Open problems session

Effective Aspects in Diophantine Approximation
Lyon, March 27 to 31, 2023

H. Cohen, V. Dimitrov, B. Salvy, W. Zudilin

1 Absolute separation of polynomials (B. Salvy)

Classically, the separation of a polynomial $P \in \mathbb{C}[x]$ is

$$\operatorname{sep}(P) = \min_{\substack{P(\alpha)=P(\beta)=0\\ \alpha\neq\beta}} \big|\alpha - \beta\big|.$$

Similarly, one defines the absolute separation of P as

$$\operatorname{abssep}(P) := \min_{\substack{P(\alpha)=P(\beta)=0,\\ |\alpha|\neq|\beta|}} \big||\alpha| - |\beta|\big|.$$

Having good lower bounds on this quantity for polynomials with integer coefficients is of interest in the asymptotic analysis of linear recurrent sequences.

In the classical case of the separation of polynomials with integer coefficients, Mahler (1964) gave the following lower bound

$$\operatorname{sep}(P) \gg H(P)^{-d+1},$$

where $H(P)$ is the height of P, i.e., the maximum of the absolute values of its coefficients, while $\gg$ hides an implicit constant that depends only on d. Even in that classical case, the tightness of the exponent $-d+1$ is still unknown, with best known upper bounds $-(2d-1)/3$ for general d and -2 for $d = 3$.

In a recent work [1], we obtained that if P has integer coefficients and α, β are two roots of P with different absolute value, then in the worst situation where neither α nor β is real (which implies $d \geq 4$), one has

$$\big||\alpha| - |\beta|\big| \gg H(P)^{-(d-1)(d-2)(d-3)/2}.$$

This cubic exponent improves on previous results. Still, we have no idea what the actual worst-case situation is. The worst examples we could construct in degree $4, 5, 6$ have exponent $-d-1$, quite far from the bound above.

This leaves lots of questions open, in particular:

Q1. Is the cubic exponent optimal?

Q2. How can one construct families of examples, even in low degree, with an exponent bigger than $-d-1$?

Q3. What if one restricts to the roots of largest (or smallest) absolute value?

[1] Yann Bugeaud, Andrej Dujella, Wenjie Fang, Tomislav Pejković, and Bruno Salvy, *Absolute root separation*, Experimental Mathematics 31 (2022), no. 3, 805–812.

2 Questions Related to Continued Fractions for π^2 and $\zeta(3)$ (H. Cohen)

Let z be a constant, and $a(n)$, $b(n)$ be polynomials in n. When I write $z = (a(n), b(n))$ I mean that the continued fraction $a(0) + b(0)/(a(1) + b(1)/(a(2) + \cdots))$ converges to $(Az + B)/(Cz + D)$ for some integers A, B, C, D with $AD - BC \neq 0$, or almost equivalently (i.e., excluding possible zeros) if there is a continued fraction (abbreviated CF) converging to z whose coefficients are equal to $a(n)$ and $b(n)$ for n sufficiently large.

2.1 Question Related to π^2

I have found *experimentally* many parametric CF for π^2, and have proved almost all. But two have resisted (in what follows $u \in \mathbb{Z}_{\geq 0}$):

$$\pi^2 = ((2u^2+8u+7)n + (2u+3)(u^2+3u+1), n(n+u)(n+u+1)(n+2u+4))$$
$$\pi^2 = ((2u^2+8u+7)n + (2u+3)(u+2)(u+3), n(n+u+3)(n+u+4)(n+2u+4))\ .$$

These two parametric CFs would follow (up to some additional work) from the following:

Conjecture 2.1. *Let u be a nonnegative integers. Up to a multiplicative constant there exists a unique polynomial P such that*

$$(x+2u+4)(x+u+1)P(x+1) - (x-1)(x+u-1)P(x-1)$$
$$= ((2u^2+8u+7)x + (2u+3)(u^2+3u+1))P(x)\ ,$$

and we have $\deg(P) = u(u+3)$. The same is true if we require

$$(x+2u+4)(x+u+4)P(x+1) - (x-1)(x+u+2)P(x-1)$$
$$= ((2u^2+8u+7)x + (2u+3)(u+2)(u+3))P(x)\ .$$

I have asked this question on Math Overflow, and what I know is the following:

1. These conjectures are equivalent.

2. If P exists it is indeed unique up to a multiplicative constant, and it has degree $u(u+3)$.

Of course this is equivalent to the vanishing of a complicated determinant, but this does not seem to help.

