ValidSDP: Coq Proofs of Polynomial Positivity

using Numerical Solvers and Floating-Point Computations

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Certified and Symbolic-Numeric Computation, Lyon

Motivation

- Polynomial inequalities in the real field are decidable (Tarski)
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⇒ Use incomplete numerical methods

  - off-the-shelf optimization solvers
  - a posteriori validation with exact rational arithmetic: state of the art (simple but costly)
  - a posteriori validation with floating-point arithmetic (more efficient but non trivial)

  ⇒ We'd like formal proofs

Sum of Squares (SOS) Polynomials

Definition (SOS Polynomial)

A polynomial p is SOS if there are polynomials $q_1, \ldots, q_m$ s.t.

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- If p SOS then $p \geq 0$
- p SOS iff there exist $z := \left[1, x_0, x_1, x_0 x_1, \ldots, x_n^d\right]$ and $Q \succeq 0$ (i.e., for all $x, x^T Q x \geq 0$) s.t.

$$p = z^T Q\, z.$$

$\Rightarrow$ SOS can be encoded as semi-definite programming (SDP).

SOS: Example

Example

Is $p(x, y) := 2x^4 + 2x^3y - x^2y^2 + 5y^4$ SOS ?

$$p(x,y) = \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$

that is

$p(x, y) = q_{11}x^4 + 2q_{13}x^3y + 2q_{23}xy^3 + (2q_{12} + q_{33})x^2y^2 + q_{22}y^4$

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For instance

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L \qquad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

hence $p(x,y) = \frac{1}{2}\left(2x^2 - 3y^2 + xy\right)^2 + \frac{1}{2}\left(y^2 + 3xy\right)^2$.

SOS: Using approximate SDP solvers

Result Q from SDP solver will only satisfy equality constraints up to some error δ

$$p = z^T Q z + z^T E z, \qquad \forall i\,j, |E_{i,j}| \leq \delta.$$

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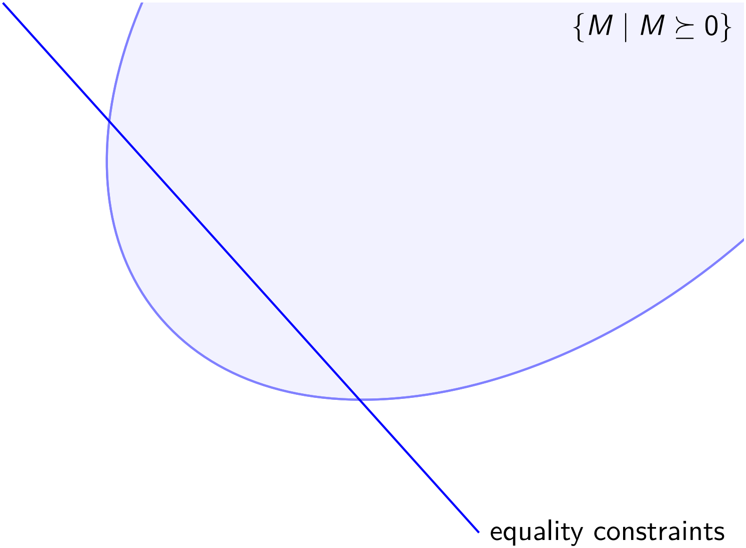
If $Q + E \succeq 0$ then $p = z^T (Q + E)\, z$ is SOS.

- Hence the validation method: given $p \simeq z^T Q\, z$
  1. Bound difference δ between coefficients of p and $z^T Q\, z$.
  2. If $Q - s\,\delta\, I \succeq 0$ ($s :=$ size of Q), then p is proved SOS.
- 1 can be done with interval arithmetic
  and 2 with a Cholesky decomposition ($\Theta(s^3)$ flops).

$\Rightarrow$ Efficient validation method using just floats.

