq 2$, equivalently f has $\overline{\Sigma^{[2]}\Pi^{[D]}}\Sigma$-circuit [although $D \approx \mathsf{WR}(f)$ can be large].

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- How is $\overline{\mathrm{CR}^{\mathrm{aff}}}$ and $\mathrm{CR}^{\mathrm{aff}}$, or other measures related, when D is polynomially bounded?

Some upper bounds and lower bounds on border Chow rank

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- Does this hold for border?

De-bordering $\overline{\Sigma^{[k]}\Pi^{[D]}\Sigma}$ circuits

Upper bound for $\overline{\mathsf{CR}}$ [Dutta-Dwivedi-Saxena'21].

Let $f \in S^d\mathbb{C}^n$, s.t. $\overline{\mathsf{CR}}(f) = s$. Then,

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Corollary. For any constant $k \geq 1$, $\overline{\Sigma^{[k]}\Pi^{[D]}\Sigma} \subseteq \mathsf{VBP}$.

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- What does work (if at all!)?

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Hierarchy Theorem [Dutta-Saxena 2022]

Fix any constant $k \geq 1$. There is an explicit n-variate and $< n$ degree polynomial f such that f can be computed by a $\overline{\Sigma^{[k+1]}\Pi^{[n]}\Sigma}$ circuit such that if f is computed by $\overline{\Sigma^{[k]}\Pi^{[D]}\Sigma}$ circuits, then $D = 2^{\Omega(n)}$.

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- Note: $\mathrm{WR}(P_d) = 2^{\Omega(d)}$ Kumar's proof establishes that P_d has a $2^{O(d)}$ product fan-in $\Sigma^{[2]}\Pi\Sigma$ circuits, showing *optimality*!

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Hierarchy Theorem [Dutta-Saxena 2022]

Fix any constant $k \geq 1$. There is an explicit n-variate and $< n$ degree polynomial f such that f can be computed by a $\Sigma^{[k+1]}\Pi^{[n]}\Sigma$ circuit such that if f is computed by $\Sigma^{[k]}\Pi^{[D]}\Sigma$ circuits, then $D = 2^{\Omega(n)}$.

- Fix $k = 2$. Define the polynomial $P_d := x_1 \cdots x_d + y_1 \cdots y_d + z_1 \cdots z_d$, a degree-$d$ polynomial on $n = 3d$-variables.
- $\mathrm{CR}(P_d) = 3$.
- [DS'22] show that P_d requires $D \geq 2^{\Omega(d)}$- product fan-in, to be computed by $\Sigma^{[2]}\Pi^{[D]}\Sigma$ circuits.
- Note: $\mathrm{WR}(P_d) = 2^{\Omega(d)}$ Kumar's proof establishes that P_d has a $2^{O(d)}$ product fan-in $\Sigma^{[2]}\Pi\Sigma$ circuits, showing *optimality*!
- Classical is about *impossibility* while in border, it is about *optimality*.

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Thank you! Questions?