2.2 Question Related to $\zeta(3)$

Consider $S = \sum_{n\geq 1} 1/\prod_{0\leq i\leq m-1}(n+i)^{e_i}$, where $e_i = 1,2,3,2,1$ according to $0 \leq i < m_1$, $m_1 \leq i < m_1+m_2$, $m_1+m_2 \leq i < m_1+m_2+m_3$, $m_1+m_2+m_3 \leq i < m_1+m_2+m_3+m_4$, $m_1+m_2+m_3+m_4 \leq i < m_1+m_2+m_3+m_4+m_5 = m$. These conditions guarantee that after suitable simplification's, the Euler CF corresponding to the series S will have $\deg(a(n)) \leq 3$ and $\deg(b(n)) \leq 6$.

S will be a linear combination of 1, $\zeta(2)$, and $\zeta(3)$, and we do not want any $\zeta(2)$ if we want CFs for $\zeta(3)$. A *sufficient* condition for this is the *symmetry* condition $m_4 = m_2$, $m_5 = m_1$, and m_3 odd, and this already gives after some additional work a *four* parameter family of CFs for $\zeta(3)$. However the symmetry condition is far from sufficient, and I have found hundreds of additional quintuples where the coefficient of $\zeta(2)$ vanishes, the simplest being $(m_1, m_2, m_3, m_4, m_5) = (0, 0, 2, 2, 0)$, and each gives a new CF. My question is: can one classify these additional quintuples, or at least give an explicit subfamily?

3 On André's algebraicity criterion (V. Dimitrov)

We have

$$\frac{1}{\sqrt{1-4x}} * \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n}^2 x^n \in \mathbb{Z}[\![x]\!]$$

as the textbook example that the Hadamard square of an algebraic function in characteristic zero is usually a transcendental holonomic function. (In positive characteristic, the algebraic functions are closed under the Hadamard product operation, by theorems of Furstenberg and Deligne.) In this case it simply is the hypergeometric function

$${}_2F_1\left[\begin{matrix}1/2 \;\; 1/2 \\ 1\end{matrix}; 16x\right] \in \mathbb{Z}[\![x]\!]$$

of parameters $\{1/2, 1/2; 1\}$ and singularities scaled to $\{0, 1/16, \infty\}$.

This is then an extremal non-example for the algebraicity criterion of André:

Theorem 3.1 (André)**.** *A formal power series $f(x) \in \mathbb{Z}[\![x]\!]$ must be algebraic as soon as there exists a meromorphic mapping*[1] $\varphi : (\mathbb{D}, 0) \to (\widehat{\mathbb{C}}, 0)$ *with $|\varphi'(0)| > 1$ and such that the composite germ $f(\varphi(z)) \in \mathbf{C}[\![z]\!]$ is also meromorphic on $\mathbb{D}$.*

Indeed we have the more precise:

[1]Here, $\mathbb{D} := \{|z| < 1\}$ denotes the complex unit disc.