Intuitively



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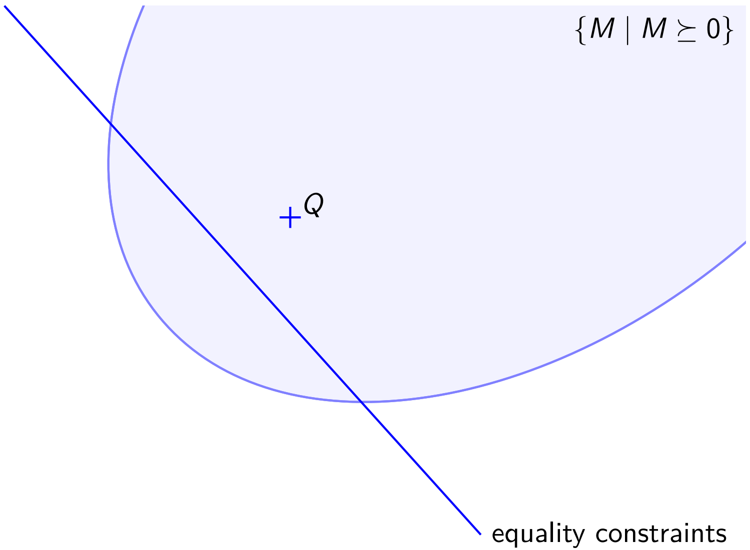

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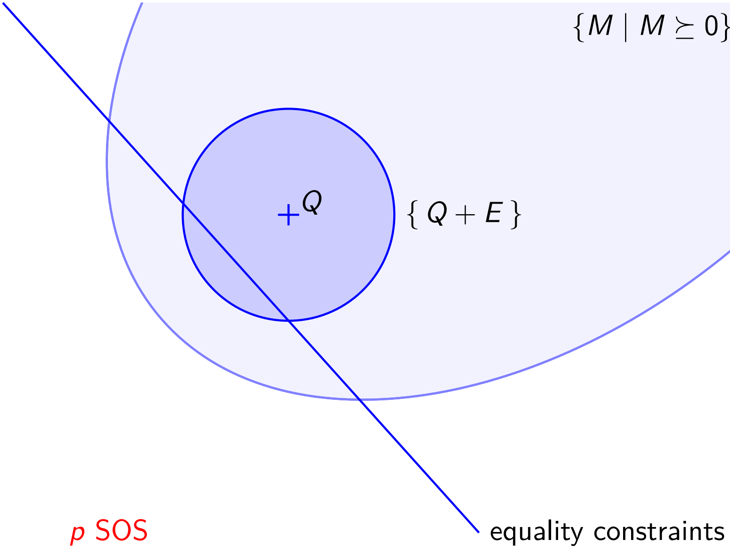

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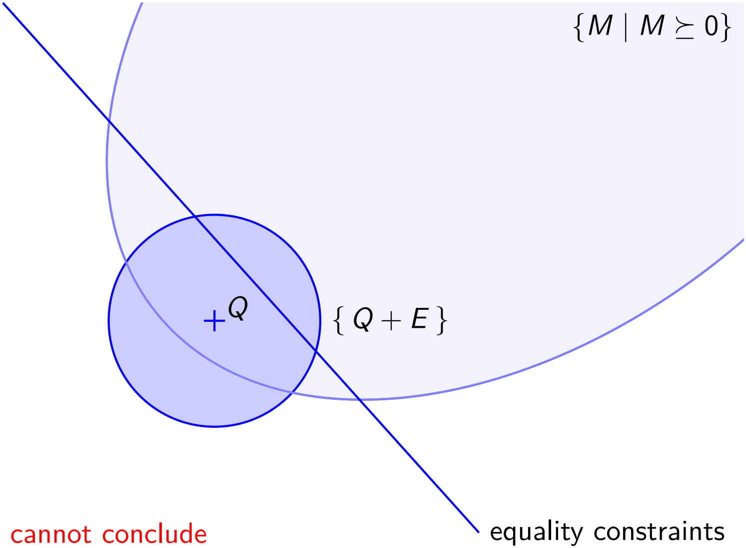

