First-Order Factors of Linear Mahler Operators

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Joint work with Th. Dreyfus, Ph. Dumas, and M. Mezzarobba

In dedication to Marko Petkovšek.

Linear Mahler Operators and Mahler Function

Linear Mahler equation

$$\ell_r(x)y(x^{b^r}) + \cdots + \ell_1(x)y(x^b) + \ell_0(x)y(x) = 0 \qquad \text{(L)}$$

for a *radix* $b \in \mathbb{N}_{\geq 2}$, an *order* $r \in \mathbb{N}_{\geq 0}$, rational functions $\ell_i \in \bar{\mathbb{Q}}(x)$.

Operator notation

In the skew algebra $\bar{\mathbb{Q}}(x)\langle M \rangle$ where $Mx = x^b M$, write

$$L := \ell_r(x)M^r + \cdots + \ell_1(x)M + \ell_0(x).$$

Action: $My(x) = y(x^b)$. (L) $\Leftrightarrow Ly(x) = 0$.

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→ Transcendence theory, Automata theory, “Divide-and-conquer” recurrences, Difference Galois theory, Computer algebra.

Mahler, Cobham, Christol, Kamae, Mendès France, Rauzy, Loxton, v. d. Poorten, Nishioka, Allouche, Shallit, Becker, Dumas, Bell, Coons, Philippon, Adamczewski, Faverjon, Dreyfus, Hardouin, Roques, Smertnig, …

Mahler-Hypergeometric Solutions and First-Order Factors

Mahler-Hypergeometric functions (w.r.t. a given base b)

The function y is *Mahler* if it satisfies some (L) of any order,
hypergeometric if it satisfies some (L) of order 1.

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Problem

Given some skew polynomial $L = L(x, M)$, several equivalent formulations:

- Find all hypergeometric solutions y of the linear Mahler equation

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- Find all first-order right-hand factors $M - u$ of L for $u \in \bar{\mathbb{Q}}(x)$.
- Find all rational solutions u of the Riccati Mahler equation

$$\ell_r(x)u(x)\cdots u(x^{b^{r-1}}) + \cdots + \ell_2(x)u(x)u(x^b) + \ell_1(x)u(x) + \ell_0(x) = 0. \qquad \text{(R)}$$

$u = \frac{My}{y}$. lhs of (R) = remainder in division of L by $M - u$.

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We provide algorithms following two algorithmic approaches.

Outline

- Motivating examples
- First approach: generalizing Petkovšek's algorithm
- An effective difference algebra for solutions
- Second approach: structured Hermite–Padé approximants
- Comparison of the approaches
  and Application to hypertranscendence

Part I

Motivating Examples

Paradigmatic Examples of Mahler Series

Thue–Morse sequence over the alphabet $\{-1, 1\}$ (2-automatic)

$$y(x) = \prod_{j\geq 0}(1 - x^{2^j})$$

fixpoint of the morphism $a \to ab$, $b \to ba$: $a.b.ba.baab.baababba\ldots$

Stern–Brocot sequence (2-regular but not 2-automatic)

$$y(x) = \prod_{j\geq 0}(1 + x^{2^j} + x^{2^{j+1}})$$

explicit bijection $\mathbb{N} \simeq \mathbb{Q}_{\geq 0}$: $n \mapsto [x^n]y/[x^{n+1}]y$

(2-Mahler but not 2-regular)

$$y(x)^{-1} = \prod_{j\geq 0}(1 - x^{2^j})^{-1}$$

expressions of $n \in \mathbb{N}$ in the form $n = n_0 + n_1 2 + n_2 2^2 + \cdots$ where $n_i \in \mathbb{N}$

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$$y(x)^{-1} = \prod_{j\geq 0}(1 - x^{2^j})^{-1} \qquad \to \quad u(x) = 1 - x$$

expressions of $n \in \mathbb{N}$ in the form $n = n_0 + n_1 2 + n_2 2^2 + \cdots$ where $n_i \in \mathbb{N}$

Ramified Mahler-Hypergeometric Solutions

Hypergeometric = infinite product + log-factor + a ramification order

$$y := (\ln x)^{\log_3 \lambda} x^{1/2} \prod_{k \geq 0} \frac{1 - 7x^{3^k}}{1 + 2x^{3^k}} \qquad (b = 3)$$

is annihilated by

$$\begin{aligned} L &:= (1 - 7x^3)M^2 + \left(2x - 14x^2 - \lambda x^3 - 2\lambda x^6\right)M + 2\lambda x^2(1 + 2x) \\ &= \left(M - 2x\right)\left((1 - 7x)M - \lambda x(1 + 2x)\right). \end{aligned}$$

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Linear equations with no ramification can need ramification to be solved.
A ramified y with unramified $u = My/y$ is possible.

