Computer-assisted proofs for elliptic problems on bounded and unbounded domains

csnc2023, Lyon, 25.5.2023

Michael Plum

Karlsruhe Institute of Technology, Germany

Consider boundary value problem

$$\begin{aligned} -\Delta u + F(x, u, \nabla u) &= 0 \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

or

$$\begin{aligned} \Delta\Delta u + F(x, u, \nabla u, \dots) &= 0 \quad \text{on} \quad \Omega \\ u = \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

$\Omega \subset \mathbb{R}^n$ domain with some regularity, F given nonlinearity with some smoothness

AIM: Derive conditions for existence of a solution in some "close" and explicit neighborhood of some approximate solution

"Conditions": either of general type, to be verified analytically, or more special, to be verified automatically on a computer

General concept:

Transformation into *fixed-point equation*

$$u = Tu$$

and computation of appropriate set U such that

$$\boxed{TU \subset U}$$

and moreover, T has certain properties (e.g. contractivity or compactness)

Application of some Fixed-Point Theorem (Banach, Schauder, ...)

$\leadsto$ *Existence* of a solution $u^* \in U$

The set U provides *enclosure*

Abstract formulation

Let $(X, \langle \cdot, \cdot \rangle_X), (Y, \langle \cdot, \cdot \rangle_Y)$ Hilbert spaces

Let $\mathcal{F} : X \to Y$ continuously (Fréchet) differentiable mapping

problem : $\boxed{u \in X,\ \mathcal{F}(u) = 0}$

Aim now (first): Existence and bounds for this abstract problem

Let $\omega \in X$ *approximate* solution,

$$L := \mathcal{F}'(\omega) : \; X \to Y \;\; (linear, bounded)$$

Suppose that constants δ and K, and a nondecreasing function $g : [0, \infty) \to [0, \infty)$ have been computed such that

a) $\|\mathcal{F}(\omega)\|_Y \leq \delta$,

b) $\|u\|_X \leq K \|Lu\|_Y$ for all $u \in X$,

c1) $\|\mathcal{F}'(\omega + u) - \mathcal{F}'(\omega)\|_{\mathcal{B}(X,Y)} \leq g(\|u\|_X)$ for all $u \in X$,

c2) $g(t) \to 0$ as $t \to 0+$

Need in addition: $L : X \to Y$ onto

$Y = X'$ *dual* space, $\Phi : X \to X'$ canonical isometric isomorphism

i.e. $(\Phi u)[v] = \langle u, v\rangle_X \quad$ for $\; u, v \in X$

Assume that $\Phi^{-1}L : X \to X$ is *symmetric* (i.e. $(Lu)[v] = (Lv)[u]$ for all $u, v \in X$)

Then $\Phi^{-1}L$ selfadjoint, one-to-one $\Rightarrow$ range $(\Phi^{-1}L)$ dense
$\Rightarrow$ range (L) *dense*

Moreover, range (L) is *closed* by b).

Transformation of $\mathcal{F}(u) = 0$ into fixed-point problem (Newton):

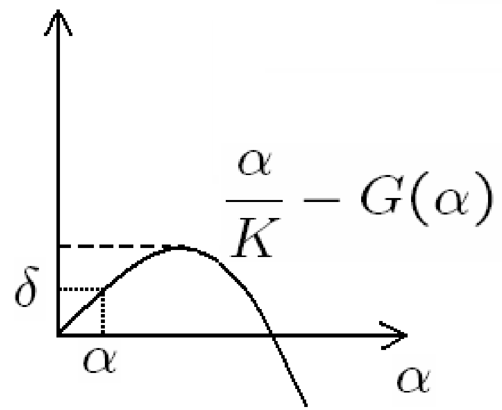
$$\mathcal{F}(u) = 0 \quad \Leftrightarrow \quad \mathcal{F}'(\omega)[u-\omega] = -\mathcal{F}(\omega) - \Big[\mathcal{F}(u) - \mathcal{F}(\omega) - \mathcal{F}'(\omega)[u-\omega]\Big]$$

$$\Leftrightarrow \underbrace{\mathcal{F}'(\omega)}_{=L}[v] = -\mathcal{F}(\omega) - \Big[\mathcal{F}(\omega+v) - \mathcal{F}(\omega) - \mathcal{F}'(\omega)[v]\Big], \; v = u - \omega$$

$$\Leftrightarrow v = -L^{-1}\Big\{\mathcal{F}(\omega) + \Big[\mathcal{F}(\omega+v) - \mathcal{F}(\omega) - \mathcal{F}'(\omega)[v]\Big]\Big\} =: Tv$$

Let $V := \{v \in X : \|v\|_X \leq \alpha\}$, $\alpha > 0$ to be chosen

Then $T(V) \subset V$ if $\boxed{\delta \leq \frac{\alpha}{K} - G(\alpha)}$, $G(t) := \int_0^t g(s)ds$



Need either i) T *compact* ($\rightsquigarrow$ Schauder's Fixed-Point Theorem)

or ii) T *contractive* ($\rightsquigarrow$ Banach's Fixed-Point Theorem)

ad ii) additional contraction condition

$$\boxed{Kg(\alpha) < 1}$$

Theorem: *For some* $\alpha \geq 0$*, let* $\boxed{\delta \leq \frac{\alpha}{K} - G(\alpha)}$ *, and let i) or ii) hold.*

Then, there exists a solution $u \in X$ *of* $\mathcal{F}(u) = 0$ *satisfying*

$$\|u - \omega\|_X \leq \alpha$$

$$\boxed{-\Delta u + F(x,u) = 0 \text{ on } \Omega,\ u = 0 \text{ on } \partial\Omega}$$

weak solutions: $\Omega \subset \mathbb{R}^n$ *Lipschitz*

$$X = \overset{\circ}{H}{}^1(\Omega),\ \langle u, v\rangle_X := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma \langle u, v\rangle_{L^2},\ Y = H^{-1}(\Omega) = X'$$

Fréchet differentiability requires *growth conditions* on F, allowing however exponential growth if $n \leq 2$.

a) $\| -\Delta\omega + F(\cdot,\omega)\|_{H^{-1}} \leq \| -div(\nabla\omega - \rho)\|_{H^{-1}} + \|div\rho - F(\cdot,\omega)\|_{H^{-1}}$

$\leq \|\nabla\omega - \rho\|_{L^2} + \widehat{c}\|div\rho - F(\cdot,\omega)\|_{L^2}$,

$\rho \in H(div;\Omega)$ approximation to $\nabla\omega$, $\|u\|_{L^2} \leq \widehat{c}\|u\|_X$ for $u \in X$

b) $Lu = -\Delta u + cu$, $c(x) = \dfrac{\partial F}{\partial u}\big(x, \omega(x)\big)$

Let $\Phi : X \to Y$, $\Phi u := -\Delta u + \sigma u$ canonical isometric isomorphism

$\Phi^{-1}L$ is *symmetric*, so

$\|u\|_X \leq K\|Lu\|_Y = K\|\Phi^{-1}Lu\|_X$ for $u \in X$

$\iff \boxed{K \geq \Big[\min\Big\{|\lambda| : \lambda \in \text{ spectrum of } \Phi^{-1}L\Big\}\Big]^{-1}}$

$\rightsquigarrow$ need bounds for essential spectrum (analytically) and

eigenvalue bounds:

$$\Phi^{-1}Lu = \lambda u \iff \boxed{-\Delta u + \sigma u = \frac{1}{1-\lambda}\big(\sigma - c(x)\big)u} \;,$$

choose $\sigma > c(x) \quad (x \in \Omega)$

Eigenvalue bounds

weak EVP $\boxed{\langle u, v\rangle_X = \lambda b(u,v) \text{ for all } v \in X}$

where b bounded, Hermitian, positive bilinear form on X

Upper eigenvalue bounds: Rayleigh-Ritz

Let $\tilde{u}_1, \ldots, \tilde{u}_N \in X$ linearly independent (approximate eigenfunctions).
Define $N \times N$-matrices

$$A_0 := (\langle \tilde{u}_i, \tilde{u}_j \rangle_X), \;\; A_1 := (b(\tilde{u}_i, \tilde{u}_j))$$

$\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_N$ eigenvalues of the matrix EVP

$$A_0 x = \Lambda A_1 x.$$

Then, if $\Lambda_N < \underline{\sigma}_{\text{ess}} := \inf\{$ essential spectrum $\}$, there are at least N eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ below $\underline{\sigma}_{\text{ess}}$, and

$$\boxed{\lambda_i \leq \Lambda_i \quad (i = 1, \ldots, N)}$$

Lower eigenvalue bounds: Temple-Lehmann

Let $\tilde{u}_1, \dots, \tilde{u}_N$ and $\Lambda_1, \dots, \Lambda_N < \underline{\sigma}_{\text{ess}}$ as before.