cannot conclude

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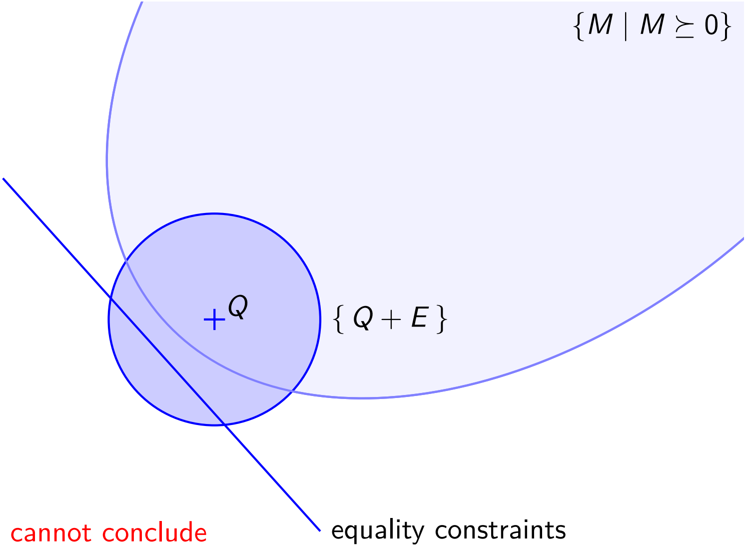

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Cholesky Decomposition

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  (since $x^T \left(R^T R\right) x = (Rx)^T (Rx) = \|Rx\|_2^2 \geq 0$).
- The Cholesky decomposition computes such a matrix R:

R **:=** 0;
for j **from** 1 **to** n **do**
  for i **from** 1 **to** $j-1$ **do**

$$R_{i,j} := \left(A_{i,j} - \sum_{k=1}^{i-1} R_{k,i} R_{k,j}\right) / R_{i,i};$$

  od

$$R_{j,j} := \sqrt{M_{j,j} - \sum_{k=1}^{j-1} {R_{k,j}}^2};$$

od

- If it succeeds (no $\sqrt{}$ of negative or div. by 0) then $A \succeq 0$.

Cholesky Decomposition (end)

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But error is bounded and for some (tiny) $c \in \mathbb{R}$:
if Cholesky succeeds on A then $A + c\,I \succeq 0$.

Hence:

Theorem

If floating-point Cholesky succeeds on $A - c\,I$ then $A \succeq 0$
holds for any $c \geq \frac{(s+1)\varepsilon}{1-(s+1)\varepsilon}\mathrm{tr}(A) + 4s\left(2(s+1) + \max_i(A_{i,i})\right)\eta$
(ε and η relative and absolute precision of floating-point format).

Proved in Coq (paper proof: 6 pages, Coq: 5.1 kloc)

Sum of Squares (SOS) Polynomials

Numerical Verification

Formalization & Reflexive Tactic

Benchmarks

Conclusion

Outline of the formalization

1. Effective multivariate polynomials
   - CoqEAL [Cano, Cohen, Dénès, Mörtberg, Rouhling, Siles]
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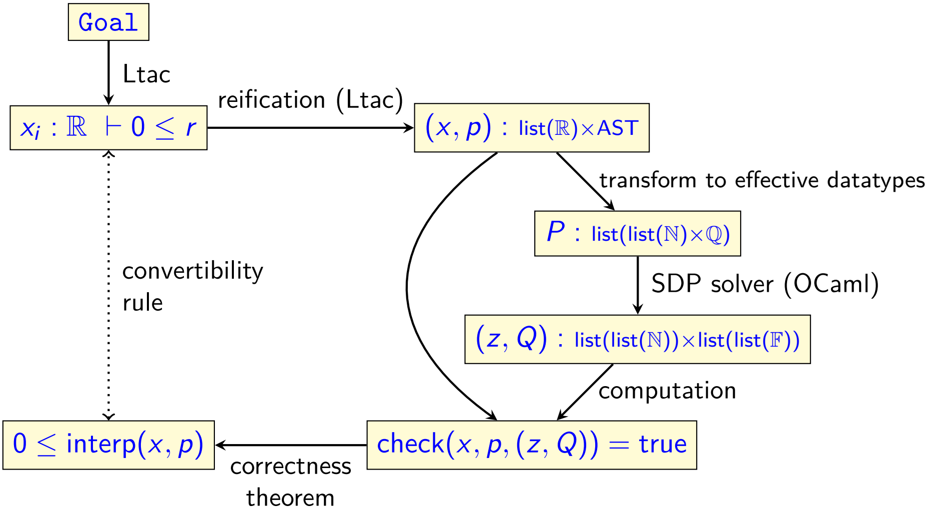
3. Reflexive tactic
   - OCaml code as a wrapper for SDP solvers
   - Some Ltac2 code