Disproving Hypergeometricity

Missing digit in ternary expansion (OEIS A005836)

$L := 3(1+x^2)^2M^2 - (1+3x+4x^2)M + x$ for $b = 2$ annihilates

$$y(x) := \sum_{n \geq 0} (n\text{-th positive integer written without 2 in base 3})\, x^n$$
$$= 1x^1 + 3x^2 + 4x^3 + 9x^4 + 10x^5 + \cdots .$$

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Unique monic right-hand first-order factor is $M - \dfrac{1}{3(1+x)}$

$\Rightarrow$ all hypergeometric solutions in $\bar{\mathbb{Q}}\dfrac{(\ln x)^{\log_2(1/3)}}{1-x}$

$\Rightarrow$ $y(x)$ is not hypergeometric.

Parametrized Mahler-Hypergeometric Solutions

Remember the differential case, $D = \dfrac{d}{dx}$:

$$Dx = xD + 1 \quad \Rightarrow \quad \forall r,\ D^2 = \left(D + \frac{1}{x+r}\right)\left(D - \frac{1}{x+r}\right),$$

in relation to: $\bar{\mathbb{Q}}\, x \oplus \bar{\mathbb{Q}}\, 1 = \bigcup_{r \in \bar{\mathbb{Q}}} \bar{\mathbb{Q}}\,(x + r)$.

Parametrized Mahler-Hypergeometric Solutions

Parities of digit repetitions in ternary expansion

Adamczewski and Faverjon (2017) introduce

$$S_a := \left\{ n \;\middle|\; \text{even number of } a\text{'s in ternary expansion of } n \right\}, \quad a = 1, 2,$$

$$y_1(x) := \sum_{n \in S_1 \cap S_2} x^n, \;\; y_2(x) := \sum_{n \in \bar{S}_1 \cap S_2} x^n, \;\; y_3(x) := \sum_{n \in S_1 \cap \bar{S}_2} x^n, \;\; y_4(x) := \sum_{n \in \bar{S}_1 \cap \bar{S}_2} x^n$$

and show

$$\boldsymbol{y}(x) = A(x)\boldsymbol{y}(x^3) \qquad \text{for} \qquad \boldsymbol{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} 1 & x & 0 & x^2 \\ x & 1 & x^2 & 0 \\ 0 & x^2 & 1 & x \\ x^2 & 0 & x & 1 \end{pmatrix}.$$

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→ Common linear Mahler equation: order 4, degree 258.

→ Hypergeometric solutions correspond to a ratio u among

$$\frac{1}{1 - x - x^2}, \qquad \frac{1}{1 + x - x^2}, \qquad \frac{g_1 + g_2 x^3}{g_1 + g_2 x} \frac{1}{1 + x^2 + x^4} \text{ for } (g_1 : g_2) \in \mathbb{P}^1(\bar{\mathbb{Q}}).$$

None of the y_i is hypergeometric.

Part II

First Approach: Generalizing Petkovšek's Algorithm

Classical Algorithms by Gosper–Petkovšek Forms

shift $x \mapsto x+1$ (Petkovšek, 1992)

$$u(x) = \eta \frac{C(x+1)}{C(x)} \frac{A(x)}{B(x)} \qquad + \text{ coprimality constraints}$$

q-shift $x \mapsto qx$ (Abramov, Paule, Petkovšek, 1998)

$$u(x) = \eta \frac{C(qx)}{C(x)} \frac{A(x)}{B(x)} \qquad + \text{ coprimality constraints}$$

Mahler (order 2) (Roques, 2018)

$$u(x^b) = \eta \frac{C(x^b)}{C(x)} \frac{A(x)}{B(x)} \qquad + \text{ coprimality constraints}$$

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All those algorithms:

- iterate on factors of A of ℓ_0 and B of ℓ_r (or slight variations),
- determine a polynomial equation on η + a degree bound on C,
- solve an auxiliary linear functional equation for C.