Let $w_1, \dots, w_N \in X$ satisfy

$$\langle w_i, v \rangle_X = b(\tilde{u}_i, v) \text{ for all } v \in X \qquad (*)$$

and let $\rho \in \mathbb{R}$ be such that

$$\Lambda_N < \rho \leq \left\{ \begin{array}{ll} \lambda_{N+1} & \text{, if } \lambda_{N+1} < \underline{\sigma}_{\text{ess}} \text{ exists} \\ \underline{\sigma}_{\text{ess}} & \text{, otherwise} \end{array} \right\} \qquad (**)$$

Define, besides A_0 and A_1,

$$A_2 := (\langle w_i, w_j \rangle_X),$$

and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N < 0$ be the eigenvalues of

$$(A_0 - \rho A_1)x = \mu(A_0 - 2\rho A_1 + \rho^2 A_2)x.$$

Then, $\boxed{\lambda_i \geq \rho\left(1 - \frac{1}{1-\mu_{N+1-i}}\right) \quad (i = 1, \dots, N)}$

$(*)$ often difficult in practice; considerable improvement by *Goerisch*

$(**)$ *homotopy method*

homotopy method for obtaining ρ such that

$$\boxed{\Lambda_N < \rho \le \lambda_{N+1}} \; .$$

Let $(b_t)_{t\in[t_0,t_1]}$ family of bilinear forms on X such that

i) for $s \le t$: $b_s(u,u) \ge b_t(u,u)$ $(u \in X)$

ii) for each t: The eigenvalue problem $\langle u,v\rangle_X = \lambda b_t(u,v)$ for all $v \in X$ (EVP$_t$) has at least $N+1$ eigenvalues $\lambda_1^{(t)} \le \cdots \le \lambda_{N+1}^{(t)}$ below its essential spectrum

iii) for $t = t_0$, the eigenvalues of (EVP_t), or at least bounds to them, are *known*

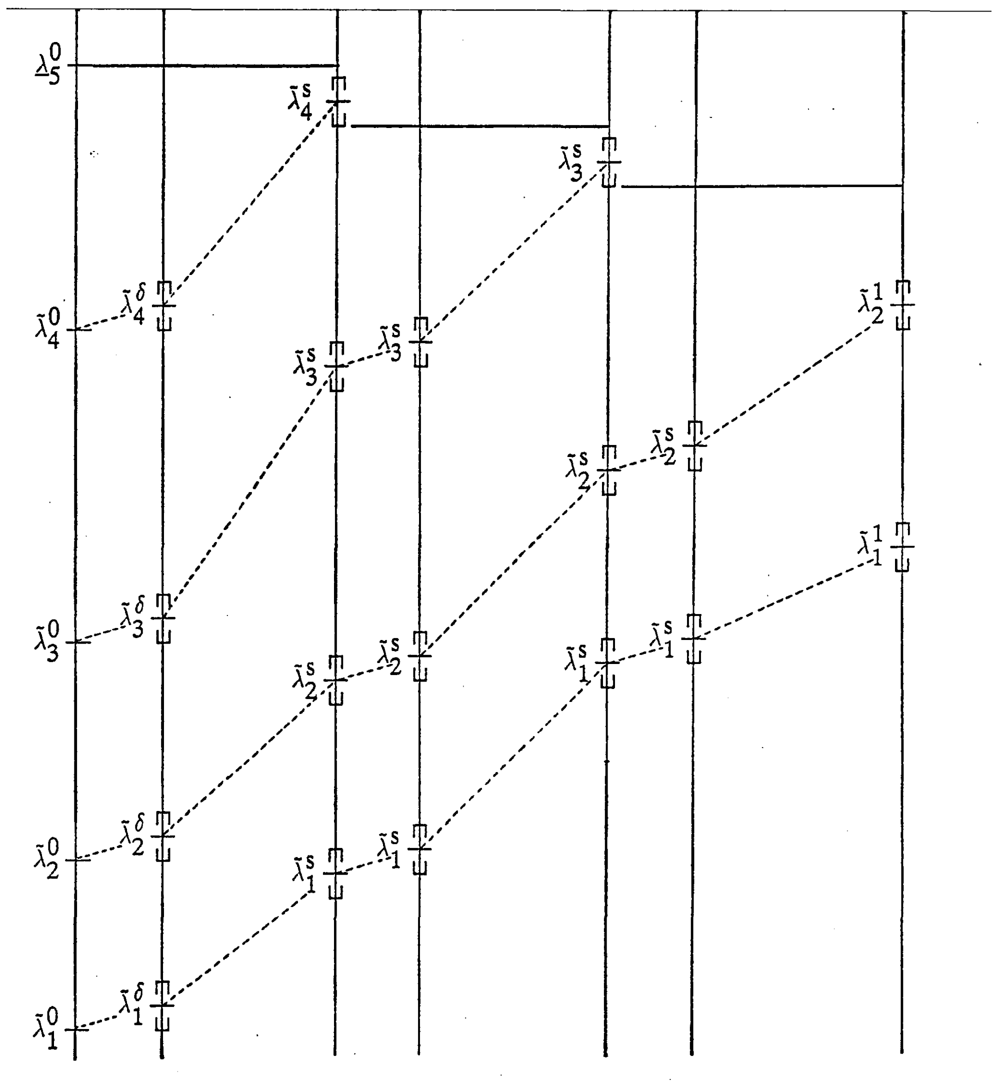
iv) for $t = t_1$, problem (EVP_t) is the *given* one

Consequences: By i), ii), and the min-max-principle $\lambda_k^{(t)}$ *increasing in* t, for each fixed $k \in \{1,\ldots,N+1\}$.

In particular, $\lambda_{N+1}^{(t_0)} \le \lambda_{N+1}^{(t_1)} = \lambda_{N+1}$.

Thus, $\boxed{\rho := \lambda_{N+1}^{(t_0)}}$ can be chosen $\boxed{\text{if } \Lambda_N < \lambda_{N+1}^{(t_0)}.}$

The last condition requires that problem (EVP_{t_0}) (solvable in closed form!) and the given one are sufficiently close.



The homotopy algorithm with $M = 5$ starting eigenvalues and $K = 3$ "dropped" eigenvalues.

Example:

with *J. McKenna* (Connecticut), *B. Breuer*

$$\boxed{\begin{array}{rcl} \Delta u + u^2 & = & \lambda \sin(\pi x)\sin(\pi y) \quad \text{on } \Omega := (0,1)\times(0,1) \\ u & = & 0 \quad \text{on } \partial\Omega \end{array}}$$

Open problem (resp. conjecture) since the 1980's:

$\boxed{\text{At least four solutions for } \lambda \text{ sufficiently large?}}$

For $\lambda = 800$:

McKenna: computation of 4 essentially different approximate solutions by numerical mountain pass method

Breuer: improvement of accuracy of the approximations by Fourier series and spectral multigrid methods

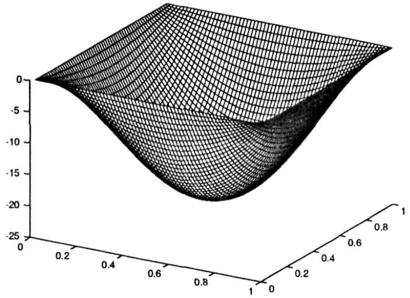
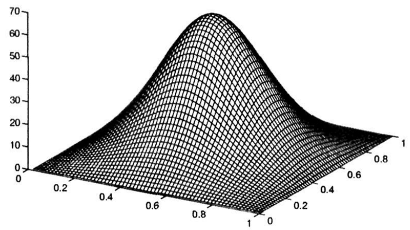
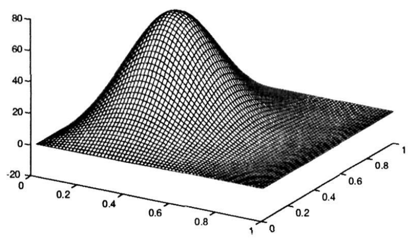
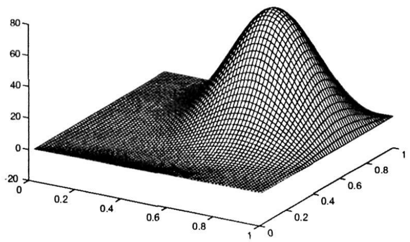
Existence and enclosure method provided indeed

$\boxed{\text{4 essentially different solutions!}}$

Two years after our result: More general analytical proof by Dancer and Yan.

