A generating function for squares of Legendre polynomials

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Legendre polynomial:

$$P_n(y) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\big((y^2-1)^n\big) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{y-1}{2}\right)^k$$

Recurrence:

$$(n+1)P_{n+1}(y) - (2n+1)yP_n(y) + nP_{n-1}(y) = 0$$

Generating functions:

$$\sum_{n=0}^{\infty} P_n(y)z^n = \frac{1}{\sqrt{1-2yz+z^2}}.$$

$$\sum_{n=0}^{\infty} P_n(y)^2 z^n = \frac{1}{\sqrt{1-2y^2z+z^2}} \cdot {}_2F_1\left(\begin{matrix}\frac14,\ \frac34\\ 1\end{matrix}\,\middle|\, \frac{4(1-y^2)^2z^2}{(1-2y^2z+z^2)^2}\right)$$

Goal: explicit expressions for the twist

$$F(y,z) = \sum_{n=0}^{\infty}\binom{2n}{n}P_n(y)^2 z^n$$

Initial hope: Write $F(y,z)$ as a product involving ${}_2F_1$ functions.

Reasons:

Experimental discoveries and conjectures by Zhi-Wei Sun, and the Barnes–Bailey identity:

$$\mathrm{F}_4\left(\begin{matrix} a,\, b \\ c_1,\, c_2 \end{matrix} \,\middle|\, x(1-y), y(1-x)\right) = {}_2F_1\left(\begin{matrix} a,\, b \\ c_1 \end{matrix} \,\middle|\, x\right) {}_2F_1\left(\begin{matrix} a,\, b \\ c_2 \end{matrix} \,\middle|\, y\right)$$

when $c_1 + c_2 = a + b + 1$.

Equations for $F(y,z)$

$$z\left(y^2-2z-\frac{1}{2}\right)\frac{\partial^2 F}{\partial y\,\partial z}+\frac{y}{2}(1-y^2)\frac{\partial^2 F}{\partial y^2}-(y^2+z)\frac{\partial F}{\partial y}+yz\frac{\partial F}{\partial z}=0,$$

$$F+\left(2z^2-\frac{z}{2}\right)\frac{\partial^2 F}{\partial z^2}+\left(5z-\frac{1}{2}\right)\frac{\partial F}{\partial z}+\left(y-\frac{1}{y}\right)\left(\left(z+\frac{1}{4}\right)\frac{\partial^2 F}{\partial y\,\partial z}+\frac{1}{2}\frac{\partial F}{\partial y}\right)=0$$

By eliminating derivatives with respect to y we get an order-4 linear differential equation $\mathrm{DE}_z \in \mathbb{Q}(y,z)[\mathrm{d}/\mathrm{d}z]$.

We can also construct DE_z by using `Maple`'s `gfun` to compute a recurrence for $\binom{2n}{n}P_n(y)^2$ and convert it to a differential equation.

Definition 1. Let $V(L)$ denote the solution space of L in a universal extension. The *symmetric product* $L_1 \circledS L_2$ is the lowest order differential equation with $y_1 \cdot y_2 \in V(L)$ for any $y_1 \in V(L_1), y_2 \in V(L_2)$.

If the initial hope is true then

(1) we expect DE_z = a symmetric product of order-2 equations
(2) we expect those to be ${}_2F_1$ solvable

Should be a 5 minute project because I have implementations for both!

$F(y,z) = \sum_{n=0}^{\infty} \binom{2n}{n} P_n(y)^2 z^n$ satisfies order-4 $\mathrm{DE}_z \in \mathbb{Q}(y,z)[\mathrm{d}/\mathrm{d}z]$.

My implementation finds:

$$\mathrm{DE}_z = L_+^z \circledS L_-^z$$

where

$$\begin{aligned} L_\pm^z := {} & z(1-4z)(1+4z)^2((1+4z)^2 - 16y^2 z)\,\frac{\mathrm{d}^2}{\mathrm{d}z^2} \\ & + (8z(48z^2+8z-3)y^2 + (1-32z^2)(1+4z)^2)(1+4z)\,\frac{\mathrm{d}}{\mathrm{d}z} \\ & + (64z^3+80z^2+4z-1)y^2 - 3z(1+4z)^3 \\ & \pm (1-8z)y\sqrt{2(1-4z)((1+4z)^2-16zy^2)}. \end{aligned}$$

Problem: The square root has degree 3 as a polynomial in z.

The software for finding ${}_2F_1$ solutions developed by my students and I is for rational function coefficients (genus 0).

