Rigorous computation of Poincaré maps

Daniel Wilczak
joint work with Tomasz Kapela and Piotr Zgliczyński

Certified and Symbolic-Numeric Computation
May 26, 2023

Motivation:

Computer Assisted Proofs in Dynamics

- existence, stability and continuation of periodic orbits (POs)
- connecting orbits between POs (ODEs, PDEs)
- invariant tori around elliptic POs
- local bifurcations of POs
- global bifurcations (homoclinic tangencies, Shilnikov orbits, Bykov cycles ...)
- symbolic dynamics (ODEs, PDEs)
- (non)uniformly hyperbolic, chaotic attractors (Tucker'2002)
- . . .

http://capd.ii.uj.edu.pl

Kapela, Mrozek, W, Zgliczyński, CAPD::DynSys: a flexible C++ toolbox for rigorous numerical analysis of dynamical systems, CNSNS'2021

Kapela, W., Zgliczyński, Recent advances in rigorous computation of Poincaré maps, CNSNS'2022

Definition

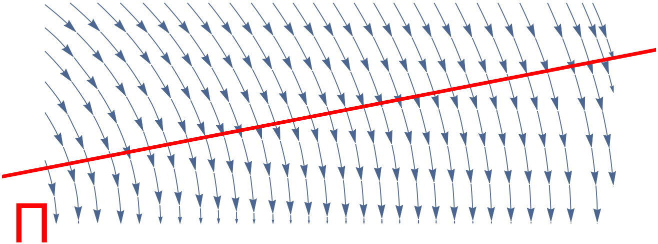
$\Pi \subset \mathbb{R}^n$ is **Poincaré section** for $x' = f(x)$ if

- Π is connected manifold of codim 1 and
- $f(x) \notin T_x\Pi$ for $x \in \Pi$

Definition

$\Pi \subset \mathbb{R}^n$ is **Poincaré section** for $x' = f(x)$ if

- Π is connected manifold of codim 1 and
- $f(x) \notin T_x\Pi$ for $x \in \Pi$



Practical description of Poincaré sections

$$\Pi = \Pi_{\alpha,\mathcal{C}} = \{x : \alpha(\mathbf{x}) = \mathbf{0} \wedge \langle \nabla\alpha(x); f(x)\rangle \neq 0 \wedge \mathcal{C}(\mathbf{x})\}$$

where

- $\alpha : \mathbb{R}^n \to \mathbb{R}$ - smooth
- zero is a **regular value** of α
- $\mathcal{C}$ **is a predicate** (additional constrains)
  - crossing direction
  - domain restriction
  - etc.

Return time (or flow time to section)

Π - Poincaré sections for $x' = f(x)$

Definition

Define $t_\Pi : \mathbb{R}^n \to \mathbb{R}$:

1. $x \in \mathrm{dom}\, t_\Pi$ iff $x(t) \in \Pi$ for some $t > 0$
2. for $x \in \mathrm{dom}\, t_\Pi$ we set

$$t_\Pi(x) = \inf \{t > 0 : x(t) \in \Pi\}$$

Return time (or flow time to section)

Π - Poincaré sections for $x' = f(x)$

Definition

Define $t_\Pi : \mathbb{R}^n \to \mathbb{R}$:

1. $x \in \mathrm{dom}\, t_\Pi$ iff $x(t) \in \Pi$ for some $t > 0$
2. for $x \in \mathrm{dom}\, t_\Pi$ we set

$$t_\Pi(x) = \inf \{t > 0 : x(t) \in \Pi\}$$

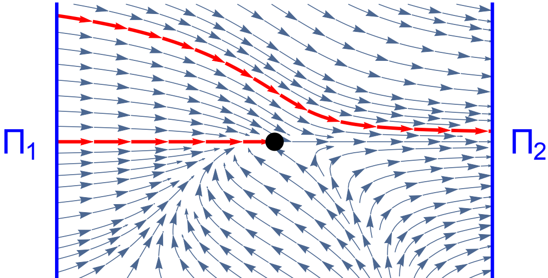
Π - Poincaré sections for $x' = f(x)$

Definition

Define $t_\Pi : \mathbb{R}^n \to \mathbb{R}$:

1. $x \in \mathrm{dom}\, t_\Pi$ iff $x(t) \in \Pi$ for some $t > 0$
2. for $x \in \mathrm{dom}\, t_\Pi$ we set

$$t_\Pi(x) = \inf\{t > 0 : x(t) \in \Pi\}$$



Π_1, Π_2 - sections

Definition

Define Poincaré map:

$$\mathcal{P} := \mathcal{P}_{\Pi_1 \to \Pi_2} : \Pi_1 \to \Pi_2$$

by

$$\mathcal{P}(x) = x(t_{\Pi_2}(x))$$

provided $t_{\Pi_2}(x)$ exists.

$t_{\mathcal{P}}$ – restriction of t_{Π_2} to Π_1

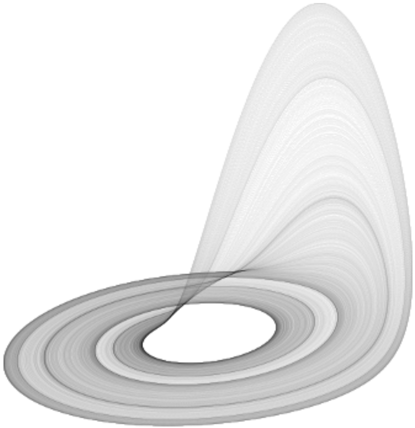
Example: attractor in the Rössler system

Example (Rössler system)

$$x' = -(y+z), \qquad y' = x + 0.2y, \qquad z' = 0.2 + z(x - 5.7)$$

Goal:

there is a compact, connected invariant set which has at least one periodic solution.



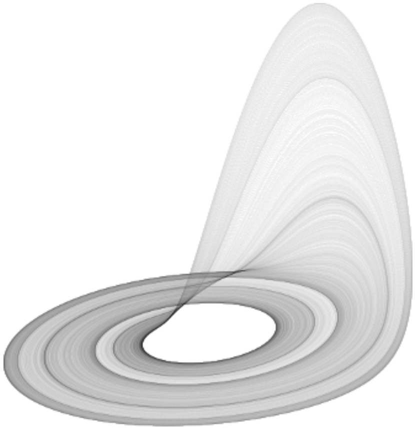
Example: attractor in the Rössler system

Example (Rössler system)

$$x' = -(y+z), \qquad y' = x + 0.2y, \qquad z' = 0.2 + z(x - 5.7)$$

Goal:

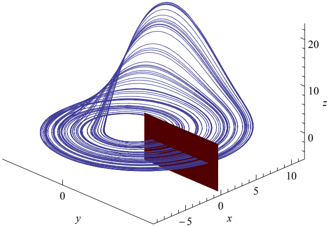
there is a compact, connected invariant set which has at least one periodic solution.



Example: attractor in the Rössler system

Settings:

$$\Pi = \{(0, y, z) : y, z \in \mathbb{R}, x' > 0\} \quad - \quad \text{Poincaré section}$$
$$P : \Pi \to \Pi \quad - \quad \text{Poincaré map}$$



Methodology: Show that there is a rectangle

$$W = [y_1, y_2] \times [z_1, z_2]$$

such that

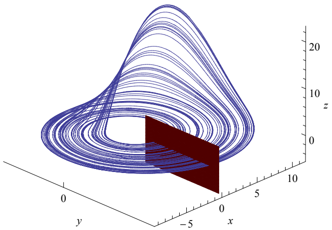
$$P(W) \subset W.$$

Then $\mathcal{A} := \bigcap_{n>0} P^n(W)$ is a compact, connected invariant set.

Example: attractor in the Rössler system

Settings:

$\Pi = \{(0, y, z) : y, z \in \mathbb{R}, x' > 0\}$ – Poincaré section
$P : \Pi \to \Pi$ – Poincaré map



Methodology: Show that there is a rectangle

$$W = [y_1, y_2] \times [z_1, z_2]$$

such that

$$P(W) \subset W.$$

Then $\mathcal{A} := \bigcap_{n>0} P^n(W)$ is a compact, connected invariant set.

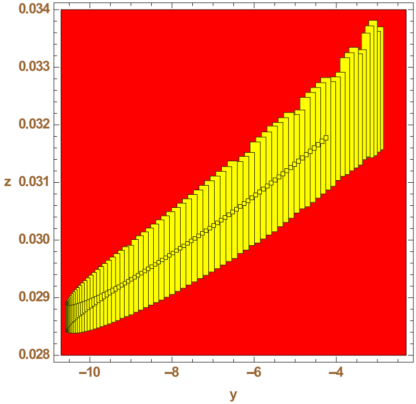
Example: attractor in the Rössler system

Data (from simulation):

$$W = [-10.7, -2.3] \times [0.028, 0.034]$$

Computations:

- subdivide $W = \bigcup_{i=1}^{200} W_i$
- check that $P(W_i) \subset W$ for $i = 1, \ldots, 200$



```cpp
#include <iostream>
#include "capd/capdlib.h"
using namespace capd;

int main(){
  IMap vf("var:x,y,z;fun:-(y+z),x+0.2*y,0.2+z*(x-5.7);");
  IOdeSolver solver(vf, 20);
  ICoordinateSection section(3, 0); // section x=0, x'>0
  IPoincareMap pm(solver, section, poincare::MinusPlus);

  // Coordinates of the trapping region
  const double B = 0.028, T = 0.034, L = -10.7, R = -2.3;

  // Subdivide the rectangle uniformly in y coordinate
  const int N = 200;
  bool result = true;
  interval p = (interval(R) - interval(L)) / N;
  for (int i = 0; i < N and result; ++i) {
    IVector x ({0., L + interval(i,i+1)*p, interval(B, T)});
    C0HOTripletonSet s(x);
    IVector u = pm(s);
    result = result and u[2]>B and u[2]<T and u[1]>L and u[1]<R;
    if(!result)
      std::cout << "Inclusion not satisfied:\n" << u << std::endl;
  }
  std::cout << "Existence of attractor: " << result << std::endl;
  return 0;
}
```

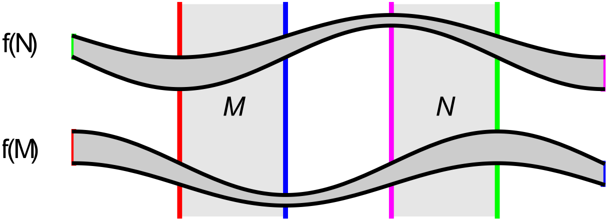
Theorem (Zgliczyński, Nonlinearity 1997)

Assume that N and M are disjoint and

$$N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} N.$$

Then

- *for every $\{S_i\}_{i\in\mathbb{Z}} \in \{N, M\}^{\mathbb{Z}}$ there is a trajectory visiting N, M in that order*
- *periodic $\{S_i\}$ lead to periodic trajectories.*



Topological tool for chaos

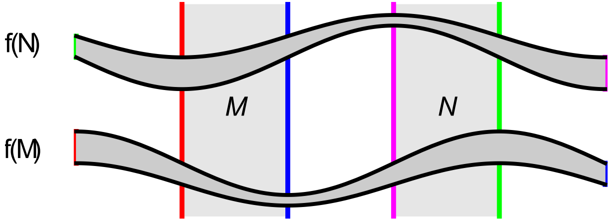
Theorem (Zgliczyński, Nonlinearity 1997)

Assume that N and M are disjoint and

$$N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} N.$$

Then

- *for every* $\{S_i\}_{i\in\mathbb{Z}} \in \{N, M\}^{\mathbb{Z}}$ *there is a trajectory visiting* N, M *in that order*
- *periodic* $\{S_i\}$ *lead to periodic trajectories.*



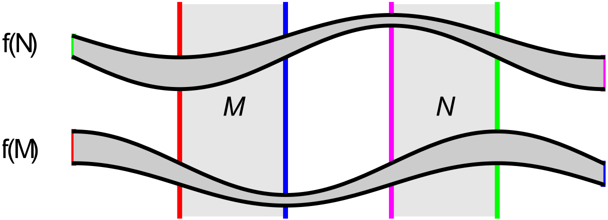
Theorem (Zgliczyński, Nonlinearity 1997)