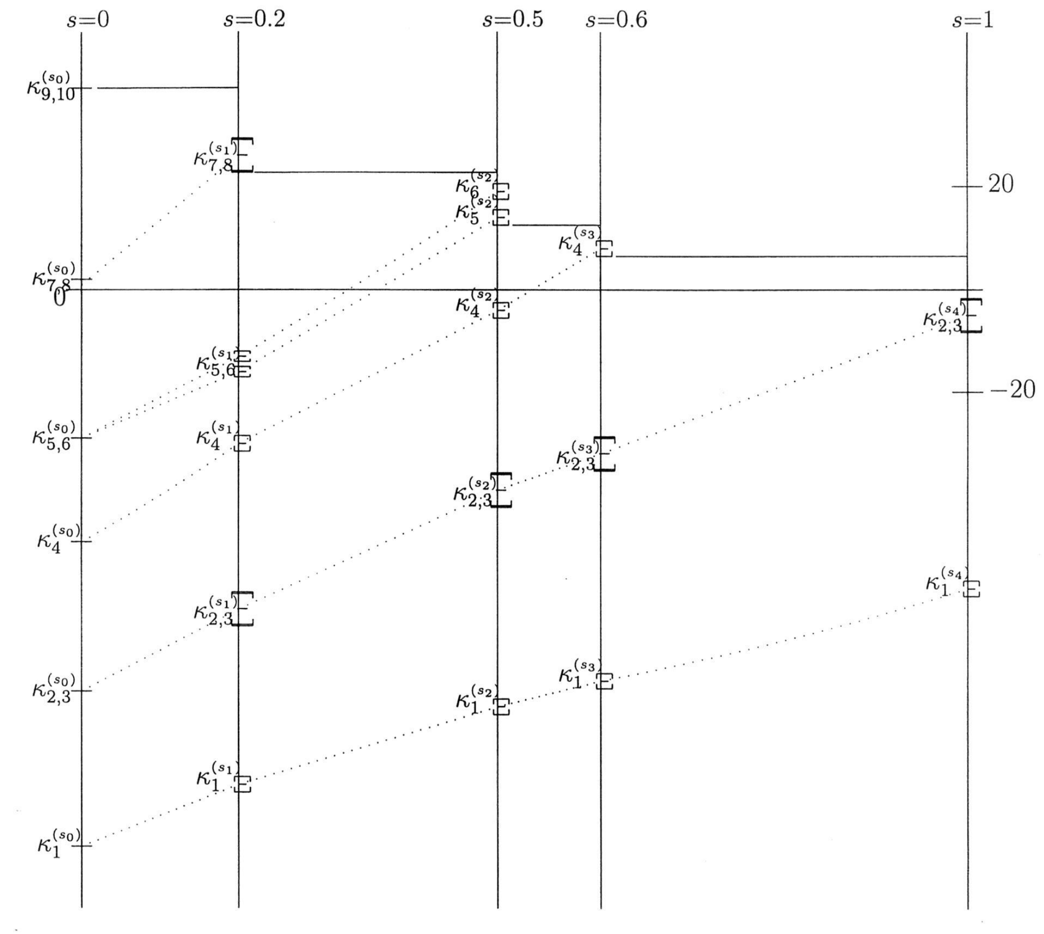
$$\boxed{\begin{aligned} -\Delta u &= u^2 - 800 \sin(\pi x)\sin(\pi y) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}}$$

where $\Omega := (0,1)\times(0,1)$.

Numerical results for the four approximate solutions $\tilde{u}_i$:

	δ	K	α
$\tilde{u}_1$	0.0023	0.2531	$5.8222 \cdot 10^{-4}$
$\tilde{u}_2$	0.0041	4.8275	0.0222
$\tilde{u}_3$	0.0059	2.8847	0.0180
$\tilde{u}_4$	0.0151	3.1436	0.0586

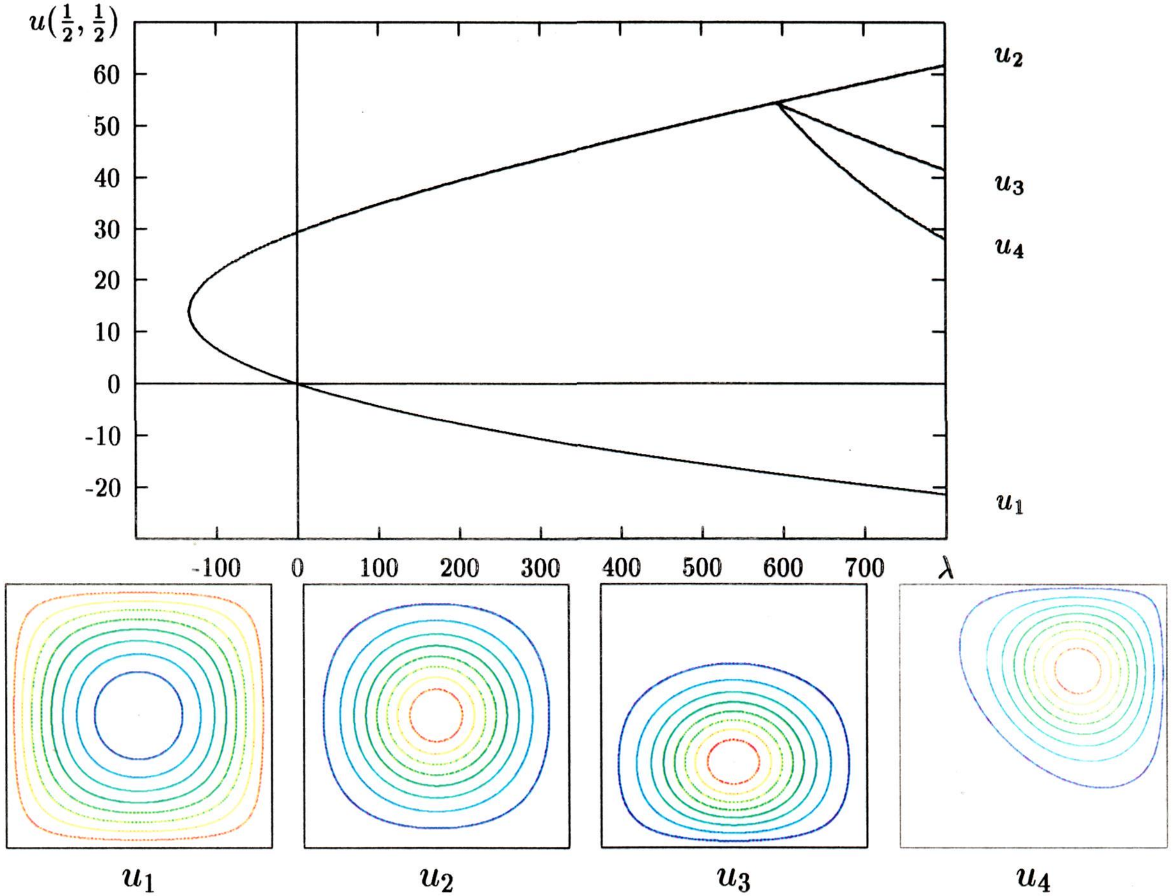


Eigenvalue homotopy for $L_2 = -\Delta - 2\omega_2$

	λ_1	λ_2	λ_3	$\underline{\lambda}_4$	K_0
ω_1	50.8_{2560}^{5453}	74.8_{5752}^{8843}	74.8_{5752}^{8843}	91.34	0.019676
ω_2	-58.9_{7504}^{6498}	-3.8_{909}^{554}	-3.8_{909}^{554}	7.01	0.25938
ω_3	-62.3_{733}^{267}	(-3.6_{801}^{369})	$[6.9301, 7.0082]$	27.65	0.144299
ω_4	$-65.{}_{2667}^{1336}$	$6.{}_{6269}^{7745}$	$6.{}_{6269}^{7745}$	12.38	0.150901