$\rightsquigarrow$ hours of adjusting software to make it work for genus 1

$\rightsquigarrow$ and still no solution???

Why spend a lot of time adjusting/testing implementation to genus 1?

Why expect a ${}_2F_1$ solution?

OEIS: many sequences $a_n \in \mathbb{Z}$. Generating function: $\sum a_n x^n$.

If differential of order 2, then closed form solutions are *very* common!

True so far:
If a_n in OEIS, $\sum a_n x^n$ positive radius of convergence, and differential equation order 2, then ${}_2F_1$ solvable.

Many order 2 arithmetic examples in OEIS. So far, all ${}_2F_1$-solvable.

Bouw Möller: counter example with $a_n \in \mathbb{Z}[\frac{1}{2}, \sqrt{17}]$.
Differential equation: order 2, arithmetic, no ${}_2F_1$ solution.

Are **"naturally occurring"** order 2 arithmetic differential equations always ${}_2F_1$-solvable?

Idea: Instead of eliminating derivatives with respect to y, we can also eliminate derivatives with respect to z.

$\rightsquigarrow$ a fourth order equation $\mathrm{DE}_y \in \mathbb{Q}(y, z)[\mathrm{d}/\mathrm{d}y]$.

Implementation finds $\mathrm{DE}_y = L_+^y \circledS L_-^y$ where

$$\begin{aligned} L_\pm^y &= 2(1-y^2)(2y^2-4z-1)^2((1+4z)^2-16zy^2)\,\frac{\mathrm{d}^2}{\mathrm{d}y^2} \\ &\quad + 4y(2y^2-4z-1)(32zy^4-(1+4z)((1+28z)y^2-4z(3+4z)))\,\frac{\mathrm{d}}{\mathrm{d}y} \\ &\quad + 16(2y^2-14z-3)zy^4+(1+4z)(1+32z+80z^2)y^2-(1+4z)^2(1+4z+8z^2) \\ &\quad \pm (2(1+8z)y^2-4(1+4z)(1-z))y\sqrt{2(1-4z)((1+4z)^2-16zy^2)}. \end{aligned}$$

This time we can rationalize the square root with a substitution. We chose:

$$y = \left(x + (t^2-1)^2\right)\sqrt{z/x}, \quad z = \frac{1}{4(1-2t^2)}.$$

After some simplifications we get the following solution for DE_y

$$\sqrt{x} \cdot F_+^x \cdot F_-^x$$

where F_+^x satisfies:

$$L_+^x = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{1}{x} - \frac{1}{x+(t+1)(t-1)^3} + \frac{2(x+t^4+2t^2-1)}{x^2+(2t^4+4t^2-2)x+(t^2-1)^4}\right)\frac{\mathrm{d}}{\mathrm{d}x}$$
$$+\frac{4x^2+(8t^4-16t^3-4t^2+20t-9)x+(4t^4-8t^3+12t^2+4t-5)(t+1)(t-1)^3}{16x(x^2+(2t^4+4t^2-2)x+(t^2-1)^4)(x+(t+1)(t-1)^3)}.$$

and where F_-^x is obtained from F_+^x by replacing t with $-t$.

Under $x \to 0$, the radius of convergence of $F(y, z)$ goes to 0.
For a functional equality, use another singularity:

$$x = -(1+\sqrt{2t^2-1})^4 = -(1+1/(2\sqrt{-z}))^4.$$

The next task is to solve L_+^x and find an explicit expression for F_+^x.
$\rightsquigarrow$ expressions for solutions $\sqrt{x} \cdot F_+^x \cdot F_-^x$ and $F(y,z)$ of DE_y.

Substituting rational numbers for t produces equations in $\mathbb{Q}(x)[\mathrm{d}/\mathrm{d}x]$.
Then all our ${}_2F_1$ programs are applicable.
A ${}_2F_1$ type solution should now be found (if it exists).

No solution found???

$F(y,z) \in \mathbb{Z}[y]((z))$ so we expect equations obtained from it to be arithmetic:

Let a_n be the n'th coefficient in the power series of F_+^x.

We expect: $\exists\, c \in \mathbb{Q}(t) - \{0\}$ with $a_n c^n \in \mathbb{Z}[t]$ for all n.

Indeed: take $c = 2^6(t^2-1)^4$.

To our surprise $a_n c^n$ was even divisible by $\binom{2n}{n}$.
Then $\sum a_n/\binom{2n}{n} x^n$ satisfies an arithmetic equation as well!
Turns out to be algebraic.

