Sparse Interpolation and Exponential Analysis going hand in hand

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* joint work with Ferre Knaepkens

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Problem statement

$$x_s = s\Delta, \quad s = 0, 1, 2, \ldots$$

Sparse interpolation:

$$\sum_{j=1}^{n} \alpha_j x_s^{k_j} = f_s, \quad n \ll \max(k_j), \quad k_j \in \mathbb{N}$$

$$\sum_{j=1}^{n} \alpha_j \cos(\phi_j x_s) = f_s, \quad \phi_j \in \mathbb{R}$$

Exponential analysis:

$$\sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s) = f_s, \quad \phi_j \in \mathbb{C}$$

$$x_s = s\Delta, \quad s = 0, 1, 2, \dots$$

Sparse interpolation:

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Exponential analysis:

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Given:

data f_s at x_s for $s = 0, \ldots, N-1$ where $x_s = s\Delta$

Regression:

$$\arg\min_{\alpha,\phi} \left\|\left(f_s - \sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s)\right)\right\|_2^2$$

choose n, compute ϕ_j and α_j

Reconstruction:

$$f_s = \sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s), \qquad s = 0, \ldots, N-1$$

detect n, extract ϕ_j and α_j

Condition:

Nyquist constraint $\Im(\phi_j) \in \left]-\pi/\Delta, \pi/\Delta\right[$

Challenges:

- numerical sensitivity [Hildebrand, 1956]

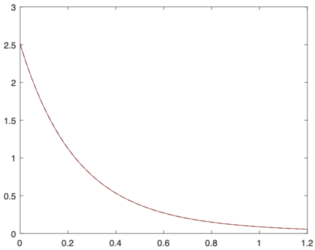
  "The slight errors in the given data here lead to completely incorrect information ... "

- ill-conditioning

  "... particularly in the case of clustered frequencies."

- ill-posed [Lanczos, 1956]

  $n = 2, \phi = (-4.45, -1.58), \alpha = (2.202, 0.305)$
  $n = 3, \phi = (-1, -3, -5), \alpha = (0.0951, 0.8607, 1.5576)$



- sub-Nyquist method [Cuyt and Lee, 2020]

  "... generalized to uniform sub-Nyquist sampling ..."

- validated implementation [Knaepkens and Cuyt, 2023]

  parallel implementation [Briani et al., 2020]

- curse of dimensionality

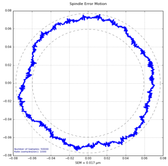
$d = 1 \longrightarrow d > 1$ [Hua, 1992] $\cdots$ [Sauer, 2018]

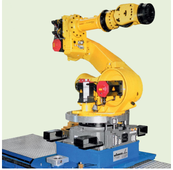
		data usage			
d	n	$\mathcal{O}(2^{d-1}n)$	$\mathcal{O}\left((d+1)n^2 \log^{2d-2} n\right)$	$\mathcal{O}\left((d+1)n\right)$	$\mathcal{O}\left(d^2n^2\right)$
3	100	4.00×10^2	1.80×10^7	4.00×10^2	9.00×10^4
3	500	2.00×10^3	1.49×10^9	2.00×10^3	2.25×10^6
3	1000	4.00×10^3	9.11×10^9	4.00×10^3	9.00×10^6
7	100	6.4×10^3	7.28×10^{12}	8.00×10^2	4.90×10^5
7	500	3.2×10^4	6.64×10^{15}	4.00×10^3	1.23×10^7
7	1000	6.4×10^4	9.44×10^{16}	8.00×10^4	4.90×10^7
12	8	1.64×10^4	8.22×10^9	1.04×10^2	9.22×10^3
12	25	5.12×10^4	1.20×10^{15}	3.25×10^2	9.00×10^4
12	100	2.05×10^5	5.07×10^{19}	1.30×10^3	1.44×10^6
25	8	1.34×10^8	3.04×10^{18}	2.08×10^2	4.00×10^4
25	25	4.19×10^8	3.81×10^{28}	6.50×10^2	3.91×10^5
25	100	1.68×10^9	1.78×10^{37}	2.60×10^3	6.25×10^6

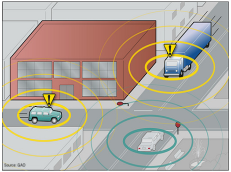
Solutions:

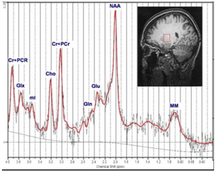
- multiscale matrix pencil formulation
  $\longrightarrow$ improve conditioning and enable sub-Nyquist
- sparse methods in computer algebra
  $\longrightarrow$ eliminate curse of dimensionality
- Froissart phenomenon in Padé approximation
  $\longrightarrow$ stabilize n and the computation of ϕ and α

Applications

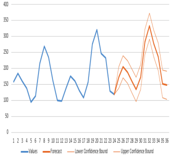

Spindle Error Motion
SEM = 0.017 µm

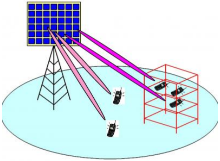



Source: GAO


NAA
Cr+PCr
Cr+PCR
Cho
Glx
mI
Glu
Gln
MM

EXPOPOWER


Values
Forecast
Lower Confidence Bound
Upper Confidence Bound



Exponential analysis in physical phenomena:

- texture analysis
- motor fault diagnosis / gearbox monitoring
- drug clearance / glucose tolerance
- magnetic resonance / infrared spectroscopy
- vibration analysis / seismic data analysis
- radar imaging / automotive radar
- repetitive robotic movements
- typed keystroke recognition
- time series analysis
- liquid (explosive) identification
- direction / angle of arrival estimation
- RFID
- ...

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Maplesoft

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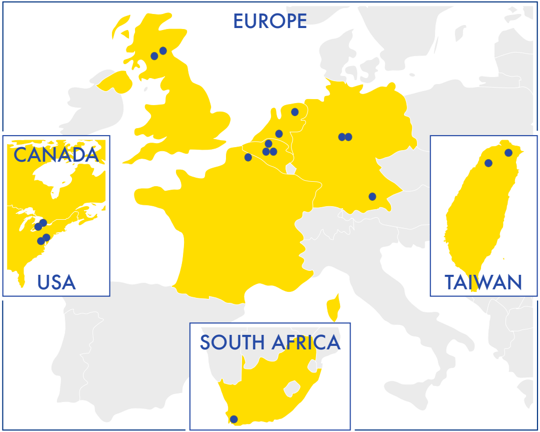
University of Waterloo

Craft Prospect

University of Delaware

ITRI Industrial Technology Research Institute

NIH National Institute on Aging



SIEMENS

KBC

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Virtonomy.io

Mahr

Exponential analysis

Exponential analysis

Gaspard Riche de Prony

Exponential analysis

interpolation problem:

$$\sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s) = f_s, \qquad s = 0, \ldots, 2n-1$$

$$x_s = s\Delta$$

$$|\Im(\phi_j)\Delta| < \pi$$

$$\Phi_j := \exp(\phi_j \Delta)$$

$$f_s = \sum_{j=1}^{n} \alpha_j \Phi_j^s, \qquad s = 0, \ldots, 2n-1$$

$$\begin{cases} \alpha_1 + \cdots + \alpha_n = f_0 \\ \alpha_1\Phi_1 + \cdots + \alpha_n\Phi_n = f_1 \\ \quad \vdots \\ \alpha_1\Phi_1^{2n-1} + \cdots + \alpha_n\Phi_n^{2n-1} = f_{2n-1} \end{cases}$$

Exponential analysis

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Exponential analysis

finding Φ_j:

Hankel matrix:

$$H_n^{(r)} = \begin{pmatrix} f_r & \dots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \dots & f_{r+2n-2} \end{pmatrix}, \qquad r = 0, 1, \dots$$

$$H_n^{(0)} = \begin{pmatrix} 1 & \dots & & 1 \\ \Phi_1 & \Phi_2 & \dots & \Phi_n \\ \vdots & & & \vdots \\ \Phi_1^{n-1} & \dots & & \Phi_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \Phi_1 & \dots & \Phi_1^{n-1} \\ \vdots & \Phi_2 & & \vdots \\ & \vdots & & \vdots \\ 1 & \Phi_n & \dots & \Phi_n^{n-1} \end{pmatrix}$$