The validsdp tactic – the big picture



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Benchmarks (1/2)

Problem	n	d	OSDP (not verified)	MonniauxC11 (not verified)	NLCertify (not verified)	ValidSDP	PVS/Bernstein	NLCertify	HOL Light/Taylor
adaptativeLV	4	4	**0.75**	2.67	1.12	5.16	14.93	**2.61**	12.31
butcher	6	4	1.58	—	**1.05**	9.40	48.44	**8.36**	15.62
caprasse	4	4	**0.41**	1.82	0.88	**5.19**	25.89	2.63	17.68
heart	8	4	**3.18**	268.75	—	**16.67**	131.13	—	26.15
magnetism	7	2	**1.11**	2.04	1.64	**5.18**	245.52	14.50	16.07
reaction	3	2	**0.81**	1.56	0.24	4.33	11.48	**1.96**	12.41
schwefel	3	4	**0.95**	2.45	2.76	**3.70**	14.72	56.13	17.46
fs260	6	4	**1.25**	—	—	**5.99**	—	—	—
fs461	6	4	**0.70**	11.18	0.87	**5.18**	621.06	7.46	22.70
fs491	6	4	**0.54**	21.81	—	**5.38**	—	—	—
fs745	6	4	0.98	11.74	**0.94**	**5.55**	623.17	6.90	22.48
fs752	6	2	**0.35**	1.81	0.90	**3.80**	54.52	7.88	13.34
fs8	6	2	**0.43**	1.53	1.48	**3.93**	52.63	6.62	13.40
fs859	6	8	—	—	—	—	—	—	—
fs860	6	4	**1.21**	10.53	1.11	**6.08**	73.65	7.34	14.28
fs861	6	4	**1.09**	10.48	1.20	**5.15**	69.74	7.87	14.28
fs862	6	4	1.27	79.25	**1.25**	**5.37**	73.54	7.58	14.14
fs863	6	2	**0.94**	1.50	—	**3.85**	—	—	13.85
fs864	6	2	**0.56**	2.05	—	**4.05**	—	—	13.28
fs865	6	2	**0.76**	2.11	—	**3.68**	—	—	13.76
fs867	6	2	**0.21**	2.09	1.74	**4.22**	—	8.04	—

Times in s with 900 s timeout

Benchmarks (2/2)

Problem	n	d	OSDP (not verified)	MonniauxC11 (not verified)	NLCertify (not verified)	ValidSDP	PVS/Bernstein	NLCertify	HOL Light/Taylor
fs868	6	4	**0.94**	—	—	**6.05**	—	—	—
fs884	6	4	—	—	—	—	—	—	—
fs890	6	4	—	**7.78**	—	—	—	—	—
ex4_d4	2	12	—	—	—	—	—	—	—
ex4_d6	2	18	—	—	—	—	—	—	—
ex4_d8	2	24	**16.99**	—	—	**82.89**	—	—	—
ex4_d10	2	30	—	—	—	—	—	—	—
ex5_d4	3	8	**1.67**	—	—	**13.63**	—	—	—
ex5_d6	3	12	**16.10**	—	—	**66.82**	—	—	—
ex5_d8	3	16	**203.06**	—	—	**353.70**	—	—	—
ex5_d10	3	20	—	—	—	—	—	—	—
ex6_d4	4	8	**16.82**	—	—	**44.99**	—	—	—
ex6_d6	4	12	—	—	—	—	—	—	—
ex7_d4	2	12	—	—	—	—	—	—	—
ex7_d6	2	18	**1.50**	—	—	**26.78**	—	—	—
ex7_d8	2	24	**15.38**	—	—	**83.47**	—	—	—
ex7_d10	2	30	—	—	—	—	—	—	—
ex8_d4	2	8	**0.87**	15.72	—	**7.52**	—	—	—
ex8_d6	2	12	—	—	—	—	—	—	—
ex8_d8	2	16	—	—	—	—	—	—	—
ex8_d10	2	20	—	—	—	—	—	—	—

Times in s with 900 s timeout

Conclusion

- Context: formal proof of multivariate polynomial positivity
- A Coq reflexive tactic
  - Input: polynomial goals with real variables and rational coefs
  - Use off-the-shelf SDP solvers as untrusted oracles
  - Numerical approach with formal floating-point arithmetic
  - Algorithm involving matrices (Cholesky)

Thank you!