First a Möbius transformation to reduce expression sizes:
Let $L^m = L_+^x\big|_{x \mapsto m}$ where $m := \frac{1}{64x} - (t+1)(t-1)^3$.

The apparent singularity moved to $x = \infty$ after $x \mapsto m$.
Solving L^m is equivalent to solving L_+^x.

L^m has solution $x^{1/2} \sum u_n x^n$ with $u_n \in \mathbb{Z}[n]$ of degree $4n$.

$$
\begin{aligned}
&(n+1)^2 u_{n+1} \\
&\quad - 2^2(16(t^4 - 6t^3 - 4t^2 + 6t - 1)(n^2 + n) + 4t^4 - 24t^3 - 12t^2 + 20t - 3)u_n \\
&\quad - 2^{11} t(t-1)^3(t+1)(8(t^2 + 2t - 1)n^2 - 2t^2 - 6t + 3)u_{n-1} \\
&\quad + 2^{18} t^2 (t-1)^6 (t+1)^2 (2n+1)(2n-3)u_{n-2} = 0 \qquad \text{and} \quad u(0) = 1.
\end{aligned}
$$

$u(n)$ is divisible by $\binom{2n}{n}$. Take $\tilde{u}_n = u_n/\binom{2n}{n}$ and $Z(x) := \sum \tilde{u}_n x^n$. Then

$$\sum_{n=0}^{\infty} u_n x^n = \sum_{n=0}^{\infty} \binom{2n}{n} \tilde{u}_n \, x^n = \frac{1}{\sqrt{1-4x}} \star Z(x).$$

where the Hadamard product $\star$ is $\sum a_n x^n \ \star \ \sum b_n x^n = \sum a_n b_n x^n$ like in yesterday's talk.

By solving a differential equation for $Z(x)$ we get

$$Z(x) = \frac{R_-^{1/8} + R_+^{1/8}}{2\sqrt{S}}$$

where

$$\begin{aligned} R_\pm &= A \pm \sqrt{A^2 - 1} \\ S &= (1 - 16(t+1)(t-1)^3 x)(1 + 2^6(t^3 + t^2 - t)x - 2^{10}(t^3 + t^2)(t-1)^3 x^2) \\ A &= 1 + 2^7(2t-1)^2 x - 2^{11}(t-1)^3(2t-1)(2t^2+5t-1)x^2 \\ &\quad + 2^{17} t(t-1)^6(2t^2+2t-1)x^3 - 2^{21}(t^3+t^2)(t-1)^9 x^4. \end{aligned}$$

In general

$$\frac{1}{\sqrt{1-4x}} \star Z(x) = \frac{1}{\pi}\int_0^{4x} \frac{Z(\xi)}{\sqrt{\xi(4x-\xi)}}\,\mathrm{d}\xi.$$

The relation between x and $Z(x)$ has genus 0, so it allows rational parametrization, however, if we allow a square root the parametrization becomes much shorter:

$$x = \frac{(t^2-v^2)}{16t(tv^4+(t+1)(t-1)^2(t^2-2v^2-t))},$$

$$Z = \frac{(t-1+v)(tv^4+(t+1)(t-1)^2(t^2-2v^2-t))}{v(v^4-2t^2v^2+(t^2-1)^2)}\sqrt{\frac{v-t}{2t(t^2+tv-1)}}.$$

Substituting gives:

$$\frac{\sqrt{2t}}{\pi}\int \frac{(t-1+v)\,\mathrm{d}v}{\sqrt{(t^2+tv-1)(t+v)(v^2-t^2+16tx(tv^4+(t+1)(t-1)^2(t^2-2v^2-t)))}}.$$

A linear transformation moves two branchpoints to $v = 0$ and $v = 1$. Combining all the various transformations gives:

Theorem 1.

$$F(y, z) = w\, I_+(4z, w^2) I_-(4z, w^2)$$

where

$$w = \sqrt{(1 + 4z)^2 - 16y^2 z} \; + \; 4y\sqrt{-z}$$

and $I_\pm(u, x) =$

$$\frac{1}{\pi} \int_0^1 \frac{1 - uv \pm v\sqrt{2u^2 - 2u}}{\sqrt{v\,(1 - v)\,\big((1 - v)(1 - u^2 v)(1 + uv)^2 + x\, v(1 - uv)^2\big)}}\, \mathrm{d}v\,.$$

$I_+(u, x)$ is a period of a hyperelliptic curve and satisfies a *second* order linear differential equation.