New Algorithm for Mahler Equations of Any Order

$$\sum_{i=0}^{r} \ell_i(x) \prod_{j=0}^{i-1} u(x^{b^j}) = 0$$

Bounded Gosper–Petkovšek forms (exist for any $u \in \mathbb{C}(x)$)

$$\begin{cases} x = t^{b^{r-1}} \\ u(t^{b^{r-1}}) = \eta \dfrac{C(t^b)}{C(t)} \dfrac{A(t^{b^{r-1}})}{B(t)} \end{cases} \qquad \begin{cases} \gcd\big(A(t^{b^{r-1}}), C(t)\big) = \gcd\big(B(t), C(t^b)\big) = 1 \\ \gcd\big(A(t^{b^i}), B(t)\big) = 1 \quad \text{for } i \in \{0, \dots, r-1\} \\ \gcd\big(A(t), B(t^b)\big) = \gcd\big(C(t), C(t^b)\big) = 1 \end{cases}$$

New Algorithm for Mahler Equations of Any Order

$$\sum_{i=0}^{r} \ell_i(t^{b^{r-1}})\, \eta^i\, C(t^{b^i}) \left(\prod_{j=0}^{i-1} A(t^{b^{r-1+j}})\right) \left(\prod_{j=i}^{r-1} B(t^{b^j})\right) = 0$$

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Sketch of new algorithm

- for all monic $A(t) \mid \ell_0(t)$, for all monic $B(t) \mid \ell_r(t)$:
  - determine potential degrees for C from the degrees of A, B, ℓ_i,
  - for all obtained candidate degrees:
    - extract the leading coefficient w.r.t. t and solve as an equation in η,
    - for all candidates η, solve equation for C by linear algebra;
- return $(\eta, A(t), B(t), C(t))$ after removing redundancy.

New Algorithm for Mahler Equations of Any Order

$$\sum_{i=0}^{r} \ell_i(t^{b^{r-1}})\, \eta^i\, C(t^{b^i}) \left(\prod_{j=0}^{i-1} A(t^{b^{r-1+j}})\right) \left(\prod_{j=i}^{r-1} B(t^{b^j})\right) = 0$$

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- return $(\eta, A(t), B(t), C(t))$ after removing redundancy.

NB: parameters in $C \to$ continuous family of u.

Efficiency Improvements

Pruning the set of (A, B)

- Factor ℓ_0 and ℓ_r into irreducible.
- Some factors of one forbid other factors of the other.
- Iterate on tuples of exponents.

Removing repetitions in the found (η, A, B, C)

Some (A, B) make other (A', B') useless.

Avoiding redundant computations of degree bounds for C

Newton polygon for different (A, B) are related.

Taking degree bounds into account

When choosing (A, B), after getting potential degrees for C.

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Number of cases to test still exponential in the degrees of the ℓ_i.

Part III

An Effective Difference Algebra for Solutions

Where to Look for Solutions of the Linear Equation?

Field of Puiseux series: $\mathcal{P} := \bigcup_{q \in \mathbb{N}_{\neq 0}} \bar{\mathbb{Q}}((x^{1/q}))$.

$$e_\lambda := (\ln x)^{\log_b \lambda}, \quad M e_\lambda = \lambda\, e_\lambda, \qquad \ell := \log_b \ln x, \quad M\ell = \ell + 1,$$

$$e_\lambda e_{\lambda'} = e_{\lambda\lambda'}. \qquad (M - 1)^2 \ell = 0.$$

Regular singular Mahler systems (Roques, 2018)

$\mathcal{U} := \mathcal{P}\left[(e_\lambda)_{\lambda \in \bar{\mathbb{Q}}_{\neq 0}}, \ell\right]$ is a universal Picard–Vessiot ring for the regular singular Mahler systems over $\mathcal{P}$: “enough” solutions, same constants.