Theorem 3.2 (André). *For a given meromorphic mapping $\varphi : (\mathbb{D}, 0) \to (\widehat{\mathbf{C}}, 0)$ as above with $|\varphi'(0)| > 1$, there is only a finite-dimensional $\mathbf{Q}(x)$-linear span of formal power series $f(x) \in \mathbb{Z}[\![x]\!]$ having the composite germ $f(\varphi(z)) \in \mathbf{C}[\![z]\!]$ also meromorphic on $\mathbb{D}$.*

A showcase of this theorem is:

Corollary 1. *Let $\alpha \in \mathbf{R}^{>0}$. Consider the $f(x) \in \mathbb{Z}[\![x]\!]$ such that there exists* some[2] *linear ODE $L(f) = 0$, with some nonzero linear differential operator $L = L_f \in \mathbb{Z}[x, d/dx]$ with singularities at most $\{0, \alpha, \infty\}$. We have then a threshold:*

- *If $\alpha > 1/16$, all such $f(x)$ are necessarily algebraic, and they have a finite-dimensional $\mathbf{Q}(x)$-linear span.*
- *For $\alpha = 1/16$, there are transcendental such $f(x)$, as well as an infinite-dimensional $\mathbf{Q}(x)$-linear span of algebraic ones.*

Proof. The condition on the singularities of the ODE implies, by Cauchy's existence theorem on linear ODE with analytic coefficients, the automatic analytic pullback under the map

$$\varphi := \alpha\lambda(z) \, : \, \mathbb{D} \to \mathbf{C} \setminus \{\alpha, \infty\},$$

where

$$\lambda(z) := 16z \prod_{n=1}^{\infty} \Big(\frac{1 + z^{2n}}{1 + z^{2n-1}} \Big)^8$$

is the modular lambda map expressed in the cusp-filling coordinate $z = q := e^{\pi i \tau}$.

As $|\varphi'(0)| = 16\alpha$, the first point follows from André's theorem.

For the second point, we have already seen the transcendental Hadamard square function above, while an infinitude of independent algebraic functions are supplied by the modular equation expressing $\lambda(q^N)/16$ formally into a function of $x := \lambda(q)/16$; this expression does indeed give a $\mathbb{Z}[\![x]\!]$ series, because $\mathbb{Z}[\![q]\!] = \mathbb{Z}[\![x]\!]$. $\square$

Some obvious examples include:

- For $\alpha > 1/4$: No examples other than just the polynomials $\mathbb{Z}[x]$.

[2]Depending on f.

- For $\alpha = 1/4$: The two-dimensional $\mathbf{Q}(x)$-vector space spanned by the functions 1 and $\sqrt{1-4x} \in \mathbb{Z}[\![x]\!]$.
- For $\alpha = 1/8$: Certainly we have the examples $(1-8x)^{1/4}$ and $(1-8x)^{3/4} \in \mathbb{Z}[\![x]\!]$.
- For $\alpha = 1/9$: Certainly we have the examples $(1-9x)^{1/3}$ and $(1-9x)^{1/3} \in \mathbb{Z}[\![x]\!]$.

Here now is the query I formulated at the Open problem session:

Question 1. *As $\alpha \to (1/16)^+$, does the list of examples blow up or stabilize?*

In other words, is there a sequence $\alpha_n \to (1/16)^+$ along which there exist n-tuples $f_1, \ldots, f_n \in \mathbb{Z}[\![x]\!]$ of $\mathbf{Q}(x)$-linearly independent algebraic power series germs with branching locus limited to the three points $\{0, \alpha_n, \infty\} \in \mathbb{P}^1$?

During my talk I also raised a related question, which here I will strip down to a still more particular form:

Question 2. *Is there any G-function with an $x = 0$ power series of the form*

$$f(x) = \sum_{n=1}^{\infty} a_n \, \frac{x^n}{[1, \ldots, n]^k}, \qquad \text{for some } k \in \mathbf{N}_0 \text{ and with all } a_n \in \mathbb{Z},$$

whose singularities are only at $\{0, 1, \infty\}$ and yet $f(x)$ does not lie in the multiple polylogarithms ring

$$\mathbb{Z}[\mathrm{Li}_{\mathbf{s}}(x) \, : \, \forall \mathbf{s}] \otimes \mathbf{Q}(x),$$

where

$$\mathrm{Li}_{(s_1, \ldots, s_k)}(x) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{x^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}} \in \mathbf{Q}[\![x]\!].$$