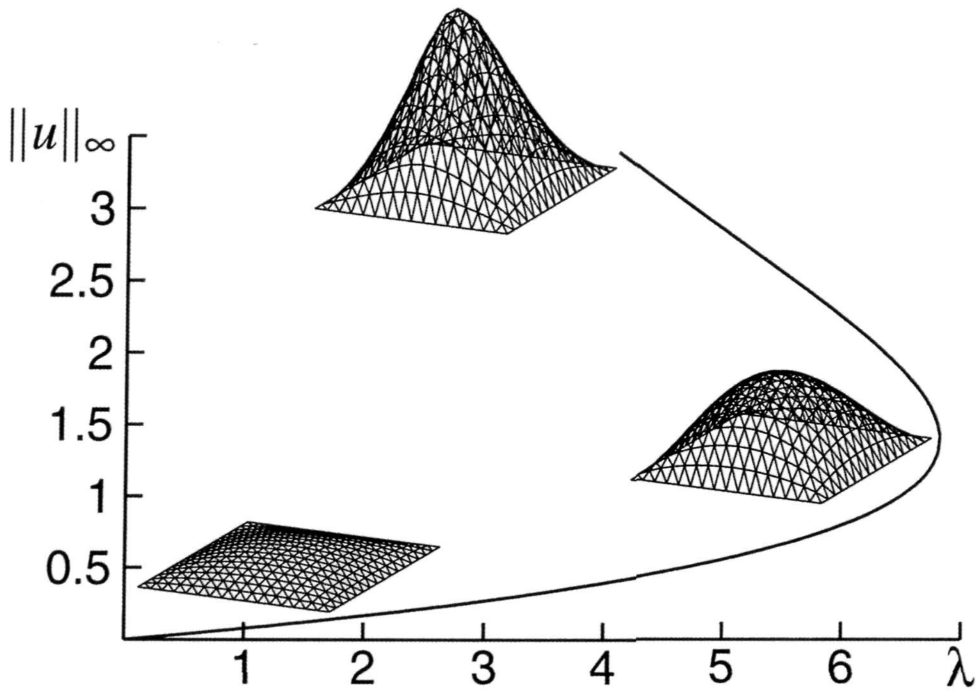
Eigenvalue enclosures

$$-\Delta u = u^2 - \lambda \sin(\pi x)\sin(\pi y)$$

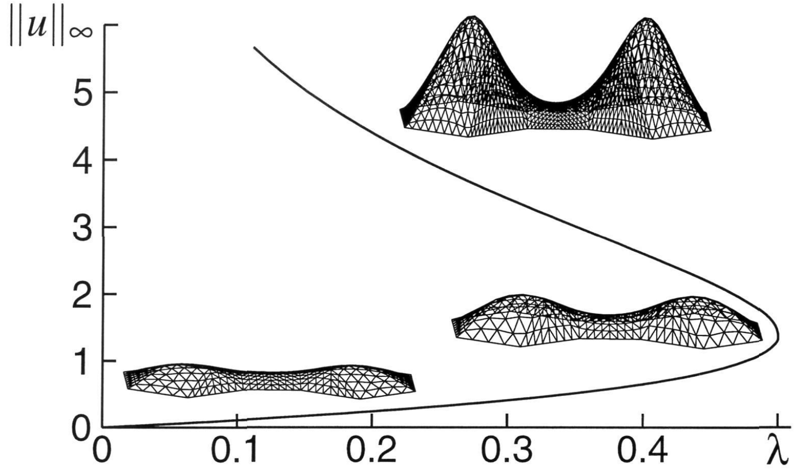


Numerical enclosure for the Gelfand problem

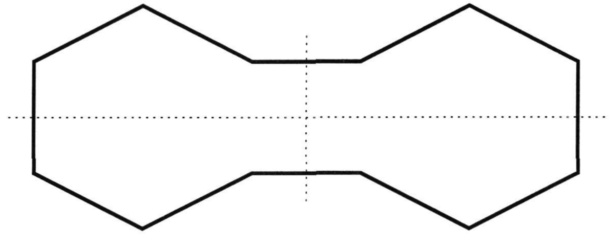
$$
\begin{aligned}
-\Delta u &= \lambda \exp(u) \quad \text{in } \Omega = (0,1)^2 \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

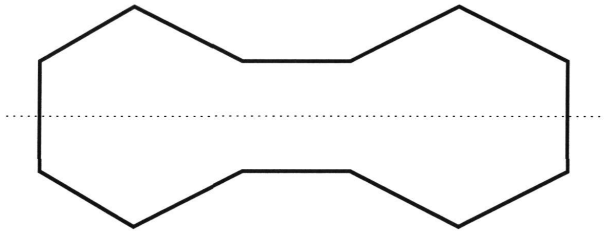


jointly with Ch. Wieners:

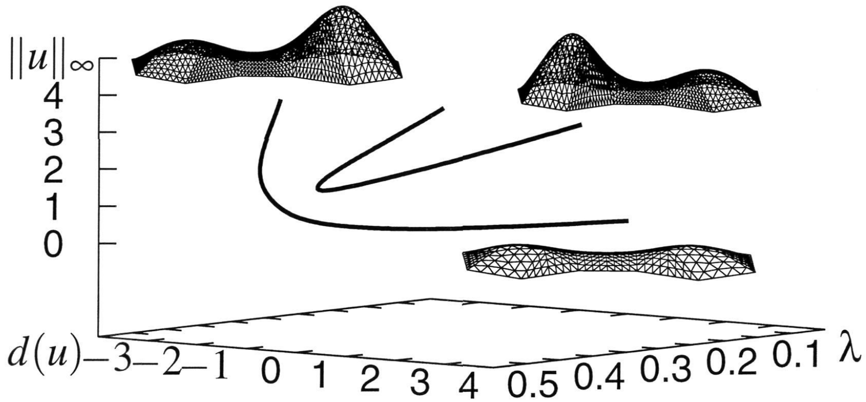


symmetric domain (two symmetry axes)

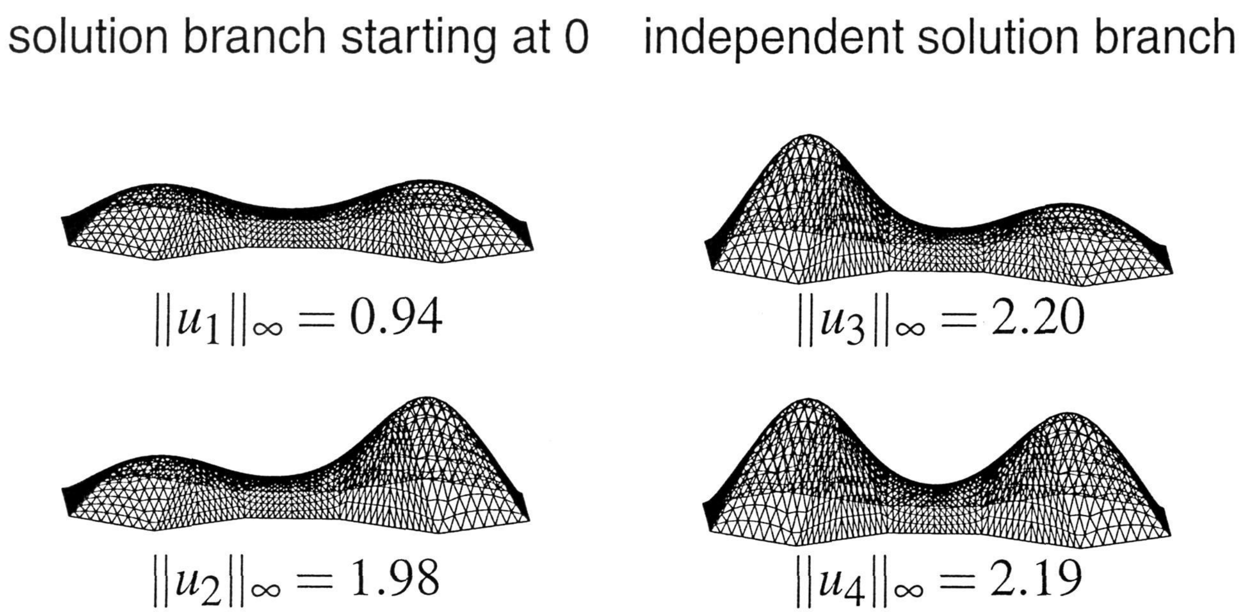




nonsymmetric domain (one symmetry axis)
symmetry breaking solution branch
independent second solution branch



Four solutions for $\lambda = 15/32$ on the unsymmetric domain



Theorem (P.-W. 2001) The existence of a continuous solution u_3 on the independent solution branch can be verified using H^1 approximations ω and $H(\mathrm{div})$ approximations σ.

Dancer obtained similar multiplicity results for a domain Ω consisting of two disjoint balls with a thin connecting corridor.

Emden's Equation on an unbounded L-shaped domain
(jointly with F. Pacella and D. Rütters)

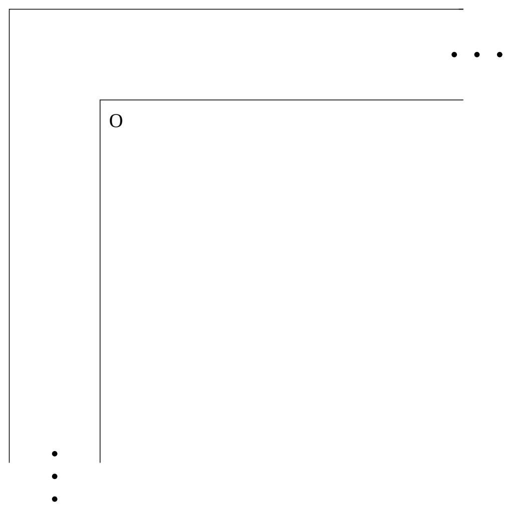
Consider the problem

$$\left\{ \begin{array}{lll} -\Delta u = u^p & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega \\ u > 0 & \text{in} & \Omega \end{array} \right\} \text{or, equivalently,} \left\{ \begin{array}{lll} -\Delta u = |u|^p & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array} \right\}$$