Field of Hahn series: $\mathcal{H} := \left\{ f \in \bar{\mathbb{Q}}^{\mathbb{Q}} \;\middle|\; \operatorname{supp} f \text{ is well-founded} \right\}$.

Local structure of Mahler systems (Roques, 2016)

Solving general systems requires $\mathcal{H}$ and solutions of all $(M - \lambda)^k y = 0$.

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What about non-regular singular systems?

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Local structure of Mahler systems (Roques, 2016)

Solving general systems requires $\mathcal{H}$ and solutions of all $(M - \lambda)^k y = 0$.

Remark: $(e_\lambda + e_{-\lambda})(e_\lambda - e_{-\lambda}) = 0$, so $\mathcal{U}$ cannot be a field.

Structure of Hypergeometric Solutions

Write: $\bar{\mathbb{Q}}((x^{1/*})) := \mathcal{P}$, $\mathfrak{D} := \mathcal{P}\left[(e_\lambda)_{\lambda\in\bar{\mathbb{Q}}_{\neq 0}}\right] = \bigoplus_{\lambda\in\bar{\mathbb{Q}}_{\neq 0}} (\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/*}))$.

similarity, hypergeometricity

- y_1 and y_2 are similar if $\exists q \in \mathbb{Q}(x)_{\neq 0},\ y_2 = q y_1$.
- y is hypergeometric if $\exists u \in \mathbb{Q}(x),\ My = uy$.

Structure of hypergeometric solutions in $\mathfrak{D}$

$$\{ \text{ hypergeometric solutions of (L) in } \mathfrak{D}\} = \{0\} \sqcup \coprod_{j=1}^{m} (\mathfrak{H}_j)_{\neq 0}$$

where:

- Each $(\mathfrak{H}_j)_{\neq 0}$ is a class of similar hypergeometric solutions.
- The vector spaces $\mathfrak{H}_j$ are in direct sum in $\mathfrak{D}$.
- The sum of the $d_j := \dim \mathfrak{H}_j$ add up to at most the order of L.
- $\mathfrak{H}_j \subset (\ln x)^{\log_b \lambda_j}\bar{\mathbb{Q}}((x^{1/*}))$ for a suitable λ_j.

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Fix: $\mathbb{Q}(x) \subset F \subset \mathfrak{D}$ with field F stable under M.

F-similarity, F-hypergeometricity

- y_1 and y_2 are F-similar if $\exists q \in F_{\neq 0},\ y_2 = q y_1$.
- y is F-hypergeometric if $\exists u \in F,\ My = uy$.

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Structure of Solutions to the Riccati Equation

$$\rho : (\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/*})) \to \bar{\mathbb{Q}}((x^{1/*})) \quad \text{is well-defined for each } \lambda.$$
$$y \mapsto My/y$$

Transport of the solution structure, given $\mathbb{Q}(x) \subset F \subset \mathfrak{D}$

$$\{\text{(some) solutions of (R)}\} = \coprod_{j=1}^{m} \mathfrak{R}_j$$

where:

- $\mathfrak{R}_j := \rho((\mathfrak{H}_j)_{\neq 0})$
- ρ induces a one-to-one parametrization of $\mathfrak{R}_j$ by $\mathbb{P}(\mathfrak{H}_j) \simeq \mathbb{P}^{d_j - 1}(\bar{\mathbb{Q}})$.

Structure of Solutions to the Riccati Equation

$$\rho : (\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/*})) \to \bar{\mathbb{Q}}((x^{1/*})) \quad \text{is well-defined for each } \lambda.$$
$$y \mapsto My/y$$

Transport of the solution structure, given $\mathbb{Q}(x) \subset F \subset \mathfrak{D}$

$$\{\text{(some) solutions of (R)}\} = \coprod_{j=1}^{m} \mathfrak{R}_j$$

where:

- $\mathfrak{R}_j := \rho((\mathfrak{H}_j)_{\neq 0})$
- ρ induces a one-to-one parametrization of $\mathfrak{R}_j$ by $\mathbb{P}(\mathfrak{H}_j) \simeq \mathbb{P}^{d_j - 1}(\bar{\mathbb{Q}})$.