$$= V_n^T D_\alpha V_n$$

Exponential analysis

$$H_n^{(r)} = V_n^T D_\alpha \begin{pmatrix} \Phi_1^r & & \\ & \ddots & \\ & & \Phi_n^r \end{pmatrix} V_n$$

$$\det\left(H_n^{(1)} - \lambda H_n^{(0)}\right) = \det\left(V_n^T D_\alpha \begin{pmatrix} \Phi_1 - \lambda & & \\ & \ddots & \\ & & \Phi_n - \lambda \end{pmatrix} V_n \right)$$
$$= 0 \text{ for } \lambda = \Phi_j, \quad j = 1, \dots, n$$

$$H_n^{(1)} v_j = \Phi_j H_n^{(0)} v_j, \quad j = 1, \dots, n$$

[Hua and Sarkar, 1990]

Exponential analysis

finding ϕ_j:

$\arg(\Phi_j)$?

$$\begin{aligned} |\Im(\phi_j)\Delta| < \pi \Rightarrow \arg(\Phi_j) &= \arg(\exp(\phi_j\Delta)) \\ &= \Im(\phi_j)\ \Delta \in\ ]-\pi, \pi[\end{aligned}$$

Exponential analysis

finding α_j:

$$\sum_{j=1}^{n} \alpha_j \Phi_j^s = f_s, \qquad s = 0, \ldots, 2n-1$$

$$\begin{pmatrix} 1 & \ldots & 1 \\ \Phi_1 & \ldots & \Phi_n \\ \vdots & & \vdots \\ \Phi_1^{2n-1} & \ldots & \Phi_n^{2n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Exponential analysis

finding n:

$$m < n : \left| H_m^{(r)} \right| \not\equiv 0, \quad r = 0, 1, \ldots$$

$$m = n : \left| H_m^{(r)} \right| \neq 0 \quad \Phi_j \neq \Phi_k \text{ for } j \neq k$$ [Kaltofen and Lee, 2003]

$$m > n : \left| H_m^{(r)} \right| \equiv 0, \quad r = 0, 1, \ldots$$

[Henrici, 1974]

$$n = \text{rank } H_n^{(r)}, \quad r \geq 0, \quad 2m - 1 \leq N - r$$

Exponential analysis

scale and shift paradigm:

$$\sigma \in \mathbb{N}, \quad \tau \in \mathbb{Z}$$

$$x_{\tau+s\sigma} = (\tau + s\sigma)\Delta$$

$$|\Im(\phi_j)\Delta| < \pi$$

$$H_{n,\sigma}^{(\tau)} = \begin{pmatrix} f_\tau & \cdots & f_{\tau+(n-1)\sigma} \\ f_{\tau+\sigma} & \cdots & f_{\tau+n\sigma} \\ \vdots & \iddots & \vdots \\ f_{\tau+(n-1)\sigma} & \cdots & f_{\tau+(2n-2)\sigma} \end{pmatrix}$$

Exponential analysis

$$H_{n,\sigma}^{(\tau)} = V_{n,\sigma}^T D_\alpha \begin{pmatrix} \Phi_1^\tau & & \\ & \ddots & \\ & & \Phi_n^\tau \end{pmatrix} V_{n,\sigma}$$

$$V_{n,\sigma} = \begin{pmatrix} 1 & \Phi_1^\sigma & \dots & \Phi_1^{(n-1)\sigma} \\ \vdots & \Phi_2^\sigma & & \vdots \\ & \vdots & & \vdots \\ 1 & \Phi_n^\sigma & \dots & \Phi_n^{(n-1)\sigma} \end{pmatrix}$$

Exponential analysis

finding Φ_j^σ and α_j:

$$\det\left(H_{n,\sigma}^{(\sigma)} - \lambda H_{n,\sigma}^{(0)}\right) = \det\left(V_{n,\sigma}^T D_\alpha \begin{pmatrix} \Phi_1^\sigma - \lambda & & \\ & \ddots & \\ & & \Phi_n^\sigma - \lambda \end{pmatrix} V_{n,\sigma}\right)$$