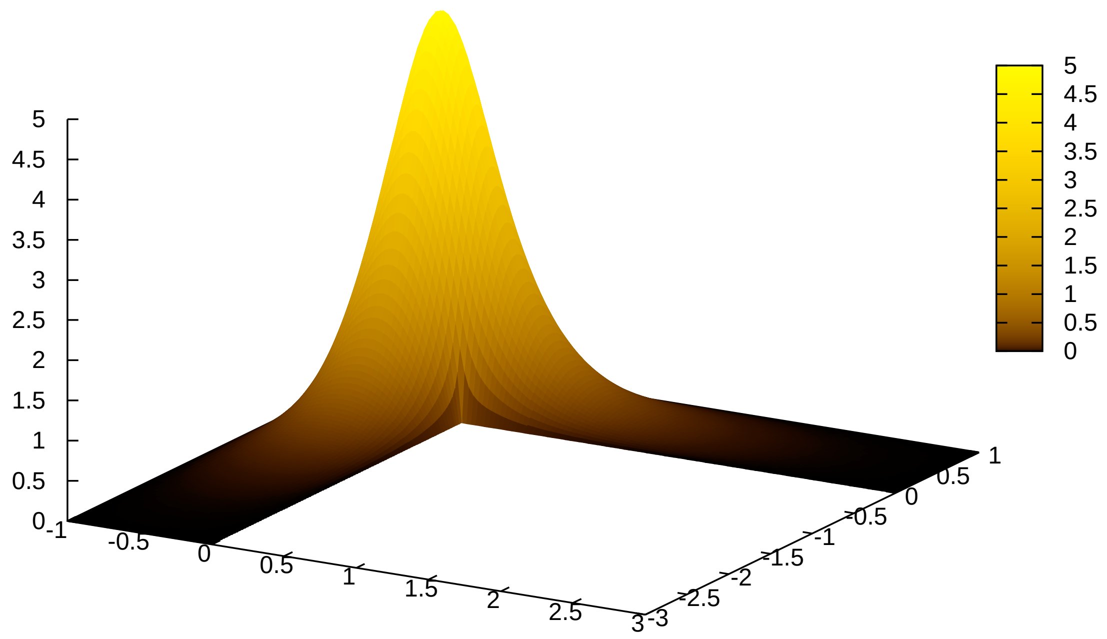
where $\Omega \subset \mathbb{R}^N$ is a domain and $1 < p < \frac{N+2}{N-2}$ ($p > 1$ in case $N = 2$).

In particular, we are interested in the unbounded L-shaped domain

$$\Omega = ((-1, \infty) \times (0, 1)) \cup ((-1, 0) \times (-\infty, 1))$$



Goal: Prove existence of a symmetric solution with one bump centered in the corner (by computer assistance)



Computation of an approximate solution

Choose $\Omega_0 \subset \Omega$ compact (cut off "far out part" of the infinite legs) and compute approximate solution $\omega_0 \in H_0^1(\Omega_0)$ of $-\Delta u = |u|^p$ in $\Omega_0,\ u = 0$ on $\partial\Omega_0$. Then use

$$\omega = \begin{cases} \omega_0 & \text{in} \quad \Omega_0 \\ 0 & \text{in} \quad \Omega\backslash\Omega_0 \end{cases}$$

as approximate solution on Ω.

For computing ω_0, we use a Newton iteration and Finite Elements; the re-entrant corner requires, in addition, use of a *corner singular function* (for accuracy reasons).

For the computation of K, we need a positive lower bound for

$$\min \{|\lambda| : \lambda \in \text{ spectrum of } \Phi^{-1}L\}.$$

The spectrum of $\Phi^{-1}L$ consists of

i) essential spectrum

ii) isolated eigenvalues of finite multiplicity

i) Essential spectrum of $\Phi^{-1}L$

- Consider $L_0 : H_0^1(\Omega) \to H^{-1}(\Omega),\ v \mapsto -\Delta v + \left(\frac{\pi^2}{\pi^2+1}\chi_{\Omega_1}\right) v$,

  where $\Omega_1 = (-1, 0) \times (0, 1)$.

- $\Phi^{-1}L_0$ is compact perturbation of $\Phi^{-1}L$ since ω and χ_{Ω_1}have compact support,

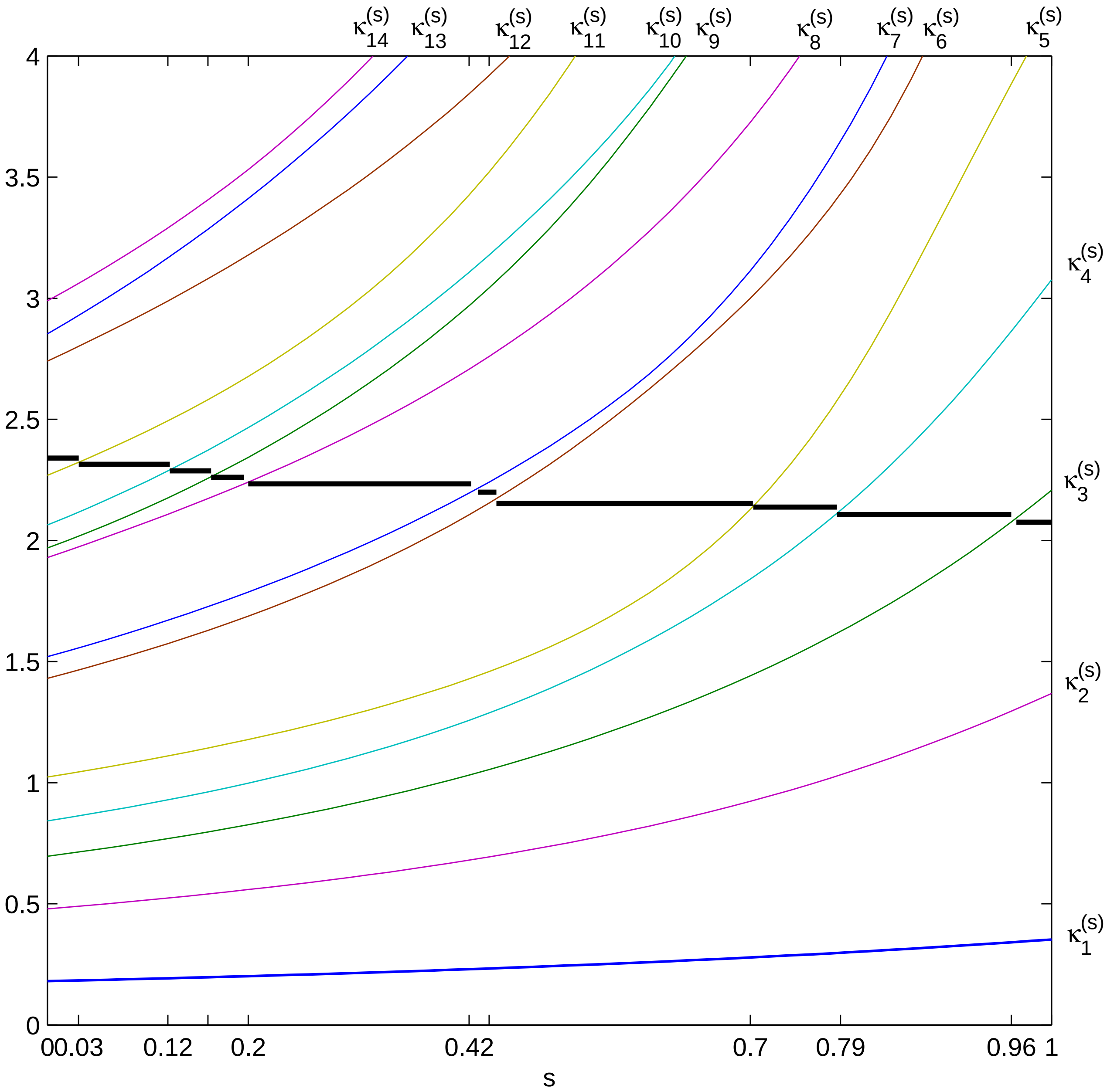
  and thus $\sigma_{ess}(\Phi^{-1}L) = \sigma_{ess}(\Phi^{-1}L_0)$

- Estimate Rayleigh quotient to obtain $\sigma(\Phi^{-1}L_0) \subset \left[\frac{\pi^2}{\pi^2+1}, \infty\right)$:

ii) Isolated eigenvalues of $\Phi^{-1}L$

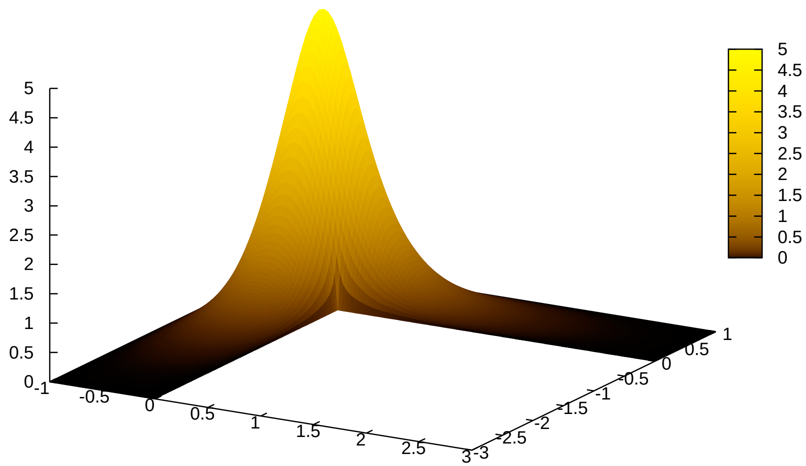
$$
\begin{aligned}
(\Phi^{-1}L)[u] = \lambda u \quad &\Longleftrightarrow \quad L[u] = \lambda \Phi[u] \\
&\Longleftrightarrow \quad -\Delta u - p|\omega|^{p-2}\omega u = \lambda(-\Delta u + u) \\
&\Longleftrightarrow \quad (1-\lambda)(-\Delta u + u) = (1 + p|\omega|^{p-2}\omega)u \\
&\overset{1-\lambda>0}{\Longleftrightarrow} \quad (-\Delta u + u) = \underbrace{\frac{1}{1-\lambda}}_{=:\kappa}(1 + p|\omega|^{p-2}\omega)u
\end{aligned}
$$

Task: Compute upper and lower bounds for eigenvalues κ neighboring 1.