Questions

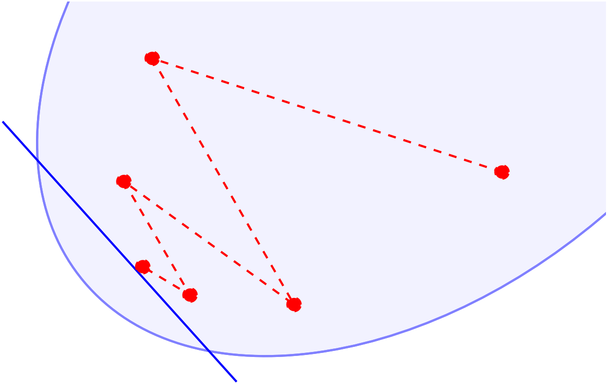
?

https://github.com/validsdp/validsdp

Inaccuracy in Solving SDPs

SDP solvers only yield approximate solutions due to

- inexact termination



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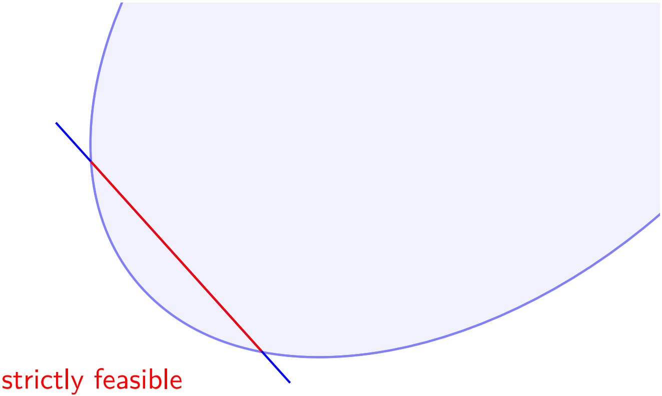

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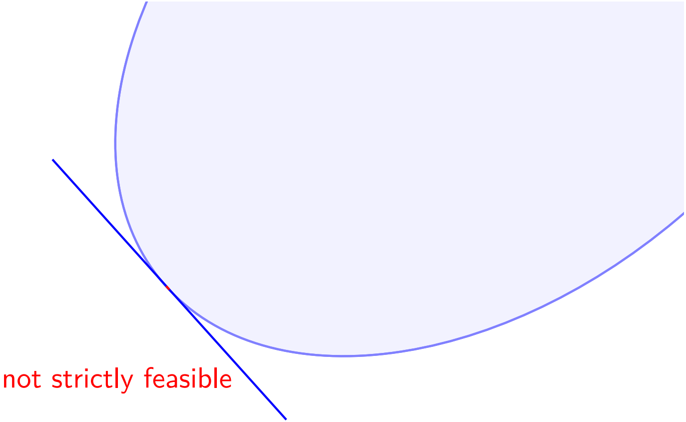

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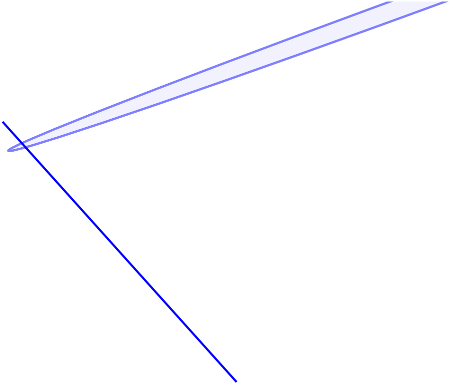