$$= 0 \text{ for } \lambda = \Phi_j^\sigma, \quad j = 1, \ldots, n$$

$$\begin{pmatrix} 1 & \ldots & 1 \\ \Phi_1^\sigma & \ldots & \Phi_n^\sigma \\ \vdots & & \vdots \\ \Phi_1^{(2n-1)\sigma} & \ldots & \Phi_n^{(2n-1)\sigma} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ f_{(2n-1)\sigma} \end{pmatrix}$$

$$|\Im(\Phi_j^\sigma)| = |\Im(\phi_j)\sigma\Delta| < \sigma\pi \quad !$$

Exponential analysis

finding Φ_j^σ and α_j:

$$\det\left(H_{n,\sigma}^{(\sigma)} - \lambda H_{n,\sigma}^{(0)}\right) = \det\left(V_{n,\sigma}^T D_\alpha \begin{pmatrix} \Phi_1^\sigma - \lambda & & \\ & \ddots & \\ & & \Phi_n^\sigma - \lambda \end{pmatrix} V_{n,\sigma}\right)$$

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$$|\Im(\Phi_j^\sigma)| = |\Im(\phi_j)\sigma\Delta| < \sigma\pi \quad !$$

Exponential analysis

finding Φ_j^τ:

$$f_{\tau+s\sigma} = \sum_{j=1}^{n} \alpha_j \exp(\phi_j(\tau+s\sigma)\Delta), \qquad s = 0,\ldots,n-1$$

$$= \sum_{j=1}^{n} \alpha_j \Phi_j^{\tau+s\sigma} = \sum_{j=1}^{n} \left(\alpha_j \Phi_j^\tau\right) \Phi_j^{s\sigma}$$

$$\begin{pmatrix} 1 & \ldots & 1 \\ \Phi_1^\sigma & \ldots & \Phi_n^\sigma \\ \vdots & & \vdots \\ \Phi_1^{(n-1)\sigma} & \ldots & \Phi_n^{(n-1)\sigma} \end{pmatrix} \begin{pmatrix} \alpha_1 \Phi_1^\tau \\ \vdots \\ \alpha_n \Phi_n^\tau \end{pmatrix} = \begin{pmatrix} f_\tau \\ \vdots \\ f_{\tau+(n-1)\sigma} \end{pmatrix}$$

$$\Phi_j^\tau = \frac{\alpha_j \Phi_j^\tau}{\alpha_j} \quad \begin{matrix} \to \text{ from } f_{\tau+s\sigma} \\ \to \text{ from } f_{s\sigma} \end{matrix}$$

Exponential analysis

finding ϕ_j:

$$\text{from } \Phi_j^\sigma : \phi_j \Delta \in S_j := \left\{ \frac{\mathrm{Log}(\exp(\phi_j \sigma \Delta)) + 2\pi \mathrm{i} \ell}{\sigma}, \quad \ell = 0, \dots, \sigma - 1 \right\}$$

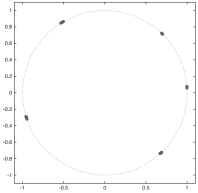
$$\text{from } \Phi_j^\tau : \phi_j \Delta \in T_j := \left\{ \frac{\mathrm{Log}(\exp(\phi_j \tau \Delta)) + 2\pi \mathrm{i} \ell}{\tau}, \quad \ell = 0, \dots, \tau - 1 \right\}$$

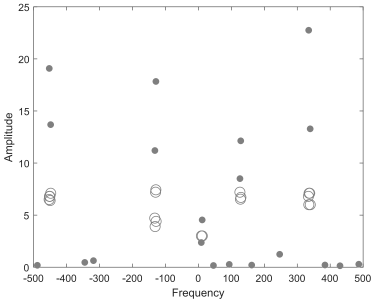
$\gcd(\sigma, \tau) = 1 \Rightarrow$ de-aliased result $S_j \cap T_j =$ singleton

scheme uses $3n$ samples when $|\Im(\phi_j \sigma \Delta)| \geq \pi$

Illustration: clustered Φ_j

- $N = 240, |\Im(\phi_j)\Delta| < 0.907\pi$
- $n = 20$ (signal space)
- $m - n = 40$ (noise space)
- SNR = 32 dB
- $\sigma = 1,\ \tau = 0$