Assume that N and M are disjoint and

$$N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} M \overset{f}{\Longrightarrow} N.$$

Then

- *for every $\{S_i\}_{i\in\mathbb{Z}} \in \{N, M\}^{\mathbb{Z}}$ there is a trajectory visiting N, M in that order*
- *periodic $\{S_i\}$ lead to periodic trajectories.*



Data (from simulation):

$$\begin{aligned} W &= [l_W, r_W] \times Z = [-10.7, -2.3] \times [0.028, 0.034] \\ M &= [l_M, r_M] \times Z = [-8.4, -7.6] \times [0.028, 0.034] \\ N &= [l_N, r_N] \times Z = [-5.7, -4.6] \times [0.028, 0.034]. \end{aligned}$$

Note: in the last example we checked $P(W) \subset \operatorname{int} W$. Therefore

$$P^2(W) \subset \operatorname{int} W \subset \mathbb{R} \times (0.028, 0.034)$$

Lemma (computer-assisted)

$$N \xRightarrow{P^2} N \xRightarrow{P^2} M \xRightarrow{P^2} M \xRightarrow{P^2} N$$

Inequalities to check:

$$\begin{aligned} \pi_y P^2(l_M \times [0.028, 0.034]) &< l_M \\ \pi_y P^2(r_M \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(l_N \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(r_N \times [0.028, 0.034]) &< l_M \end{aligned}$$

Data (from simulation):

$$\begin{aligned} W &= [l_W, r_W] \times Z = [-10.7, -2.3] \times [0.028, 0.034] \\ M &= [l_M, r_M] \times Z = [-8.4, -7.6] \times [0.028, 0.034] \\ N &= [l_N, r_N] \times Z = [-5.7, -4.6] \times [0.028, 0.034]. \end{aligned}$$

Note: in the last example we checked $P(W) \subset \operatorname{int} W$.
Therefore

$$P^2(W) \subset \operatorname{int} W \subset \mathbb{R} \times (0.028, 0.034)$$

Lemma (computer-assisted)

$$N \overset{P^2}{\Longrightarrow} N \overset{P^2}{\Longrightarrow} M \overset{P^2}{\Longrightarrow} M \overset{P^2}{\Longrightarrow} N$$

Inequalities to check:

$$\begin{aligned} \pi_y P^2(l_M \times [0.028, 0.034]) &< l_M \\ \pi_y P^2(r_M \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(l_N \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(r_N \times [0.028, 0.034]) &< l_M \end{aligned}$$

Data (from simulation):

$$\begin{aligned} W &= [l_W, r_W] \times Z = [-10.7, -2.3] \times [0.028, 0.034] \\ M &= [l_M, r_M] \times Z = [-8.4, -7.6] \times [0.028, 0.034] \\ N &= [l_N, r_N] \times Z = [-5.7, -4.6] \times [0.028, 0.034]. \end{aligned}$$

Note: in the last example we checked $P(W) \subset \operatorname{int} W$.
Therefore

$$P^2(W) \subset \operatorname{int} W \subset \mathbb{R} \times (0.028, 0.034)$$

Lemma (computer-assisted)

$$N \overset{P^2}{\Longrightarrow} N \overset{P^2}{\Longrightarrow} M \overset{P^2}{\Longrightarrow} M \overset{P^2}{\Longrightarrow} N$$

Inequalities to check:

$$\begin{aligned} \pi_y P^2(l_M \times [0.028, 0.034]) &< l_M \\ \pi_y P^2(r_M \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(l_N \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(r_N \times [0.028, 0.034]) &< l_M \end{aligned}$$

Data (from simulation):

$$\begin{aligned} W &= [l_W, r_W] \times Z = [-10.7, -2.3] \times [0.028, 0.034] \\ M &= [l_M, r_M] \times Z = [-8.4, -7.6] \times [0.028, 0.034] \\ N &= [l_N, r_N] \times Z = [-5.7, -4.6] \times [0.028, 0.034]. \end{aligned}$$

Note: in the last example we checked $P(W) \subset \operatorname{int} W$.
Therefore

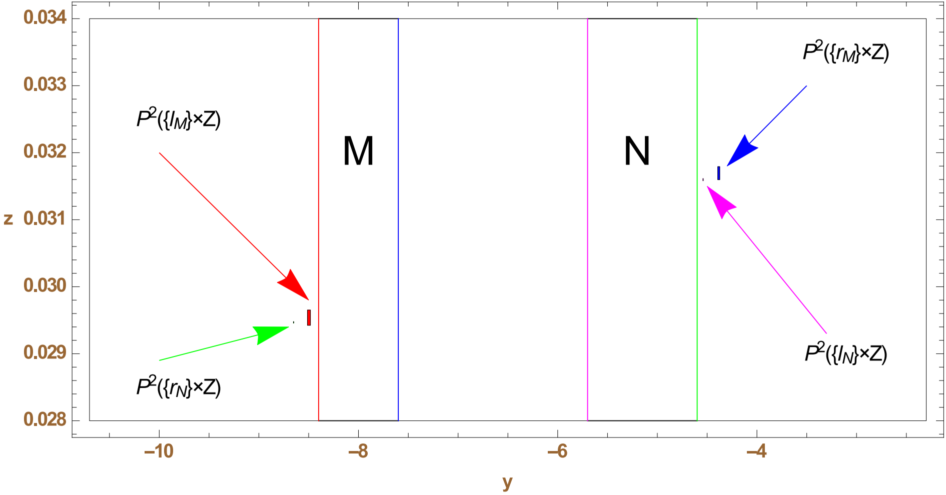
$$P^2(W) \subset \operatorname{int} W \subset \mathbb{R} \times (0.028, 0.034)$$

Lemma (computer-assisted)

$$N \stackrel{P^2}{\Longrightarrow} N \stackrel{P^2}{\Longrightarrow} M \stackrel{P^2}{\Longrightarrow} M \stackrel{P^2}{\Longrightarrow} N$$

Inequalities to check:

$$\begin{aligned} \pi_y P^2(l_M \times [0.028, 0.034]) &< l_M \\ \pi_y P^2(r_M \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(l_N \times [0.028, 0.034]) &> r_N \\ \pi_y P^2(r_N \times [0.028, 0.034]) &< l_M \end{aligned}$$



Rigorous enclosures returned by the routine

```cpp
#include <iostream>
#include "capd/capdlib.h"
using namespace capd;
using namespace std;

int main(){
  IMap vf("var:x,y,z;fun:-(y+z),x+0.2*y,0.2+z*(x-5.7);");
  IOdeSolver solver(vf, 20);
  ICoordinateSection section(3, 0); // section x=0, x'>0
  IPoincareMap pm(solver, section, poincare::MinusPlus);

  // z-coordinate of the trapping region
  interval z(0.028,0.034);
  // Coordinates of M and N
  const double lM=-8.4, rM=-7.6, lN=-5.7, rN=-4.6;

  C0HOTripletonSet LM( IVector({0.,lM,z}) );
  C0HOTripletonSet RM( IVector({0.,rM,z}) );
  C0HOTripletonSet LN( IVector({0.,lN,z}) );
  C0HOTripletonSet RN( IVector({0.,rN,z}) );

  // Inequalities for the covering relations N=>N, N=>M, M=>M, M=>N.
  cout << "P^2(LM) < lM: " << ( pm(LM,2)[1] < lM ) << endl;
  cout << "P^2(RM) > rN: " << ( pm(RM,2)[1] > rN ) << endl;
  cout << "P^2(LN) > rN: " << ( pm(LN,2)[1] > rN ) << endl;
  cout << "P^2(RN) < lM: " << ( pm(RN,2)[1] < lM ) << endl;
  return 0;
}
```

Primary goal:

Given:

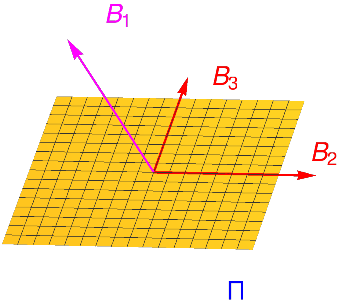
- $y \in \Pi_2$,
- matrix A,
- $X \subset \Pi_1$

enclose

$$A\left(\mathcal{P}_{\Pi_1 \to \Pi_2}(X) - y\right) \subset Y$$

Motivation (the simplest case):

- Π – a hyperplane
- $y \in \Pi_2$
- fix any matrix $B = [B_1 \dots B_n]$ such that
  1. B_1 is transverse to Π at y
  2. $\{B_2, \dots, B_n\}$ span $T_y\Pi$



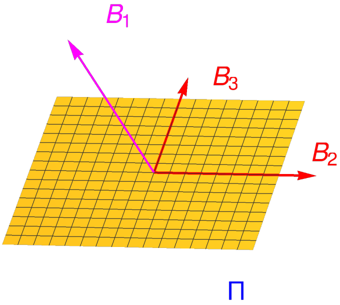
Put $A = B^{-1}$, $Y = (Y_1, \dots, Y_n) = A(P(x) - y)$

$$\mathcal{P}(X) \subset y + B(0, Y_2, \dots, Y_n)$$

- B_1 – controls direction of projection onto Π
- $\{B_2, \dots, B_n\}$ – control wrapping effect on section

Motivation (the simplest case):

- Π – a hyperplane
- $y \in \Pi_2$
- fix any matrix $B = [B_1 \dots B_n]$ such that
  1. B_1 is transverse to Π at y
  2. $\{B_2, \dots, B_n\}$ span $T_y\Pi$



Put $A = B^{-1}$, $Y = (Y_1, \dots, Y_n) = A(P(x) - y)$

$$\mathcal{P}(X) \subset y + B(0, Y_2, \dots, Y_n)$$

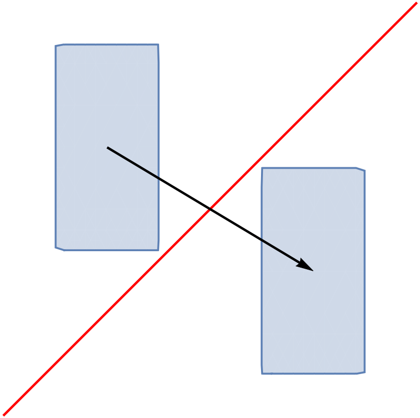
- B_1 – controls direction of projection onto Π
- $\{B_2, \dots, B_n\}$ – control wrapping effect on section

Poincaré map algorithm

Enclosing Poincaré maps

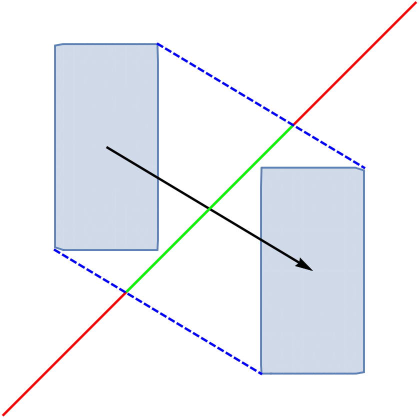
Desired properties:

- avoid subdivisions when crossing section
- reduce wrapping effect that may occur when change representation of a set to coordinates on section



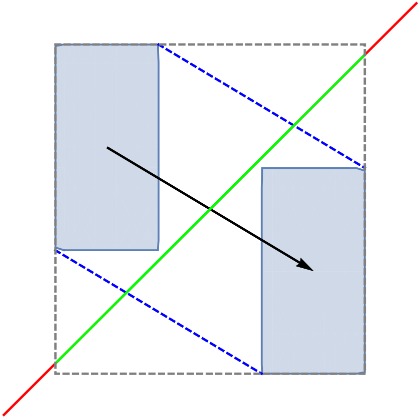
Desired properties:

- avoid subdivisions when crossing section
- reduce wrapping effect that may occur when change representation of a set to coordinates on section