$\kappa_{14}^{(s)}$ $\kappa_{13}^{(s)}$ $\kappa_{12}^{(s)}$ $\kappa_{11}^{(s)}$ $\kappa_{10}^{(s)}$ $\kappa_{9}^{(s)}$ $\kappa_{8}^{(s)}$ $\kappa_{7}^{(s)}$ $\kappa_{6}^{(s)}$ $\kappa_{5}^{(s)}$
$\kappa_{4}^{(s)}$
$\kappa_{3}^{(s)}$
$\kappa_{2}^{(s)}$
$\kappa_{1}^{(s)}$
0
0.5
1
1.5
2
2.5
3
3.5
4
0 0.03 0.12 0.2 0.42 0.7 0.79 0.96 1
s

Numerical results for $p = 3$



Bound for residuum:

$$\delta = 0.001699$$

Bound for inverse of Linearization:

$$K_{\text{sym}} = 3.73$$

$$\begin{aligned} \|\omega\|_{L^4} &\leq 3.014333 \\ C_4 &= 0.46200 \end{aligned}$$

For $\alpha = 0.006471$ we have:

$$\delta \leq \psi(\alpha) = \frac{\alpha}{K} - \gamma\alpha^2(\|\omega\|_{L^4} + C_4\alpha)$$

Thus, there exists a solution $u \in H_0^1(\Omega)$ such that $\|u - \omega\|_{H_0^1} \leq 0.006471$

Travelling wave equation on $\mathbb{R}$

(jointly with P. J. McKenna, J. Horak, B. Breuer)

Find $u \in H^2(\mathbb{R})$ such that

$$u^{(\mathrm{IV})} + c^2 u'' + e^u - 1 = 0 \text{ on } \mathbb{R}. \quad (1)$$

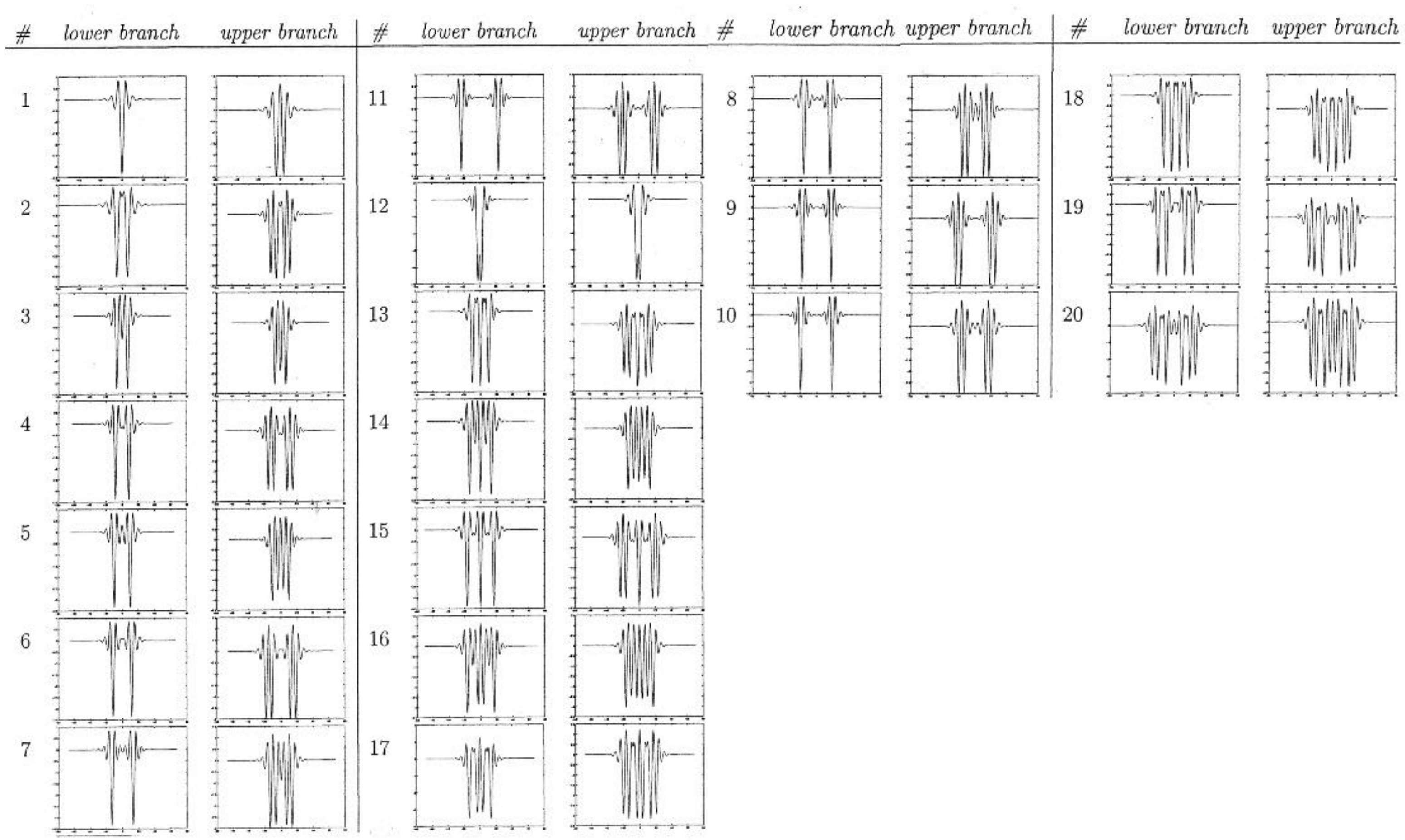
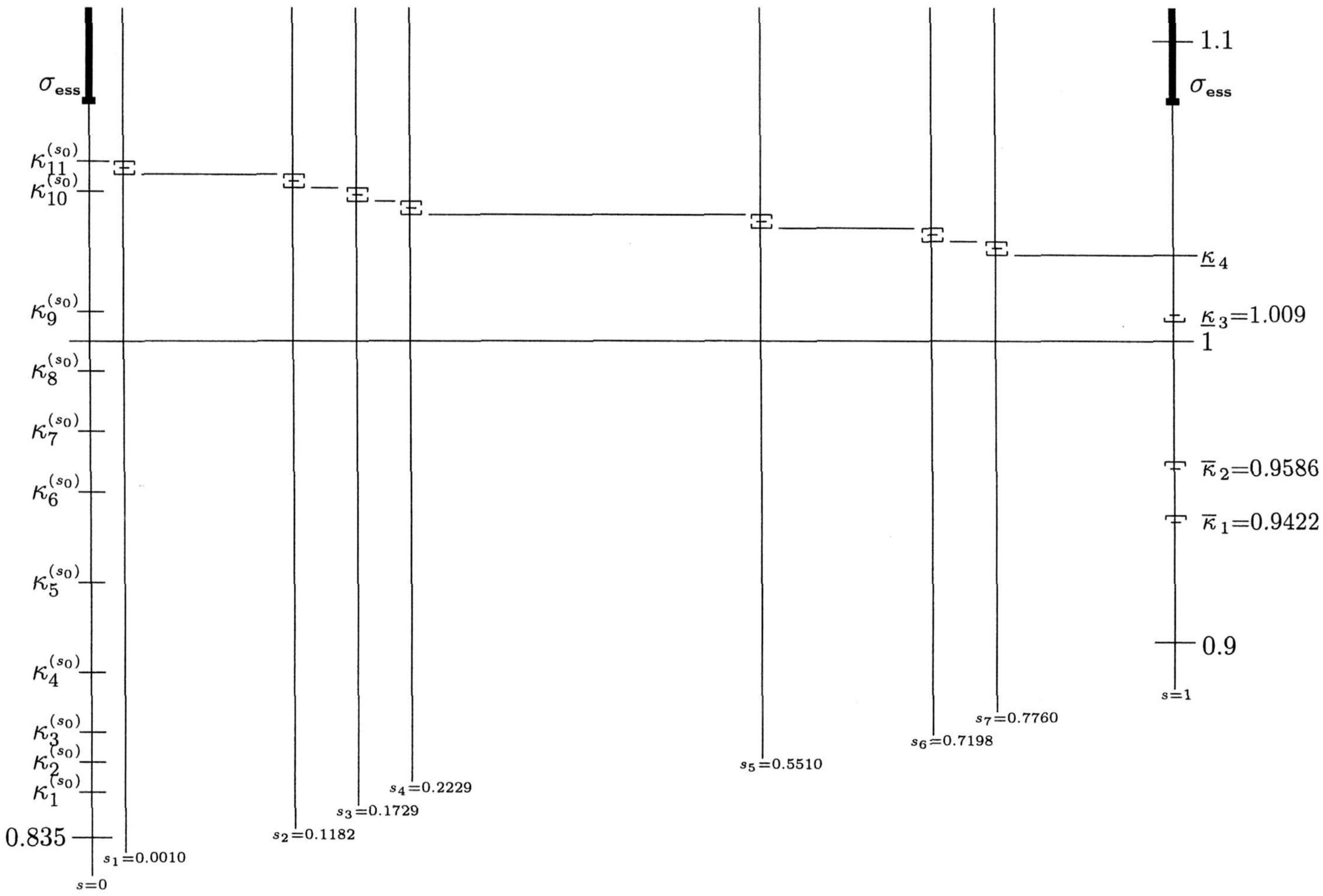