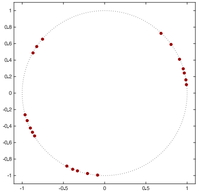


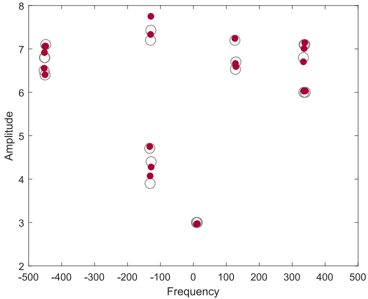


MP output $(\Im(\phi_j), |\alpha_j|)$, $j = 1, \ldots, 20$ from $f_0, \ldots, f_{239}$

Illustration: clustered Φ_j

- $N = 240, |\Im(\phi_j)\Delta| < 0.907\pi$
- $n = 20$ (signal space)
- $m - n = 40$ (noise space)
- SNR = 32 dB
- $\sigma = 11,\ \tau = 5$

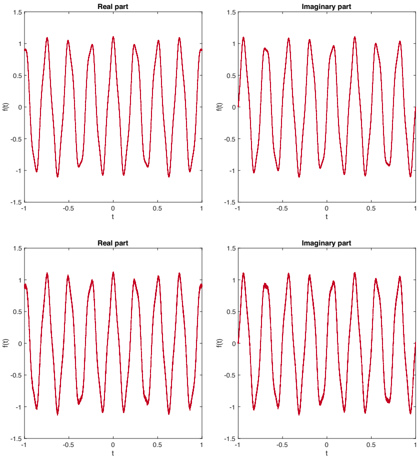




MP output $(\Im(\phi_j), |\alpha_j|)$, $j = 1, \ldots, 20$ from $f_0, f_{11}, \ldots, f_{11\times 179}$, $f_5, f_{16}, \ldots, f_{5+11\times 59}$

Illustration: superresolution

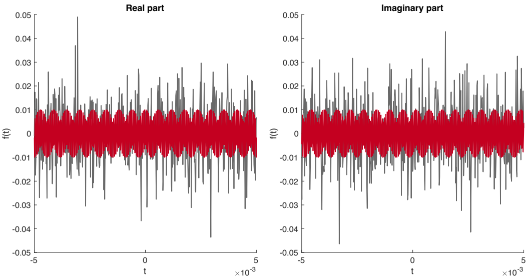
$$\phi(x) = \exp(8\pi i x) + 0.1 \exp(27\pi i x) + 0.01 \exp(2^{16}\pi i x)$$



noise free

35 dB noise

Illustration: superresolution



- third component burried under 35 dB of noise (less than 2 decimal digits of samples untouched)
- $n = 3$ components can be retrieved from 1140 samples using the multiscale matrix factorizations
- $\Delta = 1/65600, \sigma = 65, \tau = 7, (\tau, 2\tau, 3\tau, 4\tau), m = 20, N = 60,$ consecutively starting at $f_{\ell\tau}, \ell = 0, \ldots, 14$

Rational approximation

Rational approximation

Marcel Froissart

Rational approximation

$$
\begin{aligned}
f_s &= \sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s), \qquad s = 0, 1, \ldots, 2n-1, \ldots, N-1 \\
&= \sum_{j=1}^{n} \alpha_j \Phi_j^s, \qquad \Phi_j = \exp(\phi_j \Delta) \text{ with } x_s = s\Delta
\end{aligned}
$$

$$f(z) = \sum_{s=0}^{\infty} f_s z^s = \sum_{j=1}^{n} \frac{\alpha_j}{1 - z\Phi_j}$$

$$f(z) = [n-1/n](z) = p(z)/q(z)$$

$$q(z) = \prod_{j=1}^{n} (1 - z\Phi_j)$$

[Padé, 1892]

Rational approximation

$$f_s = \sum_{j=1}^{n} \alpha_j \exp(\phi_j x_s), \qquad s = 0, 1, \ldots, 2n-1, \ldots, N-1$$

$$= \sum_{j=1}^{n} \alpha_j \Phi_j^s, \qquad \Phi_j = \exp(\phi_j \Delta) \text{ with } x_s = s\Delta$$

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$$f(z) = [n-1/n](z) = p(z)/q(z)$$

$$q(z) = \prod_{j=1}^{n} (1 - z\Phi_j)$$

[Padé, 1892]