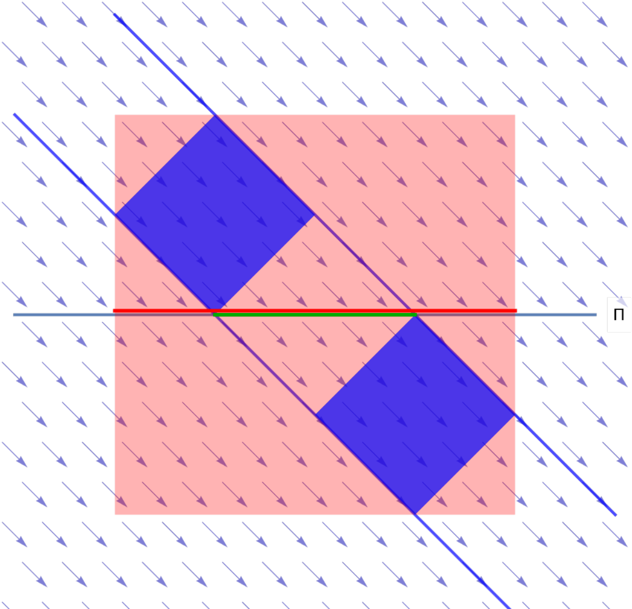


Desired properties:

- avoid subdivisions when crossing section
- reduce wrapping effect that may occur when change representation of a set to coordinates on section



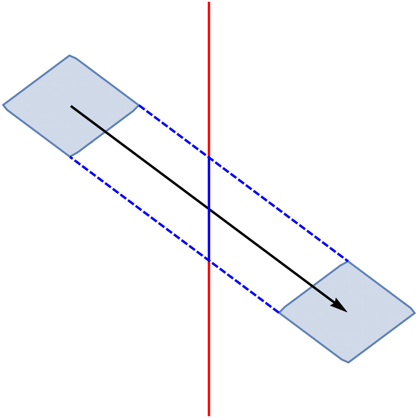
Reduce sliding effect:



Enclosing Poincaré maps

Very important:

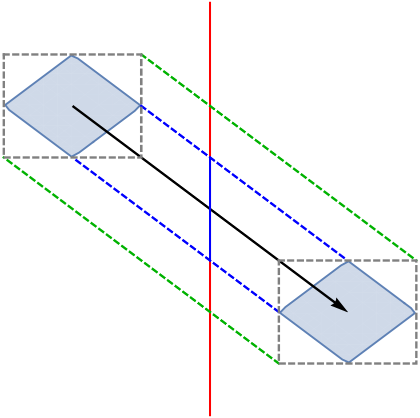
take into account internal representation of solutions in ODE solver



Enclosing Poincaré maps

Very important:

take into account internal representation of solutions in ODE solver



Abstract data structure: `RepresentableSet`

Example:

$$X = x + C\boldsymbol{r_0} + Q\boldsymbol{r}$$

Abstract (type dependent) algorithm:

Algorithm: AFFINETRANSFORM

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $A : \mathbb{R}^n \to \mathbb{R}^m$ - linear map
Input: $y \in \mathbb{R}^n$ - vector
Output: An enclosure of $A(X - y)$

Example:

$$(A(x - y + C\boldsymbol{r_0} + Q\boldsymbol{r})) \cap (A(x - y) + (AC)\boldsymbol{r_0} + (AQ)\boldsymbol{r})$$

Abstract data structure: `RepresentableSet`

Example:

$$X = x + C\boldsymbol{r_0} + Q\boldsymbol{r}$$

Abstract (type dependent) algorithm:

Algorithm: AFFINETRANSFORM

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $A : \mathbb{R}^n \to \mathbb{R}^m$ - linear map
Input: $y \in \mathbb{R}^n$ - vector
Output: An enclosure of $A(X - y)$

Example:

$$(A(x - y + C\boldsymbol{r_0} + Q\boldsymbol{r})) \cap (A(x - y) + (AC)\boldsymbol{r_0} + (AQ)\boldsymbol{r})$$

Abstract data structure: `RepresentableSet`

Example:

$$X = x + C\boldsymbol{r_0} + Q\boldsymbol{r}$$

Abstract (type dependent) algorithm:

Algorithm: AFFINETRANSFORM

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $A : \mathbb{R}^n \to \mathbb{R}^m$ - linear map
Input: $y \in \mathbb{R}^n$ - vector
Output: An enclosure of $A(X - y)$

Example:

$$(A(x - y + C\boldsymbol{r_0} + Q\boldsymbol{r})) \cap (A(x-y) + (AC)\boldsymbol{r_0} + (AQ)\boldsymbol{r})$$

Abstract data structure: `RepresentableSet`

Example:

$$X = x + C\boldsymbol{r_0} + Q\boldsymbol{r}$$

Abstract (type dependent) algorithm:

Algorithm: AFFINETRANSFORM

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $A : \mathbb{R}^n \to \mathbb{R}^m$ - linear map
Input: $y \in \mathbb{R}^n$ - vector
Output: An enclosure of $A(X - y)$

Example:

$$(A(x - y + C\boldsymbol{r_0} + Q\boldsymbol{r})) \cap (A(x - y) + (AC)\boldsymbol{r_0} + (AQ)\boldsymbol{r})$$

Abstract (type dependent) algorithm:

Algorithm: EVAL

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $g : \mathbb{R}^n \to \mathbb{R}^m$ - smooth
Output: Bound for $g(X)$

Example:

Algorithm: EVAL

Input: $x + C\boldsymbol{r_0} + Q\boldsymbol{r} \subset \mathbb{R}^n$ - doubleton
Input: $g : \mathbb{R}^n \to \mathbb{R}^m$ - smooth function
Output: Bound for $g(x + C\boldsymbol{r_0} + Q\boldsymbol{r})$

$\boldsymbol{X} \leftarrow [x + C\boldsymbol{r_0} + Q\boldsymbol{r}]_I$
$\boldsymbol{M} \leftarrow [Dg](\boldsymbol{X})$
return $[g](\boldsymbol{X}) \cap [[g](x) + (\boldsymbol{M}C)\boldsymbol{r_0} + (\boldsymbol{M}Q)\boldsymbol{r}]_I$

Abstract (type dependent) algorithm:

Algorithm: EVAL

Input: $X \subset \mathbb{R}^n$ - `RepresentableSet`
Input: $g : \mathbb{R}^n \to \mathbb{R}^m$ - smooth
Output: Bound for $g(X)$

Example:

Algorithm: EVAL

Input: $x + C\boldsymbol{r_0} + Q\boldsymbol{r} \subset \mathbb{R}^n$ - doubleton
Input: $g : \mathbb{R}^n \to \mathbb{R}^m$ - smooth function
Output: Bound for $g(x + C\boldsymbol{r_0} + Q\boldsymbol{r})$

$\boldsymbol{X} \leftarrow [x + C\boldsymbol{r_0} + Q\boldsymbol{r}]_I$
$\boldsymbol{M} \leftarrow [Dg]\,(\boldsymbol{X})$
return $[g]\,(\boldsymbol{X}) \cap [[g]\,(x) + (\boldsymbol{M}C)\boldsymbol{r_0} + (\boldsymbol{M}Q)\boldsymbol{r}]_I$

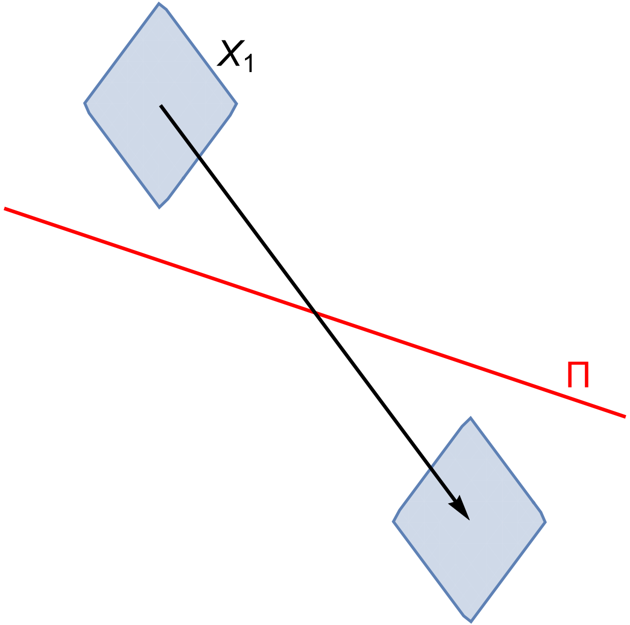
Refinement of return time

Algorithm: REFINERETURNTIME

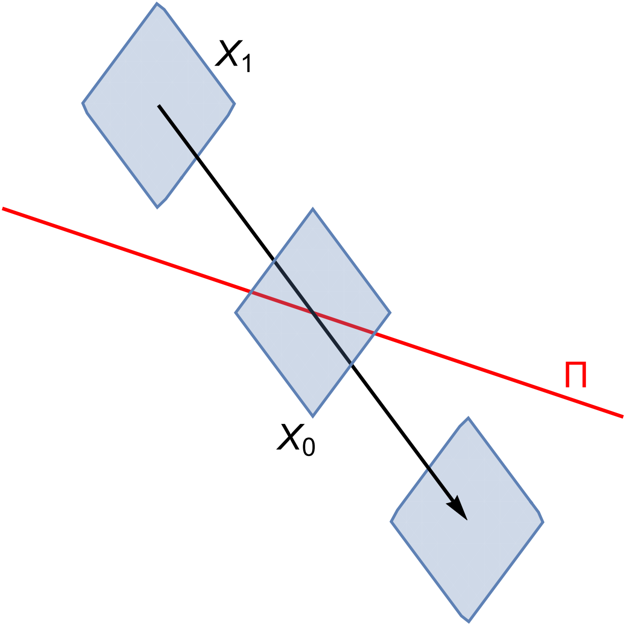
Input: $[t_1, t_2]$ an interval that encloses $t_\Pi(X)$
Input: X_1 `RepresentableSet` that encloses $\varphi(t_1, X)$
Input: α function that defines the section Π
Input: f underlying vector field
Output: Improved bound for $t_\Pi(X)$

$t_0 \leftarrow (t_1 + t_2)/2$
$X_0 \leftarrow$ `RepresentableSet` that encloses $\varphi(t_0 - t_1, X_1)$
$\boldsymbol{g_0} \leftarrow \mathtt{eval}(X_0, \alpha)$
$\boldsymbol{e} \leftarrow \varphi([0, t_2 - t_1], X_1)$
$\boldsymbol{g} \leftarrow \mathtt{eval}\,(\boldsymbol{e}, D\alpha(\cdot) \cdot f(\cdot))$
return $[t_1, t_2] \cap (t_0 - \boldsymbol{g_0}/\boldsymbol{g})$

Refinement of return time - geometry of the algorithm



Refinement of return time - geometry of the algorithm



Algorithm: COMPUTEPOINCAREMAP

Input: $[t_1, t_2]$ an interval that encloses $t_\Pi(X)$
Input: X_1 `RepresentableSet` that encloses $\varphi(t_1, X)$
Input: f a vector field
Input: y a vector
Input: A a linear map
Output: An enclosure of $A(\mathcal{P}(X) - y)$

$\boldsymbol{e} \leftarrow \varphi([0, t_2 - t_1], X_1)$
$t_0 \leftarrow (t_1 + t_2)/2$
$\boldsymbol{\Delta t} \leftarrow [t_1, t_2] - t_0$
$X_0 \leftarrow$ `RepresentableSet` that encloses $\varphi(t_0 - t_1, X_1)$
$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$
$\boldsymbol{z} \leftarrow (\boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}) \cap [A(\boldsymbol{e} - y)]_I$
return $\boldsymbol{z}$

Correctness:

Set $T = [0, t_2 - t_1]$ and use Taylor expansion:

$$\mathcal{P}(X) \subset \varphi(T, X_1) = \varphi(\mathbf{\Delta t}, X_0) \subset X_0 + f(X_0)\mathbf{\Delta t} + \frac{1}{2}Df(\boldsymbol{e})f(\boldsymbol{e})\mathbf{\Delta t}^2$$