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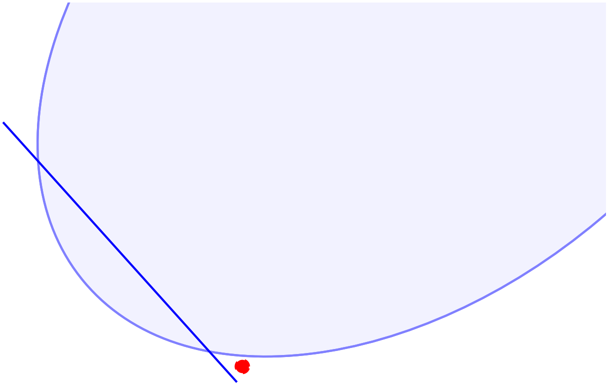

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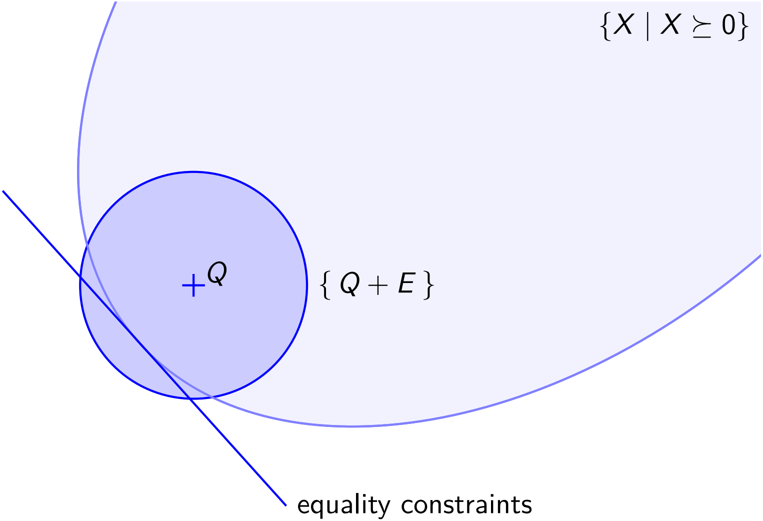
State of the art [Harrison, Peyrl and Parrilo, Monniaux and Corbineau, Kaltofen et al., Magron et al.]

- round to exact rational solution (heuristic)
- proofs in rational arithmetic (expensive).

Incompleteness: Empty Interior SDP Problems

If the interior of the feasibility set of the problem is empty
(i.e., no feasible Q s.t. every Q' in a small neighborhood is feasible)
previous method almost never works.



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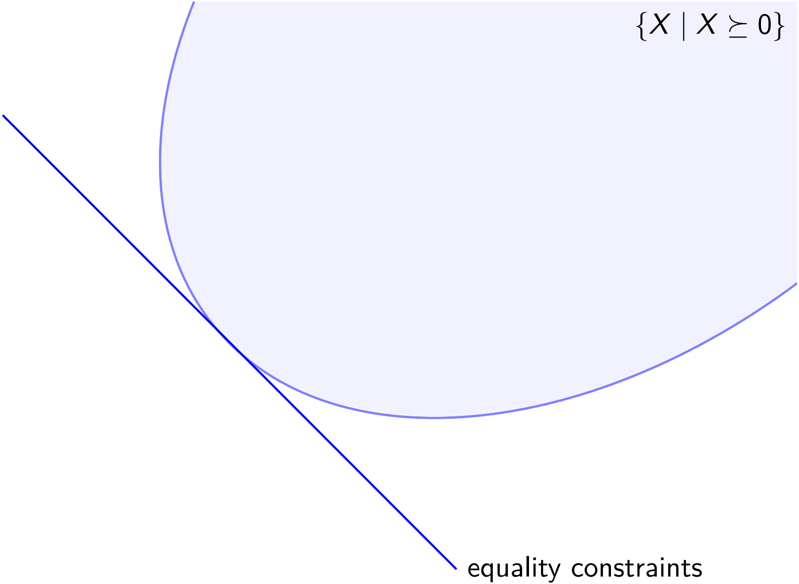