That is highly unusual! (the expected order is $2g = 4$)

Our order-2 equation is arithmetic and yet the monodromy group is dense in $\mathrm{SL}_2(\mathbb{R})$.

$\Longrightarrow$ It cannot be solved in terms of ${}_2F_1$ hypergeometric functions. Novel for an equation that "occurred naturally".

Our hyperelliptic curve $C^{u,x}$ is defined by the equation

$$Y^2 = H(x, u, v),$$

where

$$H = v\,(1-v)\,\big((1-v)(1-u^2v)(1+uv)^2 + xv(1-uv)^2\big).$$

Monodromy calculation

Fix a u to reduce our 2-parameter family of curves to 1 parameter x.

Singular members: $x = 0$, $x = \infty$, $x = u^2 - 6u + 1 \pm 4(1-u)\sqrt{-u}$.

Goal: compute the monodromy representation.

The monodromy acts on $H_1(C^{u,x}, \mathbb{Z})$, so it can be given by 4×4 integer matrices, with product 1.
In general, it is very likely for the monodromy to act irreducibly on $\mathbb{C}^4$.
$\rightsquigarrow$ irreducible differential equation of $2g = 4$.

However, our equation L_+^x has order 2!

To understand this, we computed the monodromy matrices $\in \mathrm{Sp}_4(\mathbb{Z})$.
They have two invariant subspaces defined over $\mathbb{Q}(\sqrt{2})$.
(corresponding to our order-2 equations L_+^x and L_-^x)

Implies real multiplication:

$$\mathbb{Z}[\sqrt{2}] \subset \mathrm{End}(\mathrm{Jac}(C^{u,x}))$$

(this can be verified with a formula from Humbert).

To simplify the notation and computation, first fix the value $u = 1/2$. The corresponding singularities are

$$x_1 = -7/4 + \sqrt{-2}, \quad x_2 = 0, \quad x_3 = -7/4 - \sqrt{-2}, \quad x_4 = \infty$$

We choose a base point, say

$$x_{\mathrm{BP}} = 1.$$

so that all the roots of $H(1, 1/2, v)$ are real:

$$v_1 = -3 - \sqrt{17}, \;\; v_2 = \frac{3 - \sqrt{17}}{2}, \;\; v_3 = 0, \;\; v_4 = 1, \;\; v_5 = -3 + \sqrt{17}, \;\; v_6 = \frac{3 + \sqrt{17}}{2},$$

and satisfy $v_1 < v_2 < \cdots < v_6$.

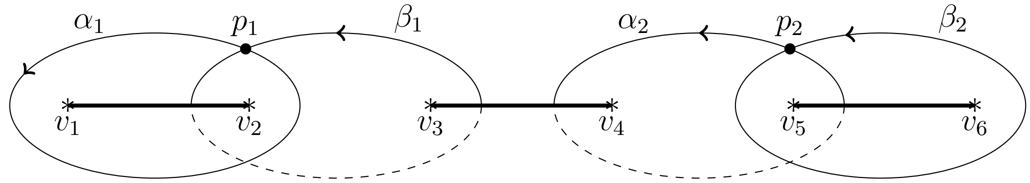

FIGURE 1. Our choice of homology basis on the curve $C^{1/2,1}$

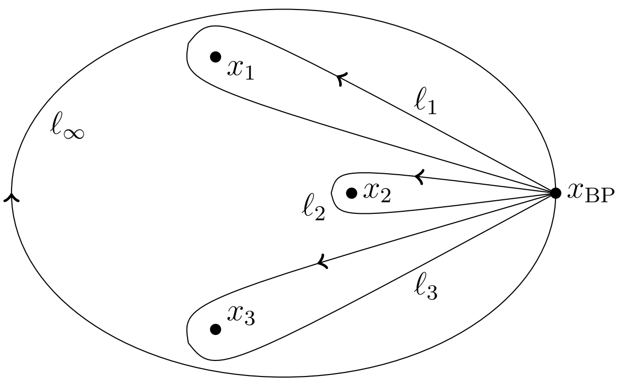

FIGURE 2. Loops around the singularities for $u = 1/2$.