Given a basis $(y_1, \ldots, y_d)$ of $\mathfrak{H} := \mathfrak{H}_j$, with dimension $d := d_j$:

$$(g_1 : \cdots : g_d) \mapsto \frac{g_1 M y_1 + \cdots + g_d M y_d}{g_1 y_1 + \cdots + g_d y_d}.$$

Structure of Solutions to the Riccati Equation

$$\rho : (\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/*})) \to \bar{\mathbb{Q}}((x^{1/*})) \quad \text{is well-defined for each } \lambda.$$
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$$x^v + \cdots \mapsto \lambda x^{(b-1)v} + \cdots$$

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Puiseux series solutions

$F := \bar{\mathbb{Q}}((x^{1/*}))$

Rational solutions

λ + Newton polygon $\to q_\lambda \in \mathbb{N}$

$q := \operatorname{lcm}_\lambda q_\lambda \to F := \bar{\mathbb{Q}}((x^{1/q}))$

Useful Solving Algorithms (old) and Bounds (new)

$$L \in \bar{\mathbb{Q}}[x]\langle M\rangle \qquad \deg_x L = d \quad \deg_M L = r$$

Arithmetic complexity of solving the linear equation (CDDM, 2018)

- Basis of polynomial solutions: $\tilde{O}\big(b^{-r}d^2 + M(d)\big)$ ops.
- Basis of approximate formal power series: $O\big(r^2 d + r^2 M(r)\big)$ ops.
- Also: rational solutions, Puiseux series solutions.

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Ramification order of Puiseux series solutions (old + new)

Each $(\ln x)^{\log_b \lambda}$ implies some $\bar{\mathbb{Q}}((x^{1/q_\lambda}))$ for q_λ read on a Newton polygon.

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Degree bounds for rational solutions u of the Riccati equation (new)

	numerators	denominators	both
$b = 2$	$\big(1 + 2^{-r}\big)(2d)$	$2d$	$O(d)$
$b \geq 3$	$\big(1 + b^{-1}\big)\dfrac{d}{b^{r-2}}$	$\dfrac{d}{b^{r-2}}$	$O(d/b^r)$

Part IV

Second Approach: Structured Hermite–Padé Approximants

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the suitable ramification order $q = q_\lambda$:

$$\rho : (\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/q})) \to \bar{\mathbb{Q}}((x^{1/q}))$$
$$y \mapsto \frac{My}{y}$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the suitable ramification order $q = q_\lambda$:

$$\rho : \{y \in (\ln x)^{\log_b \lambda} \bar{\mathbb{Q}}((x^{1/q})) \mid Ly = 0\} \to \bar{\mathbb{Q}}((x^{1/q}))$$

$$\cup \qquad \cup$$

$$\mathfrak{H}_j \to \mathfrak{R}_j \subset \bar{\mathbb{Q}}(x)$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the suitable ramification order $q = q_\lambda$:

$$\rho : \operatorname{span}\big((\ln x)^{\log_b \lambda} z_1, \dots, (\ln x)^{\log_b \lambda} z_d\big) \to \bar{\mathbb{Q}}((x^{1/q}))$$

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$$y \mapsto \frac{My}{y} \in \bar{\mathbb{Q}}(x)\ ?$$

Reformulation of the problem as structured syzygies

Parametrization of the search space

For each λ, using the suitable ramification order $q = q_\lambda$:

$$\rho : \bar{\mathbb{Q}}^t \to \bar{\mathbb{Q}}((x^{1/q}))$$
$$(a_1, \dots, a_t) \mapsto \lambda \frac{a_1 M z_1 + \cdots + a_t M z_t}{a_1 z_1 + \cdots + a_t z_t} \in \bar{\mathbb{Q}}(x) \,?$$

Reformulation of the problem as structured syzygies

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$$(a_1, \dots, a_t) \mapsto \lambda \frac{a_1 M z_1 + \dots + a_t M z_t}{a_1 z_1 + \dots + a_t z_t} \in \bar{\mathbb{Q}}(x) \,?$$

Other formulation, after renormalizing L so that $\lambda = 1$ and $z_i \in \bar{\mathbb{Q}}[[x]]$