Figure 1: Numerical solutions for $c = 1.3$ and the number of the corresponding curve in the continuation diagram in Fig. 3.

	lower branch				upper branch			
Solution	K	δ	α	Morse Index	K	δ	α	Morse Index
1	1.51e+01	5.36e-08	8.05e-07	1	2.48e+01	4.21e-08	1.05e-06	1
2	6.52e+01	4.56e-08	2.97e-06	2	1.27e+02	4.40e-08	5.59e-06	3
3	1.22e+02	2.06e-08	2.50e-06	1	6.21e+01	4.62e-08	2.87e-06	2
4	3.61e+02	4.87e-08	1.76e-05	2	8.55e+02	4.41e-08	3.80e-05	3
5	8.06e+02	5.32e-08	4.33e-05	1	1.09e+02	4.02e-08	4.37e-06	2
6	2.11e+03	5.18e-08	1.18e-04	2	5.24e+03	6.53e-11	3.42e-07	3
7	5.11e+03	4.70e-08	4.33e-05	1	3.48e+02	4.33e-08	1.51e-05	2
8	3.19e+04	1.13e-10	3.72e-06	1	2.07e+03	1.62e-10	3.34e-07	2
9	7.87e+04	1.57e-10	–	-	1.99e+05	5.37e-10	–	-
10	3.19e+04	1.57e-10	5.21e-06	1	1.30e+04	2.62e-10	3.44e-06	2
11	1.87e+06	7.69e-11	–	-	8.12e+04	3.08e-10	–	-
12	9.20e+01	5.18e-08	4.77e-06	2	1.14e+02	2.65e-08	3.02e-06	3
13	1.20e+02	4.69e-08	5.62e-06	3	2.35e+02	4.40e-08	1.04e-05	4
14	2.65e+02	2.03e-08	5.35e-06	2	1.65e+02	4.47e-08	7.35e-06	3
15	7.00e+02	5.25e-08	3.71e-05	3	1.56e+03	1.67e-08	2.61e-05	4
16	3.80e+02	4.85e-08	1.85e-05	2	2.32e+02	4.62e-08	1.07e-05	3
17	1.45e+02	4.97e-08	7.16e-06	3	2.23e+02	1.65e-08	3.65e-06	4
18	1.97e+02	2.11e-08	4.16e-06	4	3.70e+02	1.73e-08	6.38e-06	5
19	4.12e+03	5.50e-08	4.16e-06	4	6.81e+03	3.34e-09	2.37e-05	5
20	2.43e+03	7.36e-08	2.02e-04	4	2.17e+02	6.38e-10	3.34e-07	5

Verified upper bounds for the crucial constants K, α, δ. The 40 approximations are ordered as in Figure 1.



Course of homotopy algorithm for solution number 5 (upper branch).

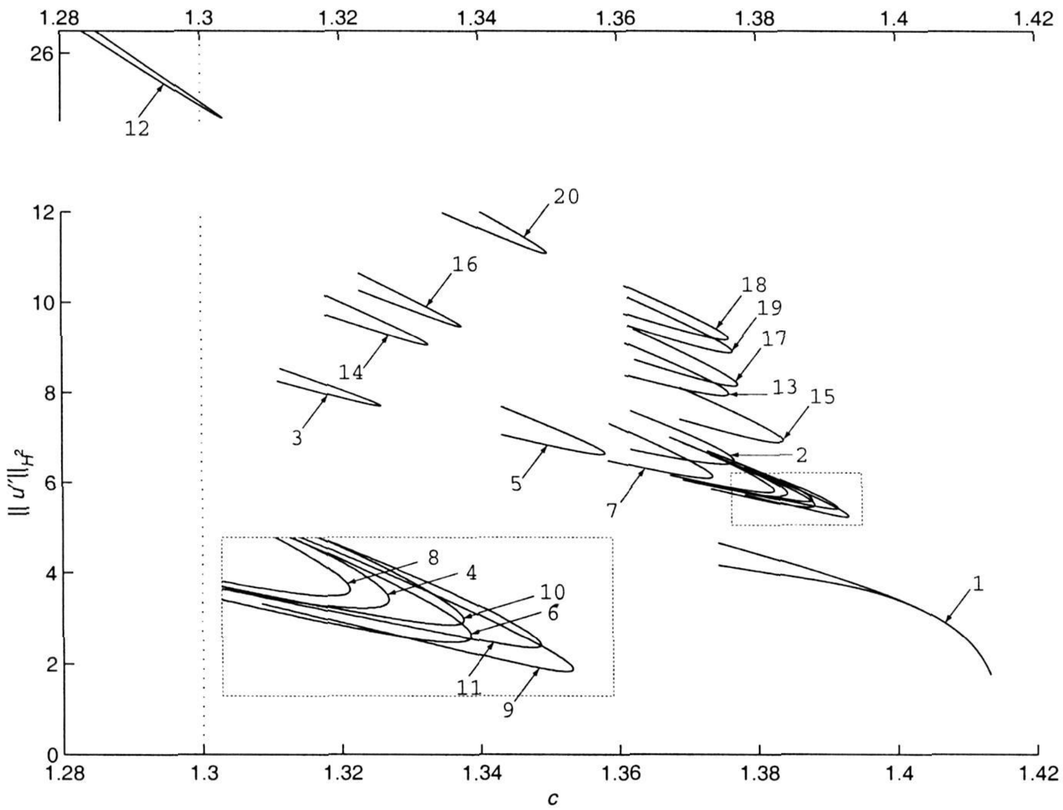
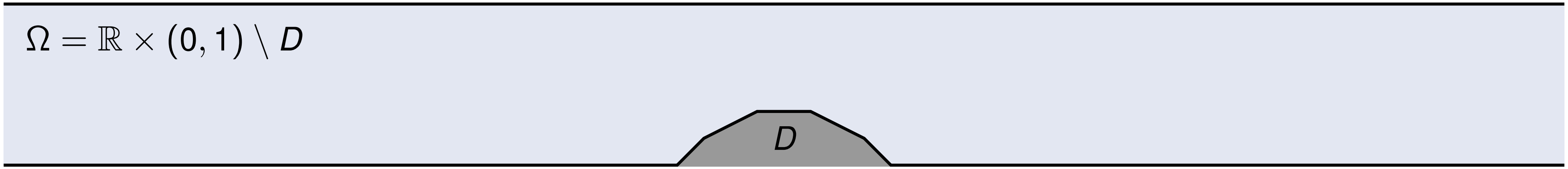


Figure 3: Continuation of numerical solutions of Fig. 1. This figure shows portions of the solution curves as c is varied above 1.3. Presumably , these branches continue as $c \to 0$. What this paper proves is that at $c = 1.3$, 36 solutions exist.

Navier-Stokes Equations
on an unbounded strip with obstacle (jointly with J. Wunderlich)

Karlsruhe Institute of Technology



Navier-Stokes Equations

$$\left.\begin{aligned} -\Delta u + Re\,[(u\cdot\nabla)u + (u\cdot\nabla)\Gamma + (\Gamma\cdot\nabla)u + \nabla p] &= g \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \begin{pmatrix}\text{transformed}\\ \text{NSE}\end{pmatrix}$$

$$u = 0 \qquad \text{on } \partial\Omega$$

- Reynolds number Re
- $\Gamma := U - V$, with Poiseuille flow U and compactly supported V to adjust boundary conditions
- $v = \Gamma + u$ solves the original Navier-Stokes problem
- Weak solutions $u \in H := \{u \in H_0^1(\Omega, \mathbb{R}^2) \colon \operatorname{div} u = 0\}$
- Inner product $\langle u, v\rangle_H := \langle \nabla u, \nabla v\rangle_{L^2} + \sigma\langle u, v\rangle_{L_2}$ $(u, v \in H)$ for some $\sigma \geq 0$
- Isometric isomorphism $\Phi\colon H \to H'$, $u \mapsto \langle u, \cdot\rangle_H$

- Norm bound for L^{-1}:

$$\|u\|_H \leq K\,\|Lu\|_{H'} = K\,\|\Phi^{-1}Lu\|_H \quad (u \in H) \tag{A1}$$

  Hence, L is injective and $\mathrm{rg}(\Phi^{-1}L)$ is closed

- Additionally assume

$$\|u\|_H \leq K^*\,\|(\Phi^{-1}L)^*u\|_H \quad (u \in H) \tag{A2}$$

  Then, using $\mathrm{rg}(\Phi^{-1}L)^\perp = \ker((\Phi^{-1}L)^*)$ and (A1), L is surjective