This gives:

$$\begin{aligned} A(\mathcal{P}(X) - y) &\subset A(X_0 - y) + Af(X_0)\mathbf{\Delta t} + \frac{1}{2}ADf(\boldsymbol{e})f(\boldsymbol{e})\mathbf{\Delta t}^2 \\ &\subset \boldsymbol{y_0} + \boldsymbol{y} + \mathbf{\Delta y}. \end{aligned}$$

Recall:

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \mathbf{\Delta t}$
$\mathbf{\Delta y} \leftarrow \frac{1}{2}A \cdot [Df](\boldsymbol{e}) \cdot [f](\boldsymbol{e}) \cdot \mathbf{\Delta t}^2$

Correctness:

Set $T = [0, t_2 - t_1]$ and use Taylor expansion:

$$\mathcal{P}(X) \subset \varphi(T, X_1) = \varphi(\boldsymbol{\Delta t}, X_0) \subset X_0 + f(X_0)\boldsymbol{\Delta t} + \frac{1}{2}Df(\boldsymbol{e})f(\boldsymbol{e})\boldsymbol{\Delta t}^2$$

This gives:

$$\begin{aligned} A(\mathcal{P}(X) - y) &\subset A(X_0 - y) + Af(X_0)\boldsymbol{\Delta t} + \frac{1}{2}ADf(\boldsymbol{e})f(\boldsymbol{e})\boldsymbol{\Delta t}^2 \\ &\subset \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}. \end{aligned}$$

Recall:

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2}A \cdot [Df](\boldsymbol{e}) \cdot [f](\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

Correctness:

Set $T = [0, t_2 - t_1]$ and use Taylor expansion:

$$\mathcal{P}(X) \subset \varphi(T, X_1) = \varphi(\boldsymbol{\Delta t}, X_0) \subset X_0 + f(X_0)\boldsymbol{\Delta t} + \frac{1}{2}Df(\boldsymbol{e})f(\boldsymbol{e})\boldsymbol{\Delta t}^2$$

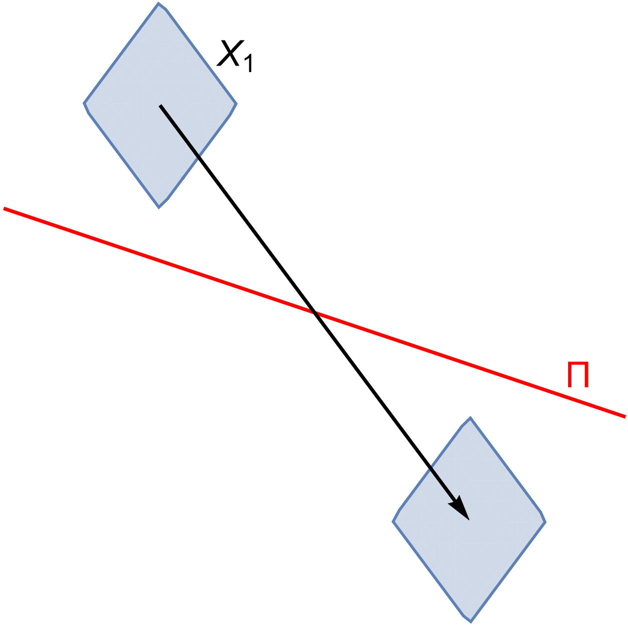
This gives:

$$\begin{aligned} A(\mathcal{P}(X) - y) &\subset A(X_0 - y) + Af(X_0)\boldsymbol{\Delta t} + \frac{1}{2}ADf(\boldsymbol{e})f(\boldsymbol{e})\boldsymbol{\Delta t}^2 \\ &\subset \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}. \end{aligned}$$

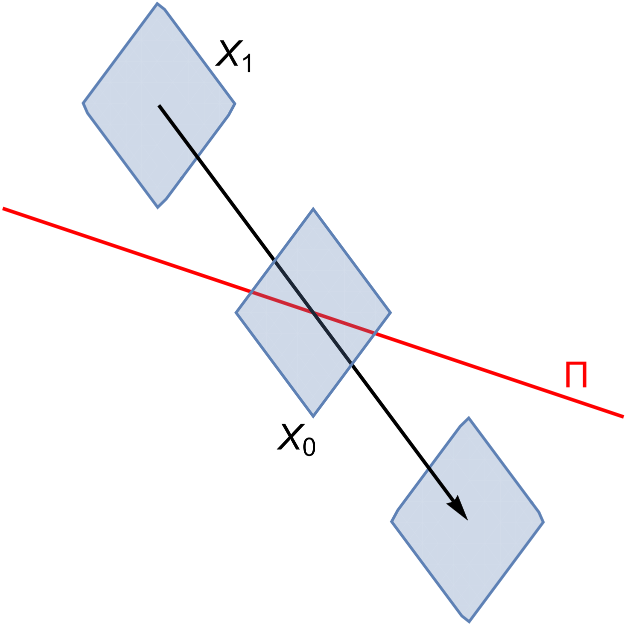
Recall:

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2}A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

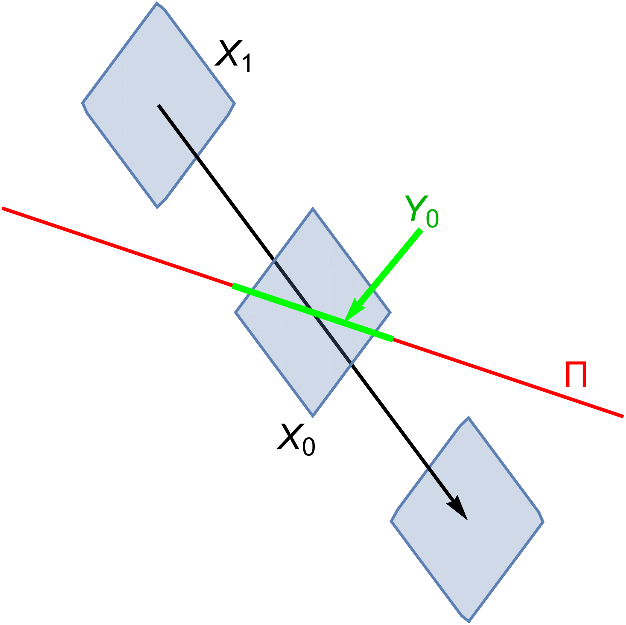
Enclosing Poincaré maps - geometry of the algorithm



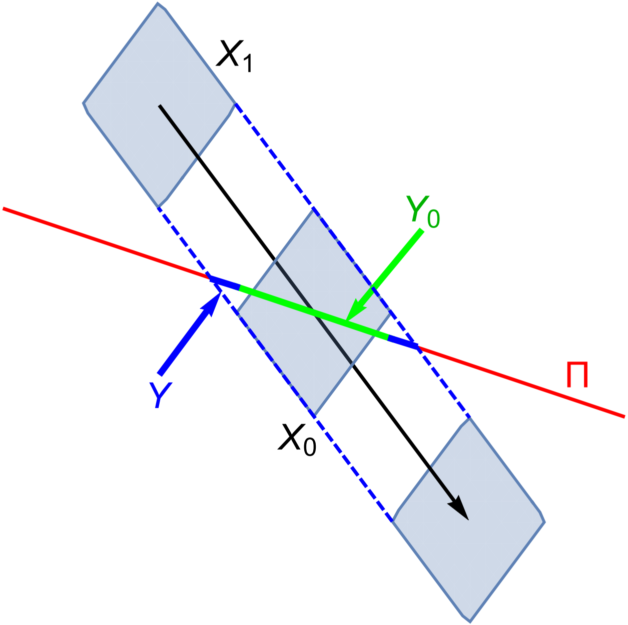
Enclosing Poincaré maps - geometry of the algorithm



Enclosing Poincaré maps - geometry of the algorithm



Enclosing Poincaré maps - geometry of the algorithm



Recall:

$$A(\mathcal{P}(X)-y) \subset \boldsymbol{y_0}+\boldsymbol{y}+\boldsymbol{\Delta y}$$

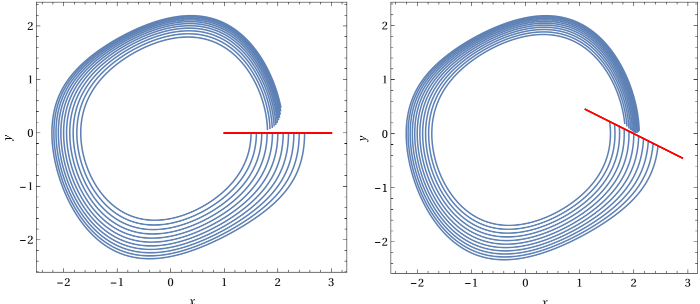
$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$

$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$

$\boldsymbol{\Delta y} \leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

Goal: minimize $\mathit{diam}(\boldsymbol{y}) \in O(\mathit{diam}(X_0)^2)$ or better

1. **Sections are fixed:** play with coordinate system A to reduce sliding effect
   $\texttt{eval}(X_0, A \circ f) \approx (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) + \mathit{small}$
2. **Sections are free to choose:** set Π so that $\boldsymbol{\Delta t}$ is small



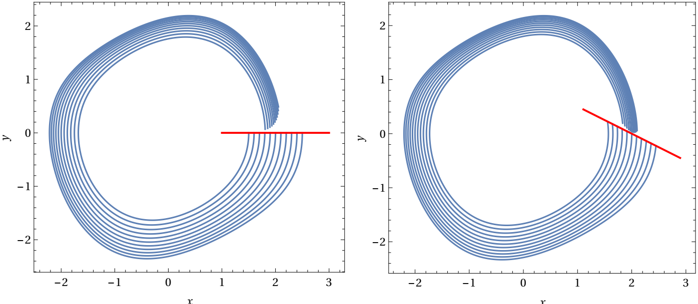
Recall:

$$A(\mathcal{P}(X) - y) \subset \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}$$

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

Goal: minimize $diam(\boldsymbol{y}) \in O(diam(X_0)^2)$ or better

1. **Sections are fixed:** play with coordinate system A to reduce sliding effect
   $\texttt{eval}(X_0, A \circ f) \approx (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) + small$
2. **Sections are free to choose:** set Π so that $\boldsymbol{\Delta t}$ is small



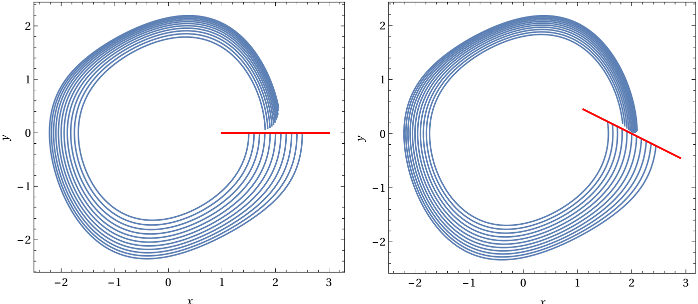
Recall:

$$A(\mathcal{P}(X) - y) \subset \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}$$

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

Goal: minimize $diam(\boldsymbol{y}) \in O(diam(X_0)^2)$ or better

1. **Sections are fixed:** play with coordinate system A to reduce sliding effect
   $\texttt{eval}(X_0, A \circ f) \approx (\mathbf{1}, \mathbf{0}, \mathbf{\ldots}, \mathbf{0}) + small$
2. **Sections are free to choose:** set Π so that $\boldsymbol{\Delta t}$ is small



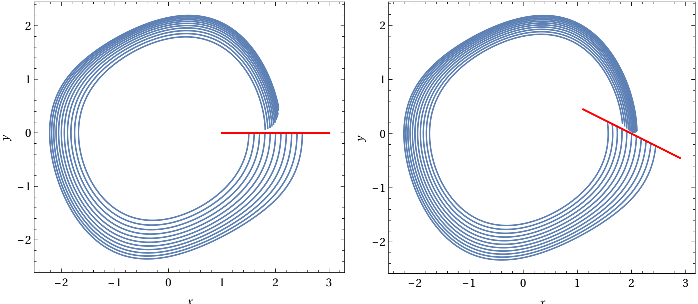
Recall:

$$A(\mathcal{P}(X) - y) \subset \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y}$$