Describe $(a_1, \dots, a_t) \neq 0$ such that $\exists P/Q \in \bar{\mathbb{Q}}(x)_{\neq 0}$,

$$(-a_1 P)\, z_1 + \dots + (-a_t P)\, z_t + (a_1 Q)\, M z_1 + \dots + (a_t Q)\, M z_t = 0.$$

Relaxation of the problem

Two-stage relaxation

Solutions

$$(-a_1P)\,z_1 + \cdots + (-a_tP)\,z_t + (a_1Q)\,Mz_1 + \cdots + (a_tQ)\,Mz_t = 0$$

are structured instances of the syzygies

$$P_1\,z_1 + \cdots + P_t\,z_t + Q_1\,Mz_1 + \cdots + Q_t\,Mz_t = 0,$$

which are approximated by approximate syzygies

$$P_1\,z_1 + \cdots + P_t\,z_t + Q_1\,Mz_1 + \cdots + Q_t\,Mz_t = O(x^\sigma).$$

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Motivation

1. For $\sigma \gg 1$, approximate syzygies of “low” degree are exact syzygies.
2. Structured syzygies are linear combinations of syzygies.

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Two-stage relaxation

Solutions

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Motivation

1. For $\sigma \gg 1$, approximate syzygies of “low” degree are exact syzygies.
2. Structured syzygies are linear combinations of syzygies.

We search for structured syzygies as recombinations of approximate syzygies.

Structure and computation of approximate syzygies

Minimal basis of approximate syzygies

Algorithms find a basis of the module of approximate syzygies to order σ:

$$\begin{pmatrix} P_{1,1},\dots,P_{1,t} & Q_{1,1},\dots,Q_{1,t} \\ \vdots & \vdots \\ P_{t,1},\dots,P_{t,t} & Q_{t,1},\dots,Q_{t,t} \\ P_{t+1,1},\dots,P_{t+1,t} & Q_{t+1,1},\dots,Q_{t+1,t} \\ \vdots & \vdots \\ P_{2t,1},\dots,P_{2t,t} & Q_{2t,1},\dots,Q_{2t,t} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_t \\ Mz_1 \\ \vdots \\ Mz_t \end{pmatrix} = \begin{pmatrix} O(x^\sigma) \\ \vdots \\ O(x^\sigma) \\ O(x^\sigma) \\ \vdots \\ O(x^\sigma) \end{pmatrix}$$

(Derksen, 1994), (Beckermann, Labahn, 1994, 2000), Neiger (2016).

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(Derksen, 1994), (Beckermann, Labahn, 1994, 2000), Neiger (2016).

Properties (module)

The module of the rows: *(i)* has rank $2t$ for all σ; *(ii)* is ultimately decreasing with σ; *(iii)* has the module of (exact) syzygies as a limit (with rank $< 2t$).

Reduction to a polynomial system

Properties (vector space)

The vector space of the rows of “low” degree: *(i)* is nonincreasing; *(ii)* has the vector space of exact syzygies of “low” degree as a limit.

W := submatrix of (independent) rows of “low” degree.

ρ := rank of the module of rows generated by W.

Reduction to a polynomial system

Properties (vector space)

The vector space of the rows of "low" degree: *(i)* is nonincreasing; *(ii)* has the vector space of exact syzygies of "low" degree as a limit.

W := submatrix of (independent) rows of "low" degree.
ρ := rank of the module of rows generated by W.

Search for structured approximate syzygies, hoping that they are exact

Given $a := (a_1, \dots, a_t) \neq 0$, the following are equivalent:

- $\exists P/Q \in \bar{\mathbb{Q}}(x)_{\neq 0}$ such that $(-aP, aQ)$ is in the module $\bar{\mathbb{Q}}[x]^{1\times\rho}W$,
- W_+ has a nontrivial left kernel, where W_+ is W stacked above

$$\begin{pmatrix} a_1, \dots, a_t & 0 \\ 0 & a_1, \dots, a_t \end{pmatrix},$$

Reduction to a polynomial system

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The vector space of the rows of “low” degree: *(i)* is nonincreasing; *(ii)* has the vector space of exact syzygies of “low” degree as a limit.