Theorem 2. *With the above choice of basis for* $H_1(C^{1/2,1},\mathbb{Z})$ *and generating paths* $\ell_i, i=1,2,3,4$ *for* $\pi_1(\mathbb{C}\setminus\{x_1,x_2,x_3\},x_{BP})$ *we have:*

$$M_1 := \begin{pmatrix} 2 & -1 & 1 & -1 \\ 2 & 0 & 1 & -2 \\ 2 & -1 & 2 & -2 \\ 1 & -1 & 1 & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 := \begin{pmatrix} 0 & -1 & 1 & 1 \\ 2 & 2 & -1 & -2 \\ 2 & 1 & 0 & -2 \\ -1 & -1 & 1 & 2 \end{pmatrix}, \quad M_\infty := \begin{pmatrix} -1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Theorem 3. *The monodromy groups of the operators* L_x^+ *is a* dense *subgroup of* $\mathrm{SL}_2(\mathbb{R})$*. In particular: no* ${}_2F_1$ *solutions.*

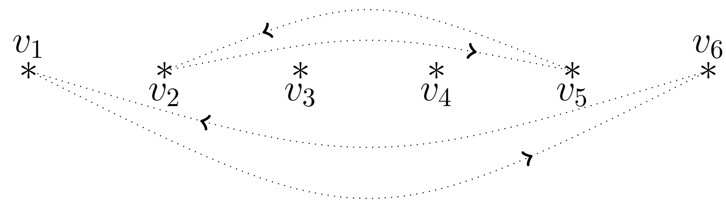

FIGURE 3. Looping along ℓ_1.

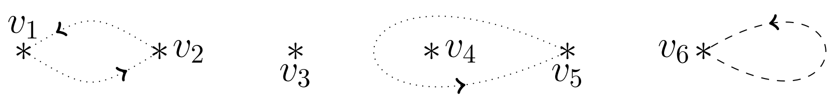

FIGURE 4. Looping along ℓ_2

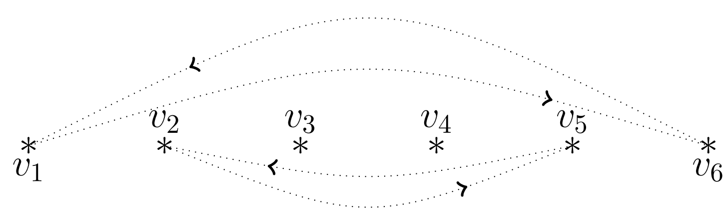

FIGURE 5. Looping along ℓ_3.

Braid action Above we picked $u = 1/2$. Now consider what happens when we change u:

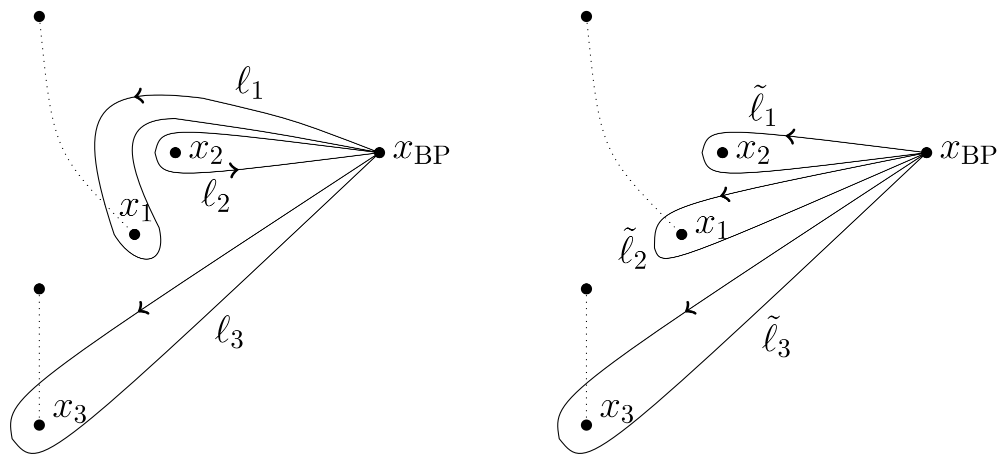

FIGURE 6. Positioning after a u-move

Cubes of Legendre polynomials

Theorem 4. *Let* $p = 1 - xy^3 + x^2(3y^2 - 2)/4$ *and*

$$H = \frac{1}{\sqrt{p}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{3}{4} \\ 1 \end{matrix} \,\middle|\, \frac{(y^2-1)^3(x^2 - \frac{1}{4}x^4)}{p^2} \right).$$

Then

$$\sum_{n=0}^{\infty} P_n(y)^3 z^n = \frac{1}{\sqrt{1+z^2}} \left(H \star \frac{1}{\sqrt{1-4x}} \right) \Bigg|_{x=\frac{z}{1+z^2}}$$

where the Hadamard product $\star$ *is with respect to variable* x.