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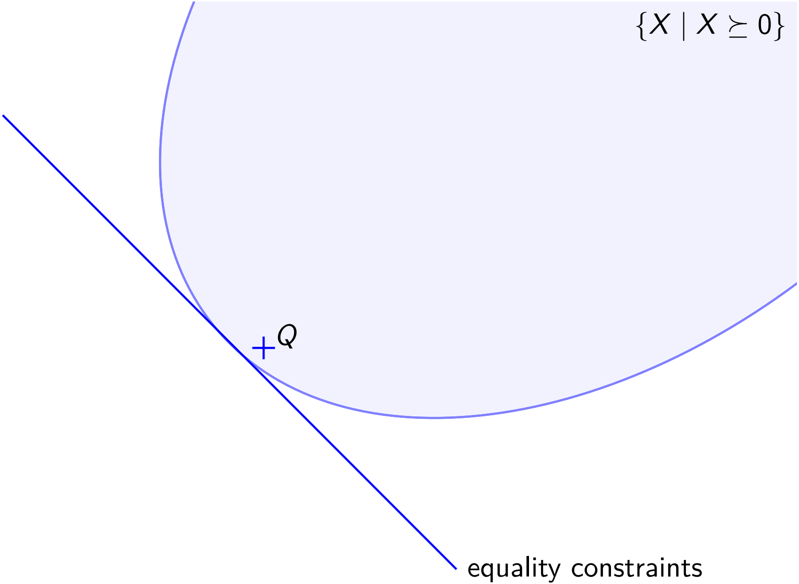

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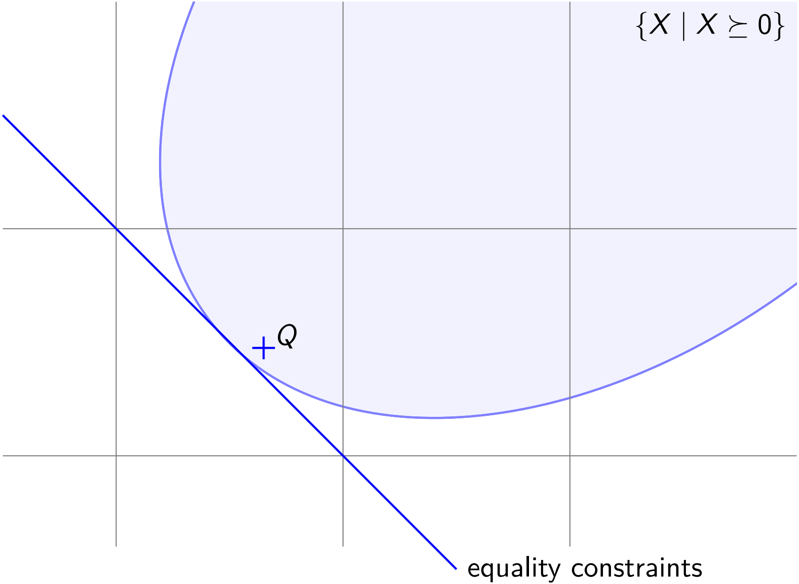

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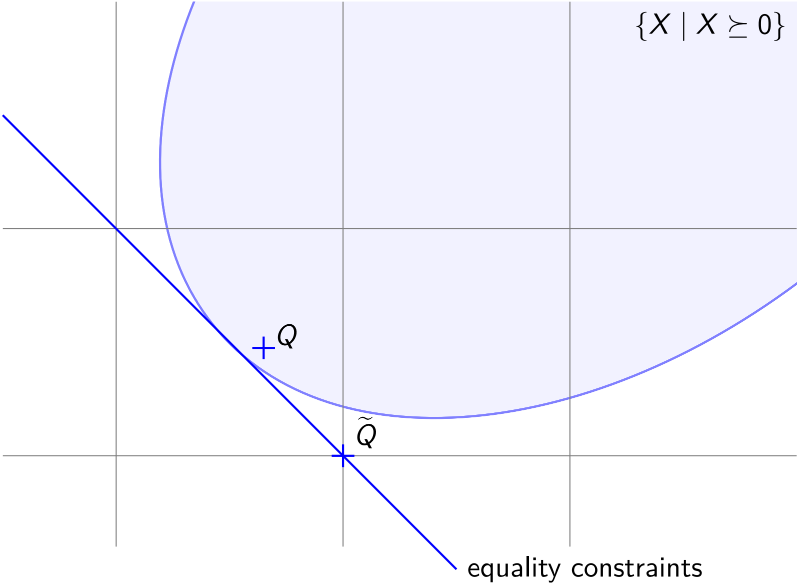