Norm bound $K > 0$ for L^{-1}

$$\|u\|_H \leq K\,\|Lu\|_{H'} = K\,\|\Phi^{-1}Lu\|_H \quad (u \in H) \tag{A1}$$

If $\Phi^{-1}L$ **were** symmetric this would lead to the eigenvalue problem $\Phi^{-1}Lu = \lambda u$, i.e., $Lu = \lambda \Phi u$

Since here $\Phi^{-1}L$ is not symmetric we square:

$$\langle u, u\rangle_H \leq K^2\,\langle \Phi^{-1}Lu, \Phi^{-1}Lu\rangle_H \quad (u \in H)$$

Need positive lower bound for the spectral points of

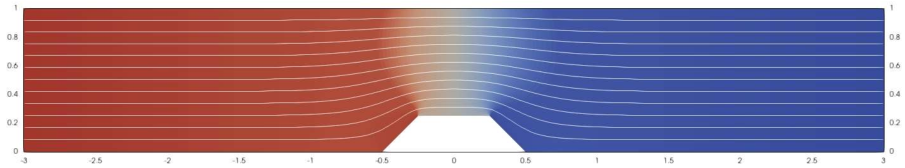
$$\langle \Phi^{-1}Lu, \Phi^{-1}L\varphi\rangle_H = \lambda\,\langle u, \varphi\rangle_H \qquad (\varphi \in H)$$

Task: Compute verified lower bounds

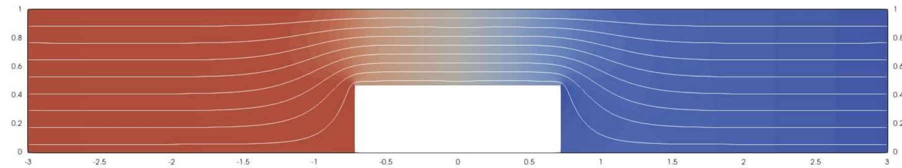
- $\underline{\sigma}_{\text{ess}} > 0$ for the essential spectrum
- $\underline{\sigma}_{\text{ev}} > 0$ for the isolated eigenvalues below $\underline{\sigma}_{\text{ess}}$, $\underline{\sigma}_{\text{ev}} = \underline{\sigma}_{\text{ess}}$ if such eigenvalues do not exist

Choose $K := \frac{1}{\sqrt{\underline{\sigma}_{\text{ev}}}}$

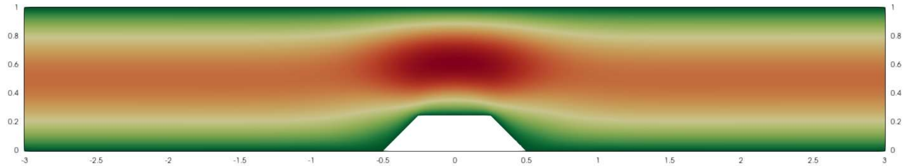
Some results



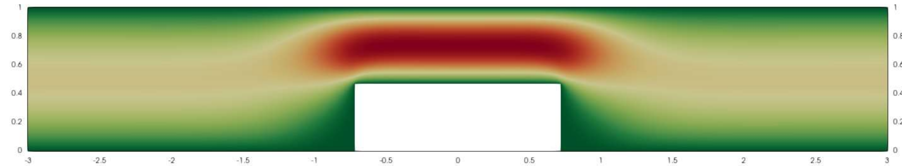
Stream lines of $\Gamma + \omega$



Stream lines of $\Gamma + \omega$



Euklidean norm of $\Gamma + \omega$

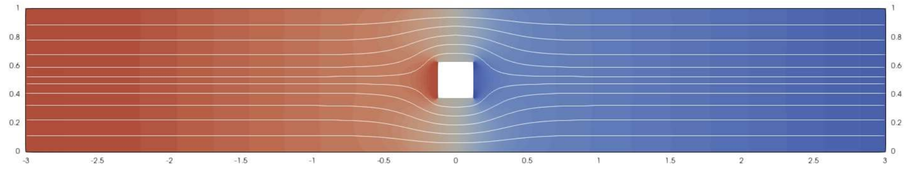


Euklidean norm of $\Gamma + \omega$

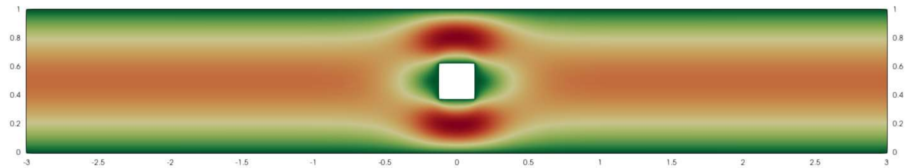
Re	δ	K	$\|\omega - u^*\|_H$	$\|\omega - u^*\|_{L^2}$
1.0	0.05472	1.27643	0.07130	0.02270
2.0	0.05472	1.76425	0.10535	0.03354

Re	δ	K	$\|\omega - u^*\|_H$	$\|\omega - u^*\|_{L^2}$
1.0	0.08452	1.46489	0.12932	0.04117
2.0	0.08454	1.84789	0.18267	0.05673

Some results

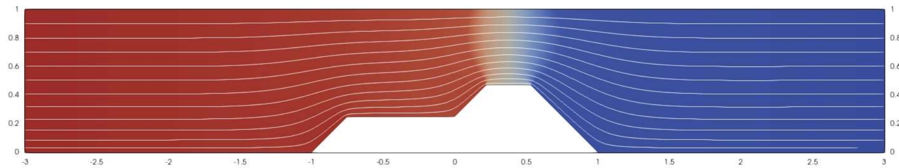


Stream lines of $\Gamma + \omega$

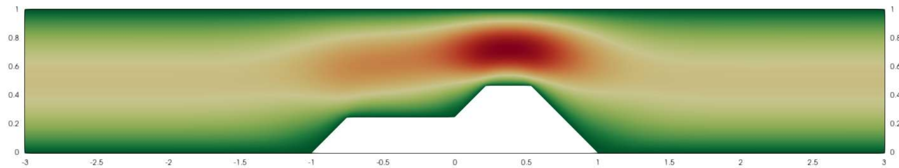


Euklidean norm of $\Gamma + \omega$

Re	δ	K	$\|\omega - u^*\|_H$	$\|\omega - u^*\|_{L^2}$
1.0	0.10203	1.31294	0.13973	0.04448
2.0	0.10205	1.91156	0.24798	0.07894

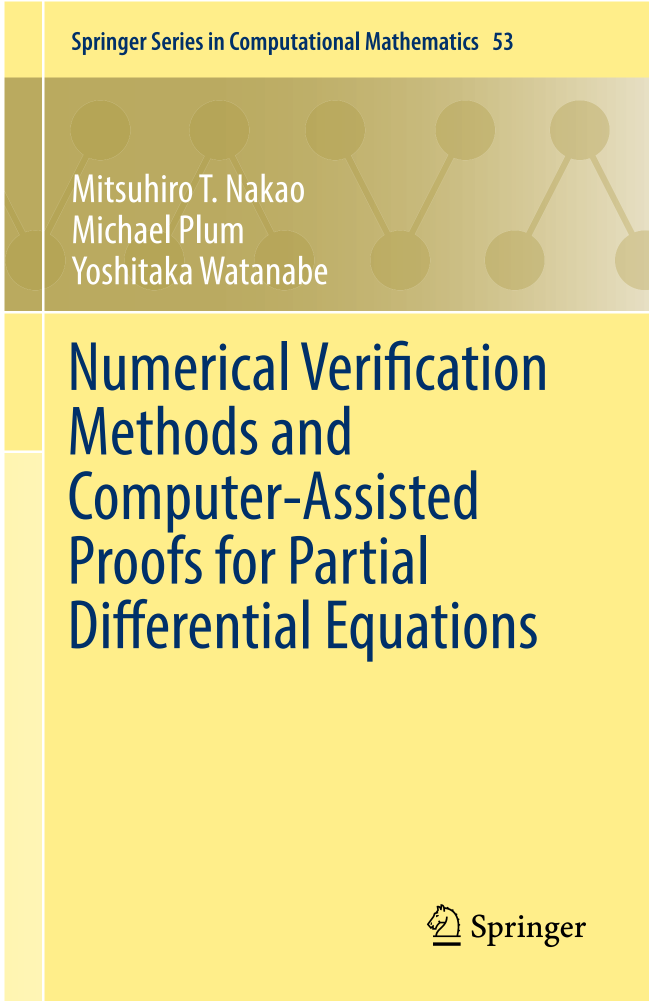


Stream lines of $\Gamma + \omega$



Euklidean norm of $\Gamma + \omega$

Re	δ	K	$\|\omega - u^*\|_H$	$\|\omega - u^*\|_{L^2}$
1.0	0.07781	1.44995	0.11732	0.03735
2.0	0.07783	2.63508	0.35232	0.11215



https://link.springer.com/book/10.1007/978-981-13-7669-6