$\boldsymbol{y_0} \leftarrow \texttt{affineTransform}(X_0, A, y)$
$\boldsymbol{y} \leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t}$
$\boldsymbol{\Delta y} \leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t}^2$

Goal: minimize $diam(\boldsymbol{y}) \in O(diam(X_0)^2)$ or better

1. **Sections are fixed:** play with coordinate system A to reduce sliding effect
   $\texttt{eval}(X_0, A \circ f) \approx (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}) + small$
2. **Sections are free to choose:** set Π so that $\boldsymbol{\Delta t}$ is small



First strategy: reduction of sliding effect

Theorem

If $y \in X_0 \cap \Pi$ and $A = B^{-1}$, where

$$B = \left[\ f(y)\ \middle|\ M\ \right]$$

and columns in M span $T_y\Pi$. Then

$$\boldsymbol{y} + \boldsymbol{\Delta y} \in (\boldsymbol{\Delta t}, 0, 0, \ldots) + O\left(diam(X_0)^2\right).$$

Corollary:

Π - hyperplane

$\boldsymbol{z} = \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y} = (\boldsymbol{z_1}, \ldots, \boldsymbol{z_n})$

Then

$$\mathcal{P}(X) \subset (y + B\boldsymbol{z}) \cap \Pi = y + B(0, \boldsymbol{z_2}, \ldots, \boldsymbol{z_n})^T.$$

where

$$(0, \boldsymbol{z_2}, \ldots, \boldsymbol{z_n}) = (0, (\boldsymbol{y_0})_2, \ldots, (\boldsymbol{y_0})_n) + O\left(diam(X_0)^2\right),$$

First strategy: reduction of sliding effect

Theorem

If $y \in X_0 \cap \Pi$ *and* $A = B^{-1}$, *where*

$$B = \left[\ f(y)\ \middle|\ M\ \right]$$

and columns in M *span* $T_y\Pi$. *Then*

$$\boldsymbol{y} + \boldsymbol{\Delta y} \in (\boldsymbol{\Delta t}, 0, 0, \ldots) + O\left(diam(X_0)^2\right).$$

Corollary:

Π - hyperplane

$\boldsymbol{z} = \boldsymbol{y_0} + \boldsymbol{y} + \boldsymbol{\Delta y} = (\boldsymbol{z_1}, \ldots, \boldsymbol{z_n})$

Then

$$\mathcal{P}(X) \subset (y + B\boldsymbol{z}) \cap \Pi = y + B(0, \boldsymbol{z_2}, \ldots, \boldsymbol{z_n})^T.$$

where

$$(0, \boldsymbol{z_2}, \ldots, \boldsymbol{z_n}) = (0, \boldsymbol{(y_0)_2}, \ldots, \boldsymbol{(y_0)_n}) + O\left(diam(X_0)^2\right),$$

First strategy: reduction of sliding effect

$$\boldsymbol{y} \subset [Af(x_0) + A[Df](X_0)\Delta X_0]_I\, \boldsymbol{\Delta t},$$

where $x_0 = \mathrm{mid}(X_0)$ and $\Delta X_0 = X_0 - x_0$. We have

$$A[Df](X_0)\Delta X_0 \boldsymbol{\Delta t} \in O\left(diam(X_0)^2\right)$$

because $diam(\boldsymbol{\Delta t}) \in O(diam(X_0))$.

$y \in X_0$ – convex:

$$Af(x_0)\boldsymbol{\Delta t} \in (Af(y))\boldsymbol{\Delta t} + [ADf(X_0)]_I(x_0 - y)\boldsymbol{\Delta t}.$$

A chosen so that $Af(y) = (1, 0, 0, \ldots)$ and thus

$$[(Af(y))\boldsymbol{\Delta} t]_I = (\boldsymbol{\Delta t}, 0, 0 \ldots).$$

Second strategy: reduction of crossing time diameter

$$\begin{aligned} \boldsymbol{y} &\leftarrow \mathtt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t} \\ \boldsymbol{\Delta y} &\leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \boldsymbol{\Delta t^2} \end{aligned}$$

$t_\Pi : \Pi_1 \to \mathbb{R}$ - return time

Observation: If

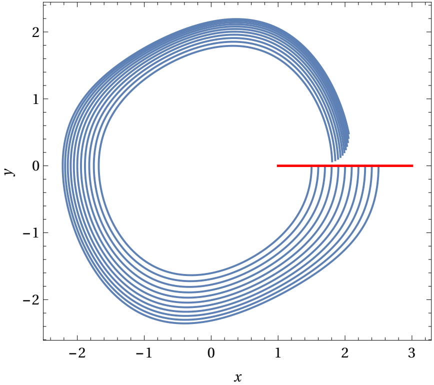
$t_\Pi \approx \mathrm{constant}$ for $x \in U \subset \Pi$

then bounds for crossing time and $\mathcal{P}$ should be tighter.

Second strategy: reduction of crossing time diameter

$$\begin{aligned} \boldsymbol{y} &\leftarrow \texttt{eval}(X_0, A \circ f) \cdot \boldsymbol{\Delta t} \\ \boldsymbol{\Delta y} &\leftarrow \frac{1}{2} A \cdot [Df](\boldsymbol{e}) \cdot [f](\boldsymbol{e}) \cdot \boldsymbol{\Delta t^2} \end{aligned}$$

$t_\Pi : \Pi_1 \to \mathbb{R}$ - return time



Observation: If

$$t_\Pi \approx \text{constant for } x \in U \subset \Pi$$

then bounds for crossing time and $\mathcal{P}$ should be tighter.

Second strategy: reduction of crossing time diameter

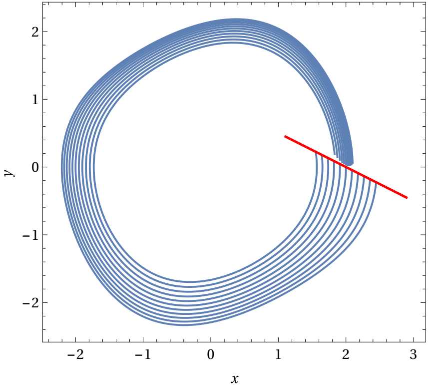
$$\begin{aligned} \boldsymbol{y} &\leftarrow \texttt{eval}(X_0, A \circ f) \cdot \color{red}{\boldsymbol{\Delta t}} \\ \boldsymbol{\Delta y} &\leftarrow \frac{1}{2} A \cdot [Df]\,(\boldsymbol{e}) \cdot [f]\,(\boldsymbol{e}) \cdot \color{red}{\boldsymbol{\Delta t^2}} \end{aligned}$$

$t_\Pi : \Pi_1 \to \mathbb{R}$ - return time

Observation: If

$t_\Pi \approx \text{constant}$ for $x \in U \subset \Pi$

then bounds for crossing time and $\mathcal{P}$ should be tighter.



Second strategy: reduction of crossing time diameter

- f – vector field
- φ – induced local flow
- $\Pi = \Pi_\alpha$ – Poincaré section

Theorem

Assume

- $\varphi(T, x_0) = x_0 \in \Pi$ *for some minimal* $T > 0$
- $\lambda = 1$ *is an eigenvalue of* $M := D_x\varphi(T, x_0)$ *of multiplicity one.*

Then

$$(\ker Dt_\Pi = T_{x_0}\Pi) \Leftrightarrow (D\alpha(x_0) \text{ is a left eigenvector of } M \text{ for } \lambda = 1)$$

In such case, $t_\Pi(x) = t_\Pi(x_0) + O(\|x - x_0\|^2)$ *for* $x \in \Pi$.

Second strategy: reduction of crossing time diameter

- f – vector field
- φ – induced local flow
- $\Pi = \Pi_\alpha$ – Poincaré section

Theorem

Assume

- $\varphi(T, x_0) = x_0 \in \Pi$ *for some minimal* $T > 0$
- $\lambda = 1$ *is an eigenvalue of* $M := D_x\varphi(T, x_0)$ *of multiplicity one.*

Then

$$(\ker Dt_\Pi = T_{x_0}\Pi) \Leftrightarrow (D\alpha(x_0) \text{ is a left eigenvector of } M \text{ for } \lambda = 1)$$

In such case, $t_\Pi(x) = t_\Pi(x_0) + O(\|x - x_0\|^2)$ *for* $x \in \Pi$.

Second strategy: reduction of crossing time diameter

- f – vector field
- φ – induced local flow
- $\Pi = \Pi_\alpha$ – Poincaré section

Theorem

Assume

- $\varphi(T, x_0) = x_0 \in \Pi$ *for some minimal* $T > 0$
- $\lambda = 1$ *is an eigenvalue of* $M := D_x\varphi(T, x_0)$ *of multiplicity one.*

Then

$$(\ker Dt_\Pi = T_{x_0}\Pi) \Leftrightarrow (D\alpha(x_0) \text{ is a left eigenvector of } M \text{ for } \lambda = 1)$$

In such case, $t_\Pi(x) = t_\Pi(x_0) + O(\|x - x_0\|^2)$ *for* $x \in \Pi$.

Second strategy: reduction of crossing time diameter

- f – vector field
- φ – induced local flow
- $\Pi = \Pi_\alpha$ – Poincaré section

Theorem

Assume

- $\varphi(T, x_0) = x_0 \in \Pi$ *for some minimal* $T > 0$
- $\lambda = 1$ *is an eigenvalue of* $M := D_x\varphi(T, x_0)$ *of multiplicity one.*

Then

$$(\ker Dt_\Pi = T_{x_0}\Pi) \Leftrightarrow (D\alpha(x_0) \textit{ is a left eigenvector of } M \textit{ for } \lambda = 1)$$

In such case, $t_\Pi(x) = t_\Pi(x_0) + O(\|x - x_0\|^2)$ *for* $x \in \Pi$.

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$

$\alpha(x) = 0$ - defines section

$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$
$$\Downarrow$$
$$D\alpha(x_0) \text{ and } Dt_\Pi(x_0) \text{ are proportional}$$
$$\Downarrow$$
$$\ker Dt_\Pi = T_{x_0}\Pi$$
$$\Downarrow$$
$$\frac{\partial t_\Pi}{\partial v}(x) = 0 \text{ for } v \in T_{x_0}\Pi$$

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$
$\alpha(x) = 0$ - defines section
$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$
$$\Downarrow$$
$D\alpha(x_0)$ and $Dt_\Pi(x_0)$ are proportional
$$\Downarrow$$
$$\ker Dt_\Pi = T_{x_0}\Pi$$
$$\Downarrow$$
$\frac{\partial t_\Pi}{\partial v}(x) = 0$ for $v \in T_{x_0}\Pi$

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$
$\alpha(x) = 0$ - defines section
$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$

$$\Downarrow$$

$D\alpha(x_0)$ and $Dt_\Pi(x_0)$ are proportional

$$\Downarrow$$

$$\ker Dt_\Pi = T_{x_0}\Pi$$

$$\Downarrow$$

$$\frac{\partial t_\Pi}{\partial v}(x) = 0 \text{ for } v \in T_{x_0}\Pi$$

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$
$\alpha(x) = 0$ - defines section
$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$

$$\Downarrow$$

$D\alpha(x_0)$ and $Dt_\Pi(x_0)$ are proportional

$$\Downarrow$$

$$\ker Dt_\Pi = T_{x_0}\Pi$$

$$\Downarrow$$

$$\frac{\partial t_\Pi}{\partial v}(x) = 0 \text{ for } v \in T_{x_0}\Pi$$

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$
$\alpha(x) = 0$ - defines section
$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$
$$\Downarrow$$
$D\alpha(x_0)$ and $Dt_\Pi(x_0)$ are proportional
$$\Downarrow$$
$$\ker Dt_\Pi = T_{x_0}\Pi$$
$$\Downarrow$$
$$\frac{\partial t_\Pi}{\partial v}(x) = 0 \text{ for } v \in T_{x_0}\Pi$$

Minimize crossing time diameter:

Case of fixed point: assume $P(x_0) = x_0$
$\alpha(x) = 0$ - defines section
$M := \frac{\partial}{\partial x}\varphi(t = t_\Pi(x_0), x_0)$

$$\alpha(\varphi(t_\Pi(x), x)) \equiv 0$$
$$\langle D\alpha(x); f(x)\rangle Dt_\Pi(x) + D\alpha(x)M \equiv 0$$

If $D\alpha(x_0)$ is left eigenvector for M for $\lambda = 1$ then

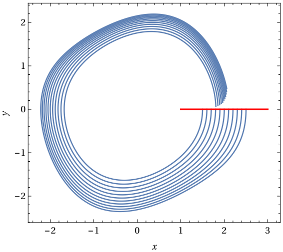
$$\langle D\alpha(x_0); f(x_0)\rangle Dt_\Pi(x_0) + D\alpha(x_0) \equiv 0$$
$$\Downarrow$$
$D\alpha(x_0)$ and $Dt_\Pi(x_0)$ are proportional
$$\Downarrow$$
$$\ker Dt_\Pi = T_{x_0}\Pi$$
$$\Downarrow$$
$\frac{\partial t_\Pi}{\partial v}(x) = 0$ for $v \in T_{x_0}\Pi$

Example: van der Pol equation

Equation:

$$x'' = 0.2x'(1 - x^2) - x$$

The section: $\Pi = \{y = 0\}$
(flowdir & normal)



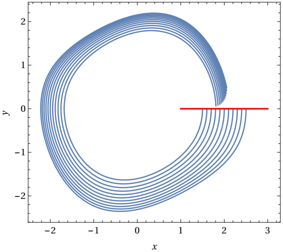
$\delta = \frac{1}{2}\mathrm{diam}(\boldsymbol{u})$	diameter of crossing time	$\pi_x\mathcal{P}(\boldsymbol{u}) - x_0$
10^{-9}	$3.6 \cdot 10^{-10}$	$[-2.83, 2.83] \cdot 10^{-10}$
10^{-8}	$3.6 \cdot 10^{-9}$	$[-2.83, 2.83] \cdot 10^{-9}$
10^{-7}	$3.6 \cdot 10^{-8}$	$[-2.83, 2.83] \cdot 10^{-8}$
10^{-6}	$3.6 \cdot 10^{-7}$	$[-2.83, 2.83] \cdot 10^{-7}$
10^{-5}	$3.6 \cdot 10^{-6}$	$[-2.83, 2.83] \cdot 10^{-6}$
10^{-4}	$3.61 \cdot 10^{-5}$	$[-2.83, 2.83] \cdot 10^{-5}$
10^{-3}	$3.64 \cdot 10^{-4}$	$[-2.84, 2.84] \cdot 10^{-4}$
10^{-2}	$3.97 \cdot 10^{-3}$	$[-2.93, 2.93] \cdot 10^{-3}$
10^{-1}	$1.18 \cdot 10^{-1}$	$[-6.5, 6.12] \cdot 10^{-2}$

Example: van der Pol equation

Equation:

$$x'' = 0.2x'(1 - x^2) - x$$

The section: $\Pi = \{y = 0\}$
(flowdir & normal)



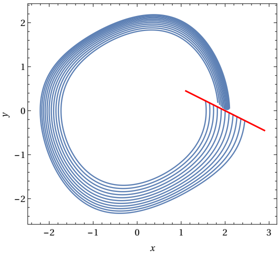
$\delta = \frac{1}{2}\mathrm{diam}(\boldsymbol{u})$	diameter of crossing time	$\pi_x\mathcal{P}(\boldsymbol{u}) - x_0$
10^{-9}	$3.6 \cdot 10^{-10}$	$[-2.83, 2.83] \cdot 10^{-10}$
10^{-8}	$3.6 \cdot 10^{-9}$	$[-2.83, 2.83] \cdot 10^{-9}$
10^{-7}	$3.6 \cdot 10^{-8}$	$[-2.83, 2.83] \cdot 10^{-8}$
10^{-6}	$3.6 \cdot 10^{-7}$	$[-2.83, 2.83] \cdot 10^{-7}$
10^{-5}	$3.6 \cdot 10^{-6}$	$[-2.83, 2.83] \cdot 10^{-6}$
10^{-4}	$3.61 \cdot 10^{-5}$	$[-2.83, 2.83] \cdot 10^{-5}$
10^{-3}	$3.64 \cdot 10^{-4}$	$[-2.84, 2.84] \cdot 10^{-4}$
10^{-2}	$3.97 \cdot 10^{-3}$	$[-2.93, 2.93] \cdot 10^{-3}$
10^{-1}	$1.18 \cdot 10^{-1}$	$[-6.5, 6.12] \cdot 10^{-2}$

Example: van der Pol equation

Equation:

$$x'' = 0.2x'(1 - x^2) - x$$

The section: minimizes crossing time diameter



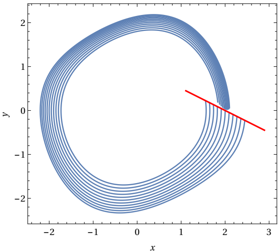
$\delta = \frac{1}{2}\mathrm{diam}(\boldsymbol{u})$	diameter of crossing time	$\pi_{x_2}\mathcal{P}(\boldsymbol{u})$
10^{-9}	$3.46 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-10}$
10^{-8}	$3.46 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-9}$
10^{-7}	$6.39 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-8}$
10^{-6}	$2.99 \cdot 10^{-12}$	$[-2.83, 2.83] \cdot 10^{-7}$
10^{-5}	$2.96 \cdot 10^{-10}$	$[-2.83, 2.83] \cdot 10^{-6}$
10^{-4}	$2.96 \cdot 10^{-8}$	$[-2.83, 2.83] \cdot 10^{-5}$
10^{-3}	$2.97 \cdot 10^{-6}$	$[-2.83, 2.83] \cdot 10^{-4}$
10^{-2}	$3.11 \cdot 10^{-4}$	$[-2.89, 2.89] \cdot 10^{-3}$
10^{-1}	$6.26 \cdot 10^{-2}$	$[-4.66, 4.78] \cdot 10^{-2}$

Example: van der Pol equation

Equation:

$$x'' = 0.2x'(1 - x^2) - x$$

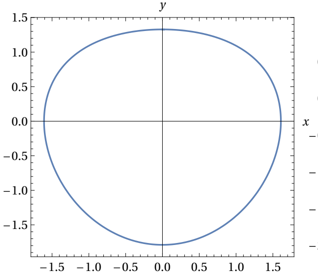
The section: minimizes crossing time diameter



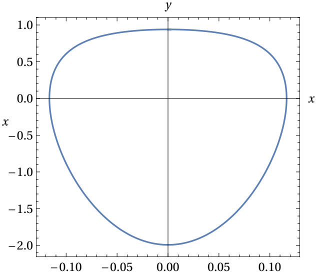
$\delta = \frac{1}{2}\mathrm{diam}(\boldsymbol{u})$	diameter of crossing time	$\pi_{x_2}\mathcal{P}(\boldsymbol{u})$
10^{-9}	$3.46 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-10}$
10^{-8}	$3.46 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-9}$
10^{-7}	$6.39 \cdot 10^{-14}$	$[-2.83, 2.83] \cdot 10^{-8}$
10^{-6}	$2.99 \cdot 10^{-12}$	$[-2.83, 2.83] \cdot 10^{-7}$
10^{-5}	$2.96 \cdot 10^{-10}$	$[-2.83, 2.83] \cdot 10^{-6}$
10^{-4}	$2.96 \cdot 10^{-8}$	$[-2.83, 2.83] \cdot 10^{-5}$
10^{-3}	$2.97 \cdot 10^{-6}$	$[-2.83, 2.83] \cdot 10^{-4}$
10^{-2}	$3.11 \cdot 10^{-4}$	$[-2.89, 2.89] \cdot 10^{-3}$
10^{-1}	$6.26 \cdot 10^{-2}$	$[-4.66, 4.78] \cdot 10^{-2}$

Experiments:

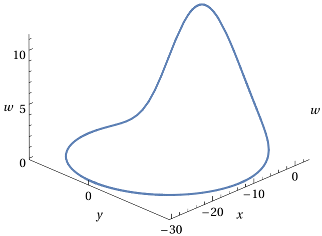
(u_M)



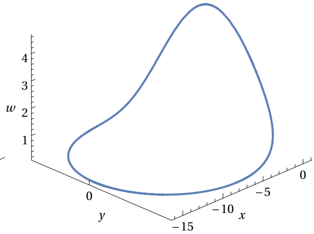
(u_{FS})



(u_{R_h})



$(u_{R_{\mathrm{pd}}})$



Experiments:

- **Michelson system**

$$x' = y, \quad y' = z, \quad z' = 0.8^2 - y - \frac{1}{2}x^2$$

$$u_M \approx (0, 1.3282586610856929025\,8, 0)$$

$$\lambda_{M_1} \approx -21.57189303583905, \quad \lambda_{M_2} \approx -0.046356617768258279$$

- **Falkner-Skan equation**

$$x' = y, \quad y' = z, \quad z' = 250(y^2 - 1) - xz$$

$$\approx (0, 0.93971220877967247627\,5, 0)$$

$$\lambda_{FS_1} \approx -3.1255162015308575, \quad \lambda_{FS_2} \approx -0.31994714969329141.$$

- **Rössler system**

$$x' = -y - w, \quad y' = x + ay + z, \quad z' = dy + cw, \quad w' = xw + b$$

$u_{R_h} \approx (-29.841563300389689, 0, 15.047757539453583, 0.10059818458161384)$

$a = 0.25$, $b = 3$, $c = -0.5$, $d = 0.05$

$\lambda_{R_{h1}} \approx -2.9753618617897111, \quad \lambda_{R_{h2}} \approx 1.11933293616997, \quad \lambda_{R_{h3}} \approx -2{\cdot}10^{-18}$

- **Rössler system**

$u_{R_{pd}} \approx (-16.051468914417546, 0, 8.362179513564907, 0.18738588995067224)$

$a = 0.25$, $b = 3$, $c = -0.397617541005413$, $d = 0.05$

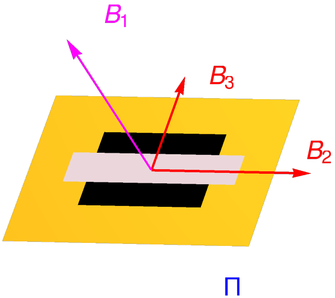
$$\lambda_{R_{pd1}} \approx 1.2039286263296654, \quad \lambda_{R_{pd2}} \approx -1, \quad \lambda_{R_{pd3}} \approx -6 \cdot 10^{-17}$$

Settings:

- Section always linear $x = 0$ or $y = 0$
- $B_2, \ldots, B_n$ – eigenvectors of $DP(u)$

Goal:

Compute $\mathcal{P}(u + \frac{1}{2}s[-1,1]B_2 + \ldots + \frac{1}{2}s[-1,1]B_n)$ in coordinate system $B_2, \ldots, B_n$



Three strategies:

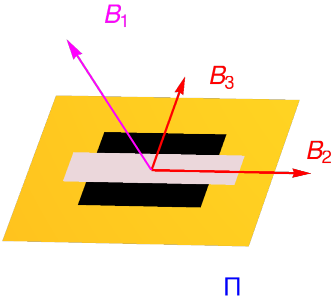
- **cartessian** : $A = Id$
- **diag+normal** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - normal to Π
- **diag+flowdir** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - flow direction

Settings:

- Section always linear $x = 0$ or $y = 0$
- $B_2, \ldots, B_n$ – eigenvectors of $DP(u)$

Goal:

Compute $\mathcal{P}(u + \frac{1}{2}s[-1,1]B_2 + \ldots + \frac{1}{2}s[-1,1]B_n)$ in coordinate system $B_2, \ldots, B_n$



Three strategies:

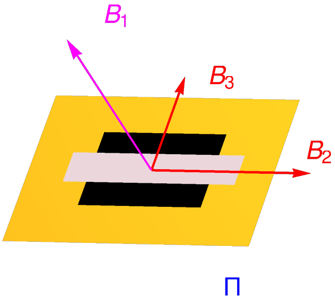
- **cartessian** : $A = Id$
- **diag+normal** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - normal to Π
- **diag+flowdir** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - flow direction

Settings:

- Section always linear $x = 0$ or $y = 0$
- $B_2, \ldots, B_n$ – eigenvectors of $DP(u)$

Goal:

Compute $\mathcal{P}(u + \frac{1}{2}s[-1,1]B_2 + \ldots + \frac{1}{2}s[-1,1]B_n)$ in coordinate system $B_2, \ldots, B_n$



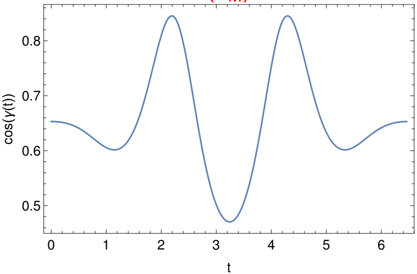
Three strategies:

- **cartesian** : $A = Id$
- **diag+normal** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - normal to Π
- **diag+flowdir** $A^{-1} = B = [B_1, \ldots, B_n]$, B_1 - flow direction

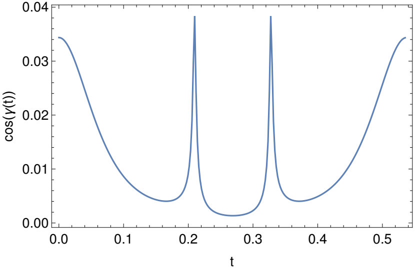
Strategies:

- **orthogonal**
- **CTO** - Crossing-Time Optimal Section
- **max angle CTO**

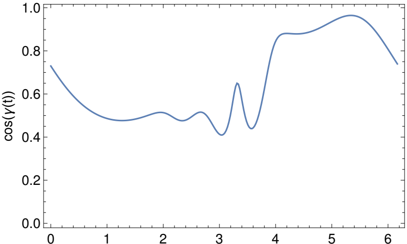
(u_M)



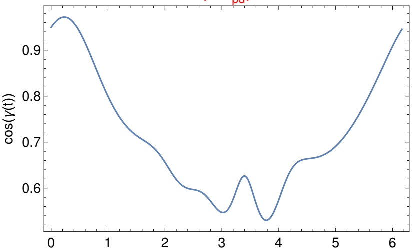
(u_{FS})

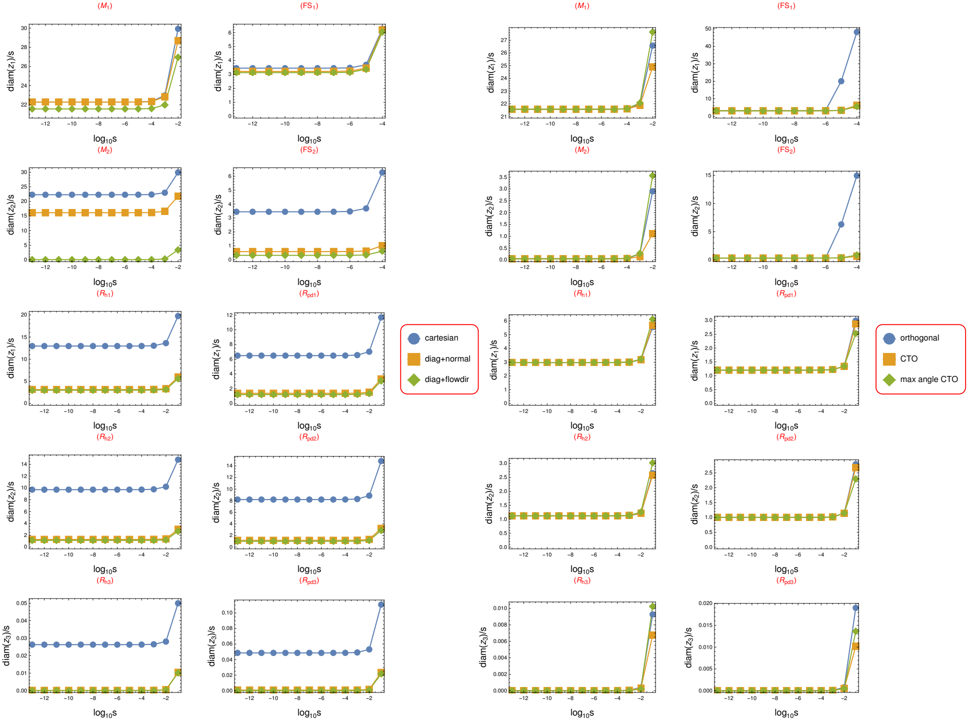


(u_{R_h})



$(u_{R_{pd}})$





(M1)
(FS1)
(M2)
(FS2)
(Rh1)
(Rpd1)
(Rh2)
(Rpd2)
(Rh3)
(Rpd3)
diam(z1)/s
diam(z2)/s
diam(z3)/s
log10s
cartesian
diag+normal
diag+flowdir
orthogonal
CTO
max angle CTO

$\log_{10} s$	**diag+normal**	**diag+flowdir**	**orthogonal**	**CTO**	**max angle CTO**
			$\lambda_{M_1} \approx -21.57189303583905$		
-13	22.288770093234	21.571893035879	21.571893035877	21.571893035868	21.571893035886
-12	22.288770093715	21.571893036244	21.571893036220	21.571893036127	21.571893036325
-11	22.288770098524	21.571893039893	21.571893039664	21.571893038726	21.571893040654
-10	22.288770146617	21.571893076394	21.571893074103	21.571893064721	21.571893083996
-9	22.288770627515	21.571893441397	21.571893418486	21.571893324667	21.571893517420
-8	22.288775436860	21.571897091434	21.571896862320	21.571895924126	21.571897851656
-7	22.288823530421	21.571933591888	21.571931300742	21.571921918754	21.571941194109
-6	22.289304477585	21.572298605637	21.572275692728	21.572181868412	21.572374627014
-5	22.294115104218	21.575949664009	21.575720389491	21.574781702701	21.576709794306
-4	22.342338639296	21.612552589485	21.610245257920	21.600813874537	21.620145477068
-3	22.836610909239	21.988082379753	21.963506140123	21.864579188760	22.064023271359
-2	28.684735372216	26.950444290822	26.572537443087	24.899391229219	27.640671971912
			$\lambda_{M_2} \approx -0.046356617768258279$		
-13	16.134915640465	0.046356617787156	0.046356617784308	0.046356617774995	0.046356617788210
-12	16.134915640887	0.046356617957266	0.046356617928783	0.046356617835662	0.046356617967803
-11	16.134915645110	0.046356619658364	0.046356619373536	0.046356618442332	0.046356619763732
-10	16.134915687348	0.046356636669347	0.046356633821066	0.046356624509024	0.046356637723029
-9	16.134916109700	0.046356806779235	0.046356778296425	0.046356685175975	0.046356817316206
-8	16.134920333587	0.046358507884187	0.046358223056044	0.046357291848356	0.046358613269082
-7	16.134962572563	0.046375519540821	0.046372671255224	0.046363358859580	0.046376574908491
-6	16.135384972877	0.046545696834452	0.046517213563835	0.046424057719320	0.046556402492512
-5	16.139610031950	0.048253558313519	0.047968685063875	0.047033927584654	0.048375923353681
-4	16.181967982389	0.065957180118245	0.063105333370683	0.053427383868315	0.067766038947423
-3	16.616844425767	0.27530605365471	0.24486689683707	0.13335710379016	0.26868430398964
-2	21.778676674834	3.3796717087139	2.8985584015404	1.1081551960197	3.5608490573157

Michelson system: computed ratio $diam(z_i)/s$ for various choices o section.

$\log_{10} s$	**diag+normal**	**diag+flowdir**	**orthogonal**	**CTO**	**max angle CTO**
			$\lambda_{FS_1} \approx -3.1255162015308699$		
-13	3.2116142444263	3.1255162038258	3.1255162269739	3.1255162039846	3.1255162077022
-12	3.2116142655580	3.1255162244797	3.1255164559613	3.1255162260679	3.1255162249376
-11	3.2116144768748	3.1255164310196	3.1255187458357	3.1255164469009	3.1255164321661
-10	3.2116165900434	3.1255184964189	3.1255416446378	3.1255186552321	3.1255185078836
-9	3.2116377217797	3.1255391504587	3.1257706385001	3.1255407385979	3.1255392650706
-8	3.2118490440011	3.1257456956048	3.1280611613276	3.1257615776825	3.1257468381526
-7	3.2139627521408	3.1278116218805	3.1510249035516	3.1279705113076	3.1278226902171
-6	3.2351484960973	3.1485184344710	3.3866083502860	3.1501142089676	3.1485933519525
-5	3.4519428524253	3.3604105774357	20.073719793613	3.3770717466865	3.3575308921411
-4	6.1918831644386	6.0382345907118	48.192050322691	6.2929366399838	5.5793036988113
			$\lambda_{FS_2} \approx -0.31994714969328985$		
-13	0.58904797067828	0.31994714992826	0.31994715230733	0.31994714994447	0.31994715050329
-12	0.58904797368178	0.31994715204298	0.31994717583371	0.31994715220505	0.31994715370896
-11	0.58904800371685	0.31994717319017	0.31994741109760	0.31994717481089	0.31994718904359
-10	0.58904830406741	0.31994738466211	0.31994976374250	0.31994740086931	0.31994754319639
-9	0.58905130758169	0.31994949938640	0.31997329079895	0.31994966145907	0.31995108473517
-8	0.58908134341692	0.31997064711589	0.32020862211667	0.31997226791244	0.31998650120326
-7	0.58938177100241	0.32018217307783	0.32256802070380	0.32019838802569	0.32034077392439
-6	0.59239298018272	0.32230230643042	0.34678075911647	0.32246515567799	0.32389432028560
-5	0.62320851091933	0.34399811694236	6.3011376294349	0.34569814280737	0.36052693835901
-4	1.0129131821593	0.61828066926388	14.959060502378	0.64422908261053	0.84497758478011

Falkner-Skan system: computed ratio $diam(z_i)/s$ for various choices o section.

$\log_{10} s$	**diag+normal**	**diag+flowdir**	**orthogonal**	**CTO**	**max angle CTO**
			$\lambda_{R_{h1}} \approx -2.9753618617896986$		
-13	3.1269112296727	2.9753618617987	2.9753618617991	2.9753618617973	2.9753618617953
-12	3.1269112296767	2.9753618618023	2.9753618618019	2.9753618618042	2.9753618618071
-11	3.1269112299772	2.9753618620903	2.9753618620850	2.9753618620637	2.9753618620608
-10	3.1269112327992	2.9753618647951	2.9753618647424	2.9753618645294	2.9753618645006
-9	3.1269112610131	2.9753618918436	2.9753618913169	2.9753618891866	2.9753618888981
-8	3.1269115432163	2.9753621623288	2.9753621570618	2.9753621357587	2.9753621328741
-7	3.1269143652214	2.9753648671818	2.9753648145118	2.9753646014813	2.9753645726348
-6	3.1269425854365	2.9753919158704	2.9753913891678	2.9753892588373	2.9753889703684
-5	3.1272248040375	2.9756624185652	2.9756571512700	2.9756358454017	2.9756329603539
-4	3.1300486358109	2.9783690272406	2.9783163272842	2.9781030120969	2.9780741257411
-3	3.1551299308307	3.0024276270455	3.0019370205452	3.0053562752534	3.0006911559382
-2	3.3312447246999	3.1612927762392	3.1579216074761	3.1824418580936	3.2139664668233
-1	5.9748947614907	5.5919232064410	5.5499757238534	5.6828287198362	6.1202554505770
			$\lambda_{R_{h2}} \approx 1.1193329361699592$		
-13	1.2750081236836	1.1193329361772	1.1193329361780	1.1193329361755	1.1193329361756
-12	1.2750081236814	1.1193329361751	1.1193329361749	1.1193329361760	1.1193329361771
-11	1.2750081238407	1.1193329363308	1.1193329363269	1.1193329362917	1.1193329363020
-10	1.2750081253204	1.1193329377781	1.1193329377391	1.1193329373876	1.1193329374899
-9	1.2750081401109	1.1193329522510	1.1193329518609	1.1193329483465	1.1193329493692
-8	1.2750082880882	1.1193330969806	1.1193330930796	1.1193330579353	1.1193330681626
-7	1.2750097678299	1.1193345442775	1.1193345052677	1.1193341538239	1.1193342560970
-6	1.2750245653503	1.1193490173467	1.1193486272464	1.1193451127757	1.1193461355106
-5	1.2751725508514	1.1194937580732	1.1194898568592	1.1194547088893	1.1194649366501
-4	1.2766534361840	1.1209421693982	1.1209031360826	1.1205513299068	1.1206536486960
-3	1.2904231607513	1.1343998401486	1.1340331850761	1.1369430453153	1.1333056956081
-2	1.3698919414549	1.2014426463159	1.1996643242106	1.2152067812507	1.2538154992768
-1	2.9997723907998	2.6705679634689	2.6493196773279	2.5783568485068	3.0211083048782

Rössler system (hyperbolic orbit): computed ratio $diam(z_i)/s$ for various choices o section.