Rational approximation

$$f(z) = \sum_{s=0}^{\infty} f_s z^s$$

$$\frac{p(z)}{q(z)}, \quad p(z) = \sum_{j=0}^{n-1} a_j z^j, \quad q(z) = \sum_{j=0}^{n} b_j z^j$$

$$\left(\sum_{s=0}^{\infty} f_s z^s \right) q(z) - p(z) = \sum_{j \geq 2n} c_j z^j$$

Rational approximation

$$\left(\sum_{s=0}^{\infty} f_s z^s\right) q(z) - p(z) = \sum_{j \geq 2n} c_j z^j$$

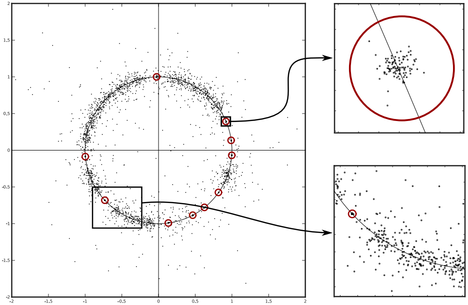
$$\begin{cases} f_0 b_0 = a_0 \\ f_1 b_0 + f_0 b_1 = a_1 \\ \quad \vdots \\ f_{n-1} b_0 + \cdots + f_0 b_{n-1} = a_{n-1} \end{cases}$$

$$\begin{cases} f_n b_0 + \cdots + f_0 b_n = 0 \\ \quad \vdots \\ f_{2n-1} b_0 + \cdots + f_{n-1} b_n = 0 \end{cases}$$

Rational approximation

$$[m-1/m](z) \approx [n-1/n](z) \times \prod_{j=1}^{m-n} \frac{(x-\xi_{j,m}-\delta_{j,m})}{(x-\xi_{j,m})}, \qquad m > n$$

[Gilewicz and Pindor, 1999] [Knaepkens and Cuyt, 2023]



Froissart doublets: $n = 10$, $m = 30$, various $\varepsilon(z) = \sum_{s=0}^{\infty} \varepsilon_s z^s$

Rational approximation

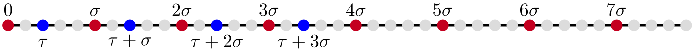
$f_s, \quad s = 0, 1, \ldots, N-1$

$\sigma \in \mathbb{N}, \quad \tau \in \mathbb{Z}, \quad \gcd(\sigma, \tau) = 1$

$F_k := \{f_{k+s\sigma} \; : s = 0, \ldots, \}, \quad k = 0, \ldots, \sigma - 1$

each F_k suffers independent realization of the noise

$F_k \to \Phi_j^\sigma, \alpha_j, \Phi_j^\tau, S_j \cap T_j = \{\Phi_j\}$



after all σ runs: $\begin{cases} \#\text{clusters} = n \\ \#\{\Phi_j\text{-cluster}\} = \sigma \end{cases}$

- automatic estimation of n
- validation of the Φ_j
- parallellism in the algorithm

Rational approximation

$$f_s, \quad s = 0, 1, \ldots, N-1$$

$$\sigma \in \mathbb{N}, \quad \tau \in \mathbb{Z}, \quad \gcd(\sigma, \tau) = 1$$

$$F_k := \{f_{k+s\sigma} \ : s = 0, \ldots, \}, \quad k = 0, \ldots, \sigma - 1$$

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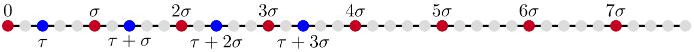

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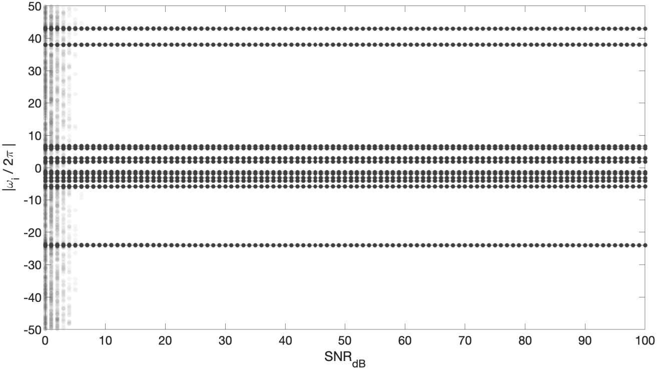
Illustration: validation