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- W_+ has a nontrivial left kernel, where W_+ is W stacked above

$$\begin{pmatrix} a_1, \dots, a_t & 0 \\ 0 & a_1, \dots, a_t \end{pmatrix},$$

- a is a solution of the quadratic homogeneous polynomial system

$$\Sigma := \left\{ \text{coefficients w.r.t } x \text{ of the minors of size } \rho + 2 \text{ of } W_+ \right\} \subset \bar{\mathbb{Q}}[a].$$

A Polynomial System with a Linear Variety of Solutions

$$V(\Sigma) = \bigcup_j I_j \qquad (I_j = \text{irreducible component})$$

Properties

When σ increases, $V(\Sigma)$ stabilizes. At the limit:

- each I_j is a subspace of $\bar{\mathbb{Q}}^t$,
- the I_j are in direct sum,
- each I_j parametrizes a subset of rational solutions of (R),
- the images of the I_j form a partition of the rational solutions of (R).

Adjust the precision σ to be able to solve

- Primary decomposition: obtain Gröbner bases for prime ideals $\mathfrak{p}_j$ s.t.
  $$\sqrt{(\Sigma)} = \bigcap_j \mathfrak{p}_j \subset \bar{\mathbb{Q}}[a_1, \dots, a_t].$$
  (Gianni, Trager, Zacharias, 1988): implementation over $\bar{\mathbb{Q}}$ in Singular.
- If any Gröbner basis contains a nonlinear element, σ is too small.

Sketch of the algorithm (for a given λ)

Obtain all rational $u = \lambda x^\alpha + \cdots$ s.t. $M - u$ is a right-hand factor of L:

- Renormalize L so as to reduce the computation of the solutions of L in $(\ln x)^{\log_b \lambda}\bar{\mathbb{Q}}((x^{1/q_\lambda}))$ to solutions of some L_λ in $\bar{\mathbb{Q}}[[x]]$.
- Compute a basis of truncated series solutions $(z_1, \dots, z_t)$ to some initial order σ_0.
- For σ in a geometric sequence $\phi^k \sigma_0$:
  - Prolong the basis to order σ.
  - Compute a minimal basis of the module of approximate syzygies.
  - Extract the "low"-degree rows into a matrix W of rank $0 \leq \rho \leq 2t$.
  - $\rho \in \{0, 2t-1, 2t\}$ are special cases dealt with separately.
  - Compute minors of W_+, then their coefficients to obtain Σ.
  - Compute the primary decomposition $\sqrt{(\Sigma)} = \bigcap_j \mathfrak{p}_j$ over $\bar{\mathbb{Q}}$.
  - If any $\mathfrak{p}_j$ shows a nonlinear polynomial, increase σ.
  - For each j:
    - Solve $\mathfrak{p}_j$ to get a matrix S and a parametrization $a = Sg$ for g in some $\bar{\mathbb{Q}}^v$.
    - Solve for the left kernel of W_+ at $a = Sg$. If incompatible result, increase σ.
    - Get a candidate P/Q (with param. g) from the basis element of the kernel.
    - If degrees of $u := P/Q$ are too high, or if u does not satisfy (R), increase σ.
  - Convert all obtained u from solutions of L_λ into solutions of L.
  - Quit and return the solutions.

Part V

Comparison of the Approaches and Application to Hypertranscendence

Benchmark (preliminary Maple implementation by Dumas)