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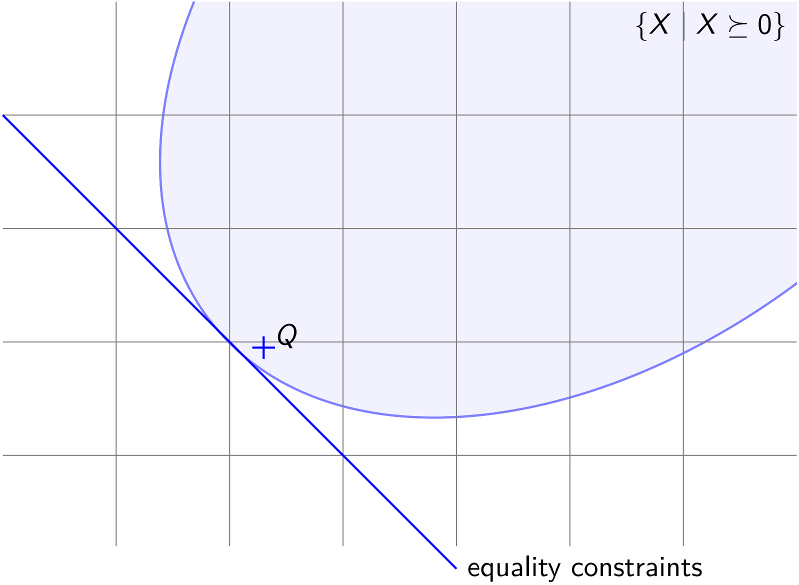

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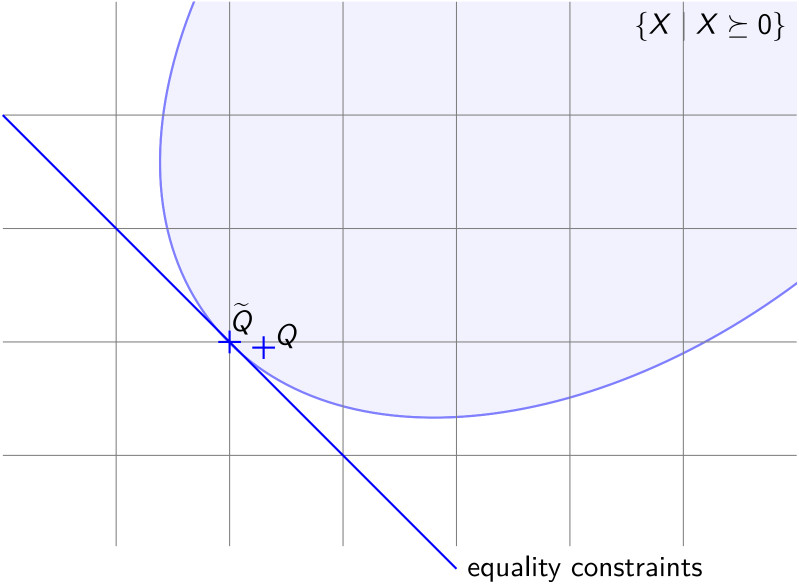

Positivstellensatz

We want to prove that

$$p_1(x_1, \ldots, x_n) \geq 0 \wedge \ldots \wedge p_m(x_1, \ldots, x_n) \geq 0$$

is not satisfiable.

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Sufficient condition: there exist $r_i \in \mathbb{R}[x]$ s.t.

$$-\sum_i r_i \, p_i > 0 \quad \text{and} \quad \forall i, r_i \geq 0$$

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$$-\sum_i r_i \, p_i > 0 \quad \text{and} \quad \forall i, r_i \geq 0$$

- equivalence under hypotheses (Putinar's Positivstellensatz)
- no practical bound on degrees of $r_i \Rightarrow$ will be arbitrarily fixed