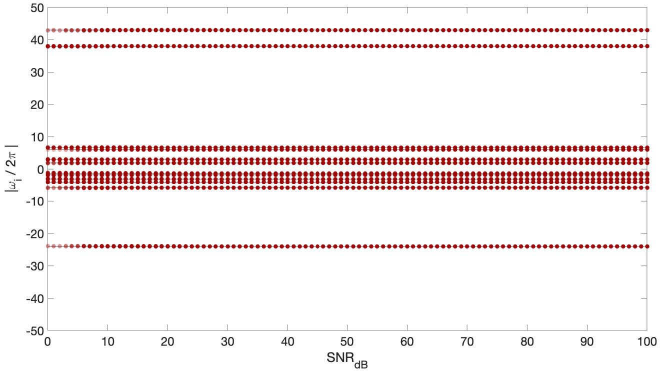
- $N = 300, |\Im(\phi_j)\Delta| < 0.865\pi$
- $n = 12$ (signal space, n is passed)
- $m - n = 88$ (noise space)
- SNR $\in [0, 100]$ dB (repeated 500 times)
- $\sigma = 1,\ \tau = 0$



MP output $(\mathrm{SNR}, \Im(\phi_j)), j = 1, \ldots, 12$

Illustration: validation

- $N = 300, |\Im(\phi_j)\Delta| < 0.865\pi$
- $n = 12$ (signal space)
- $m - n = 15$ or 14 (noise space)
- SNR $\in [0, 100]$ dB (repeated 500 times)
- $\sigma = 7$, $\tau = 0, \ldots, \sigma - 1$



VEXPA output $(\mathrm{SNR}, \Im(\phi_j)), j = 1, \ldots, 12$

Illustration: robustness

- $n = 3, \Delta = 0.001, N = 300$
- 30 dB noise
- MP without and with validation ($\sigma = 7, \tau = 11$)
- RMSE=0.008 for both

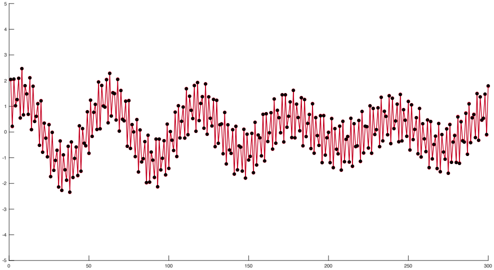
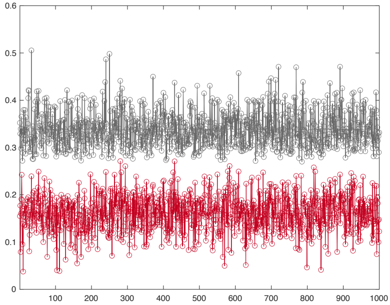


Illustration: robustness

- $n = 3, \Delta = 0.001, N = 300$
- 30 dB noise
- 2 random (real) outliers ($15 \leq |\cdot| \leq 25$) at random locations
- MP without (grey) and with (red) validation ($\sigma = 7, \tau = 11$)



RMSE: 1000 repetitions

d-Dimensional analysis

d-Dimensional

$$f(t_1,\ldots,t_d) = \sum_{j=1}^{n} \alpha_j \exp\left(\phi_{j1} t_1 + \cdots + \phi_{jd} t_d\right) = \sum_{j=1}^{n} \alpha_j \exp(\langle \phi_j, t\rangle),$$

$$(d+1)n \text{ parameters } \alpha_j, \phi_{j\ell} \in \mathbb{C}$$

$$f_{s1} := f(s\Delta_1), \quad |\Im(\langle \phi_j, \Delta_1\rangle)| < \pi, \quad \Delta_1 \in \mathbb{R}^d, \quad s = 0, \ldots 2n-1$$

$$f_{s\ell} := f(s\Delta_1 + \Delta_\ell), \quad |\Im(\langle \phi_j, \Delta_\ell\rangle)| < \pi, \quad \ell = 2, \ldots, d, \quad \Delta_\ell \in \mathbb{R}^d$$