			IP	HP					
example	*b r*	*d*	tot	fst	dim	σ	rfn	syz	tot
Baum_Sweet	2 2	1	0.10	0.14	(1, 1)	(6, 6)	0.04	0.03	0.22
Rudin_Shapiro	2 2	1	0.08	0.11	(1, 0)	(6, †)	0.02	0.02	0.22
Stern_Brocot_b2	2 2	4	0.22	0.12	(1)	(21)	0.02	0.10	0.25
no_2s_in_3_exp	2 2	4	0.25	0.16	(1, 1)	(33, 9)	0.04	0.18	0.39
Dilcher_Stolarsky	4 2	4	0.11	0.09	(2)	(43)	0.07	0.27	0.48
Stern_Brocot_b4	4 2	26	6.3	0.15	(1)	(63)	0.03	0.23	0.42
Katz_Linden	2 4	14	2.5	0.14	(0, 1, 0, 0)	(†, 69, †, †)	0.14	0.36	0.65
Adamczewski_Faverjon	3 4	258	707	0.31	(4)	(163)	0.59	2.0	3.3
lclm_3rat_1log	3 3	121	275	0.12	(3)	(140)	0.31	2.8	3.4
lclm_3rat_2log	3 3	122	281	0.14	(2, 1)	(88, 52)	0.17	0.65	1.0
lclm_2rat_trunc_sl0	2 4	56	569	0.16	(4)	(294)	1.8	12	14
lclm_2rat_trunc_sl1	2 4	61	965						>2 d
lclm_3rat_trunc_sl1	3 5	1260	>2 d	0.36	(3, 2)	(574, 268)	9.3	47	56
lclm_4pow_b2	2 7	107	>2 d	0.37	(1, 4)	(429, 739)	0.20	3.5	4.1
lclm_4pow_b3	3 6	727	>2 d	0.85	(1, 4)	(108, 174)	1.1	0.82	2.9
lclm_4pow_b4	4 5	989	>2 d	0.47	(4)	(223)	0.72	0.91	2.2
lclm_4pow_b5	5 5	3103	>2 d	14	(1, 4)	(44, 289)	21	1.2	37
lclm_5pow_b4	4 7	17270	>2 d	84	(1, 5)	(274, 1326)	129	8.3	226
dft_Baum_Sweet	4 2	6	0.15	0.09	(2)	(124)	0.10	0.56	0.79
dft_Rudin_Shapiro	4 2	7	6.3	0.07	(1, 0)	(88, †)	0.04	0.29	0.40
dft_Stern_Brocot_b2	4 2	24	3.3	0.13	(1)	(59)	0.10	0.14	0.39
dft_no_2s_in_3_exp	4 2	20	11	0.09	(1, 1)	(85, 33)	0.08	0.78	0.96
dft_Dilcher_Stolarsky	16 2	50	4275	0.23	(2)	(666)	0.14	4.6	5.0
dft_Stern_Brocot_b4	16 2	348	43213	0.26	(1)	(239)	0.17	2.2	2.7
rmo_2_1	2 3	19	6.1						>2 d
rmo_3_1	3 3	37	16	0.10	(3)	(111)	0.24	517	518
rmo_2_2	2 3	44	17						>2 d
rmo_3_2	3 3	82	46	0.12	(3)	(247)	2.1	10100	10102
rmo_2_3	2 3	69	31						>2 d
rmo_3_3	3 3	127	85	0.12	(3)	(386)	6.8	60102	60109
rmo_2_4	2 3	94	49						>2 d
rmo_3_4	3 3	172	131						>2 d
rmo_2_5	2 3	119	70						>2 d
rmo_3_5	3 3	217	194						>2 d

Caveat:

- timings with a heuristic for absolute decomposition,
- ongoing work: calling Singular from Maple.

Differentially Algebraic Independence

Hypertranscendence (a.k.a. differential transcendence)

$f \in \mathbb{C}((x))$ is hypertranscendental over $\mathbb{C}(x)$ $\Leftrightarrow$
f admits no polynomial differential equation over $\mathbb{C}(x)$

Corollary of a criterion (Roques, 2018) on the difference Galois group of L

Assume:

- $y(x^{b^2}) + A(x)y(x^b) + B(x)y(x) = 0$ admits a nonzero solution $f \in \bar{\mathbb{Q}}[[x]]$.
- No rational function $u(x)$ is solution of one of the Riccati equations

$$u(x)u(x^b) + A(x)u(x) + B(x) = 0,$$

$$u(x)u(x^{b^2}) + \left(\frac{B(x^{b^2})}{A(x^{b^2})} - A(x^b) + \frac{B(x^b)}{A(x)}\right) u(x) + \frac{B(x)B(x^b)}{A(x)^2} = 0.$$

Then, f and Mf are differentially algebraically independent.
In particular, f is hypertranscendental, which was already proven in (Adamczewski, Dreyfus, and Hardouin, 2021).

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Independence for the Baum–Sweet, Rudin–Shapiro, and Dilcher–Stolarsky examples!