$\log_{10} s$	**diag+normal**	**diag+flowdir**	**orthogonal**	**CTO**	**max angle CTO**
			$\lambda_{R_{pd1}} \approx 1.2039286263296685$		
-13	1.3744854888330	1.2039286263323	1.2039286263322	1.2039286263322	1.2039286263316
-12	1.3744854888497	1.2039286263484	1.2039286263480	1.2039286263474	1.2039286263453
-11	1.3744854890396	1.2039286265322	1.2039286265287	1.2039286265228	1.2039286265000
-10	1.3744854909230	1.2039286283550	1.2039286283202	1.2039286282607	1.2039286280334
-9	1.3744855097563	1.2039286465824	1.2039286462353	1.2039286456404	1.2039286433675
-8	1.3744856980897	1.2039288288571	1.2039288253857	1.2039288194367	1.2039287967079
-7	1.3744875814245	1.2039306516056	1.2039306168909	1.2039305574015	1.2039303301127
-6	1.3745064136666	1.2039488792229	1.2039485320727	1.2039479371714	1.2039456642602
-5	1.3746947443819	1.2041311685904	1.2041276967474	1.2041217470111	1.2040990156302
-4	1.3764909757232	1.2058681783762	1.2058331605620	1.2057750473739	1.2056335194386
-3	1.3923461796892	1.2211702025606	1.2208174545532	1.2202191822506	1.2201485891365
-2	1.5245370044750	1.3482398006188	1.3444079718526	1.3378502373169	1.3452772133533
-1	3.3189752159528	3.0595461571246	2.9899841238967	2.8651359578762	2.5268610111985
			$\lambda_{R_{pd2}} \approx -1$		
-13	1.1486171722293	1.0000000000020	1.0000000000020	1.0000000000019	1.0000000000015
-12	1.1486171722484	1.0000000000190	1.0000000000186	1.0000000000181	1.0000000000159
-11	1.1486171724575	1.0000000002049	1.0000000002015	1.0000000001971	1.0000000001738
-10	1.1486171745303	1.0000000020486	1.0000000020146	1.0000000019712	1.0000000017376
-9	1.1486171952589	1.0000000204855	1.0000000201457	1.0000000197118	1.0000000173756
-8	1.1486174025442	1.0000002048552	1.0000002014574	1.0000001971178	1.0000001737558
-7	1.1486194753993	1.0000020485537	1.0000020145757	1.0000019711789	1.0000017375590
-6	1.1486402030893	1.0000204856702	1.0000201458870	1.0000197119133	1.0000173756918
-5	1.1488474901282	1.0002048700441	1.0002014718711	1.0001971314897	1.0001737670547
-4	1.1508203440202	1.0019607036312	1.0019263987058	1.0018836409982	1.0017386848276
-3	1.1681410842692	1.0174040964627	1.0170577983291	1.0165945694829	1.0165327959562
-2	1.3108850142655	1.1452467277951	1.1414685208050	1.1359979815714	1.1438008444065
-1	3.2096656391944	2.8618925282316	2.7920556484803	2.6772944104801	2.2936430842608

Rössler system (period-doubling orbit): computed ratio $diam(z_i)/s$ for various choices o section.

Applications

Kuramoto-Sivashinsky equations

$$u_t = 2uu_x - u_{xx} - \nu u_{xxxx}$$

2π-periodic, odd

$$u(t,x) = -2\sum_{k=1}^{\infty} a_k(t)\sin(kx)$$

Infinite dimensional ODE

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right)$$

Kuramoto-Sivashinsky equations

$$u_t = 2uu_x - u_{xx} - \nu u_{xxxx}$$

2π-periodic, odd

$$u(t,x) = -2\sum_{k=1}^{\infty} a_k(t)\sin(kx)$$

Infinite dimensional ODE

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right)$$

Kuramoto-Sivashinsky equations

$$u_t = 2uu_x - u_{xx} - \nu u_{xxxx}$$

2π-periodic, odd

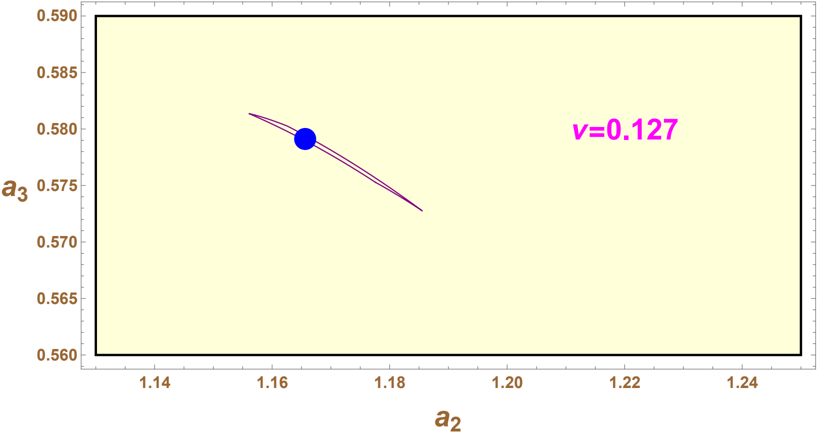
$$u(t,x) = -2\sum_{k=1}^{\infty} a_k(t)\sin(kx)$$

Infinite dimensional ODE

$$a_k' = k^2(1-\nu k^2)a_k - k\left(\sum_{n=1}^{k-1} a_n a_{k-n} - 2\sum_{n=1}^{\infty} a_n a_{n+k}\right)$$

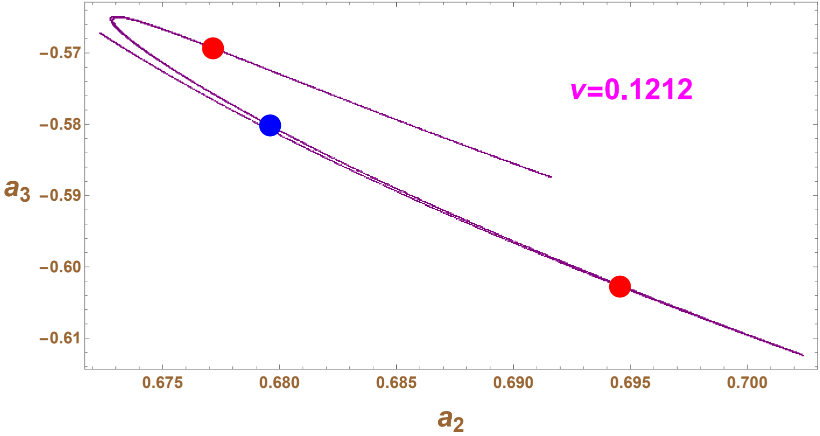
Proof of stable periodic orbit

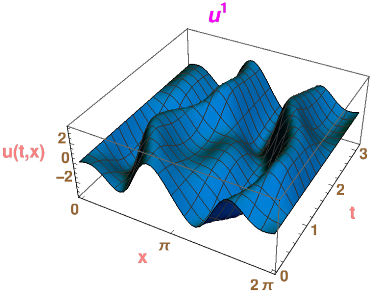
Result reproduced from Zgliczyński FoCM'2004



u_i	$P_i(u)$	λ_i
$[-1,1]\cdot 10^{-5}$	$[-5.45, 5.45]\cdot 10^{-6}$	0.5258
$[-1,1]\cdot 10^{-5}$	$[-9.85, 9.81]\cdot 10^{-7}$	0.0903
$[-1,1]\cdot 10^{-5}$	$[-5.86, 4.67]\cdot 10^{-9}$	$3.5\cdot 10^{-8}$
$[-1,1]\cdot 10^{-5}$	$[-6.61, 4.32]\cdot 10^{-9}$	$1.65\cdot 10^{-8}$
$[-1,1]\cdot 10^{-5}$	$[-8.02, 5.65]\cdot 10^{-9}$	$-3.77\cdot 10^{-9}$
$[-1,1]\cdot 10^{-5}$	$[-6.62, 8.19]\cdot 10^{-9}$	$-4.01\cdot 10^{-11}$
$[-1,1]\cdot 10^{-5}$	$[-7.30, 9.62]\cdot 10^{-9}$	$-8.94\cdot 10^{-10}$
$[-1,1]\cdot 10^{-5}$	$[-2.15, 1.53]\cdot 10^{-9}$	$-6.69\cdot 10^{-11}$
$\dots$	$\dots$	
$k > 23$	$k > 23$	
$10^{-5}(1.5)^{-k}$	$5.01\cdot 10^{-8}(1.5)^{-k}$	

Unstable periodic orbit for $\nu = 0.1212$





u1
u(t,x)
2
0
–2
0
x
π
2 π
0
1
2
3
t
u2
u(t,x)
2
0
–2
0
x
π
2 π
0
2
4
6
t

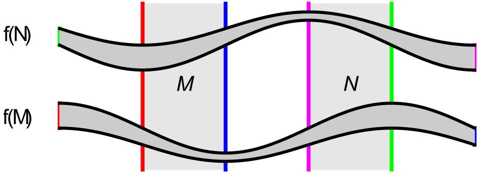
Data from the proof of blue fixed point

u_i	$P_i(u)$	λ_i
$3.8[-1,1]\cdot 10^{-6}$	$[-8.09, 8.09]\cdot 10^{-6}$	-1.7704
$1.9[-1,1]\cdot 10^{-7}$	$[-4.33, 4.59]\cdot 10^{-8}$	-0.06511
$1.9[-1,1]\cdot 10^{-7}$	$[-2.35, 1.68]\cdot 10^{-8}$	$-2.92\cdot 10^{-16}$
$1.9[-1,1]\cdot 10^{-7}$	$[-0.718, 1.13]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-0.982, 1.40]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-1.33, 2.04]\cdot 10^{-8}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.86, 3.64]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-2.75, 1.67]\cdot 10^{-9}$	≈ 0
$1.9[-1,1]\cdot 10^{-7}$	$[-3.64, 4.31]\cdot 10^{-10}$	≈ 0
$\dots$	$\dots$	
$k > 23$	$k > 23$	
$1.9\cdot 10^{-7}(1.5)^{-k}$	$2.64\cdot 10^{-9}(1.5)^{-k}$	

Theorem (PZ & DW, JDE '2020)

Fix $\nu = 0.1212$. There is an invariant set $\mathcal{H}$ such that

- *the system on $\mathcal{H}$ is chaotic (symbolic dynamics)*
- *$\mathcal{H}$ possesses countable infinity of periodic orbits of arbitrary large principal periods*



Theorem (PZ & DW 2021)

- *there are a hyperbolic periodic orbit $\mathbf{a}^1, \mathbf{a}^2 \subset \mathcal{H}$*
- *there is countable infinity of connecting orbits between $\mathbf{a}^1$ and $\mathbf{a}^2$ in $\mathcal{H}$*

Thank you for your attention!