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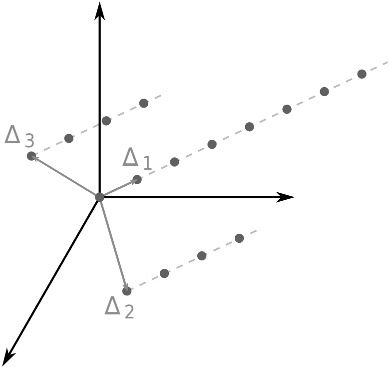

$d = 3, n = 4$

$\Delta_1 = (\Delta_{11}, \Delta_{12}, \Delta_{13})$
$\Delta_2 = (\Delta_{21}, \Delta_{22}, \Delta_{23})$
$\Delta_3 = (\Delta_{31}, \Delta_{32}, \Delta_{33})$

$s\Delta_1, s = 0, \ldots, 2n-1$
$s\Delta_1 + \Delta_\ell, s = 0, \ldots, n-1$

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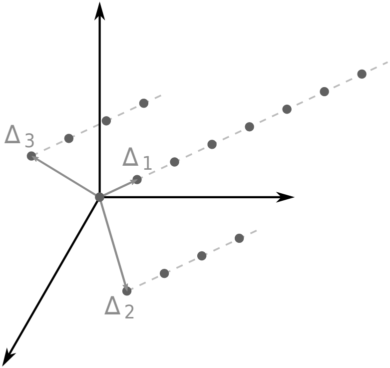

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fill $H_n^{(0)}$ and $H_n^{(1)}$ with f_{s1}:

$\exp(\langle \phi_j, \Delta_1 \rangle)$ is a generalized eigenvalue of $H_n^{(1)} v_j = \lambda_j H_n^{(0)} v_j$

consider $f_{s\ell}$, $\ell = 2, \ldots, d$:

ratio of $\dfrac{\alpha_j \exp(\langle \phi_j, \Delta_\ell \rangle)}{\alpha_j}$ $\begin{array}{l} \rightarrow \text{ from } f_{s\ell} \\ \rightarrow \text{ from } f_{s1} \end{array}$

$\Delta_1, \Delta_2, \ldots, \Delta_d$ linearly independent:

$\exp(\langle \phi_j, \Delta_1 \rangle)$ and $\exp(\langle \phi_j, \Delta_\ell \rangle), \ell = 2, \ldots, d$

$\rightarrow \phi_{j\ell}, j = 1, \ldots, n, \ell = 1, \ldots, d$

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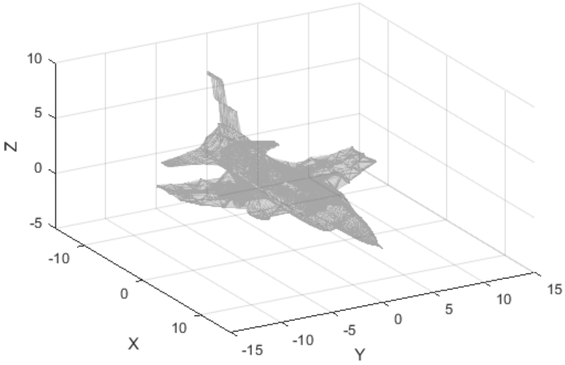
Radar imaging

- 90000 samples (here 30000 per $\Delta_{1,2,3}$) $\ll$ literature (only 4%)
- in 11 snapshots, SNR=20 dB
- cluster radii gradually increased from 10^{-5} to 10^{-2}
- validation = 9/11
- 934/1000 identified within 20cm



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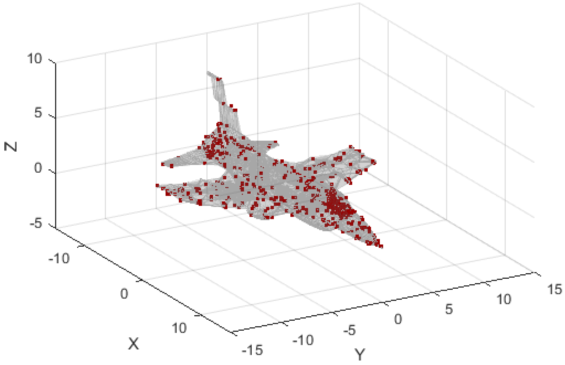

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