The FFT and computer assisted proofs for nonlinear functional equations

Workshop on Certified and Symbolic-Numeric Computation
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This talk is built on, rather than being about, the interval FFT.

G. Liu and V. Kreinovich, "Fast convolution and Fast Fourier Transform under interval and fuzzy uncertainty," *Journal of Computer and System Sciences*, vol. 76, no. 1, pp. 63–76, Feb. 2010. [Online]. Available: https://linkinghub.elsevier.com/retrieve/pii/S0022000009000452

S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, Developments in Reliable Computing, pages 77-104. Kluwer Academic Publishers, Dordrecht, 1999.

MR4103639 Brisebarre, Nicolas; Joldeş, Mioara; Muller, Jean-Michel; Naneş, Ana-Maria; Picot, Joris
Error analysis of some operations involved in the Cooley-Tukey fast Fourier transform.
ACM Trans. Math. Software 46 (2020), no. 2, Art. 11, 27 pp.

Computer assisted proofs based on DFT/FFT go back to the early 1990's

MR1008096 de la Llave, R.; Rana, David
Accurate strategies for small divisor problems.
Bull. Amer. Math. Soc. (N.S.) 22 (1990), no. 1, 85–90.

MR1156419 Reviewed de la Llave, Rafael
A renormalization group explanation of numerical observations of analyticity domains.
J. Statist. Phys. 66 (1992), no. 5-6, 1631–1634.

A fantastic general reference is the paper of Jordi:

MR3709329 Reviewed Figueras, J.-Ll.; Haro, A.; Luque, A.
Rigorous computer-assisted application of KAM theory: a modern approach.
Found. Comput. Math. 17 (2017), no. 5, 1123–1193.

Today's talk is inspired by/builds on thees, with a slight change of emphasis...

sequence space versus functions space

Question of The Hour:

consider the Banach algebra "little ell nu one"

$$\ell^1_\nu = \{a = \{a_n\}_{n\in\mathbb{Z}} \,:\, a_n \in \mathbb{C} \text{ and } \|a\|_\nu < \infty\}$$

where $\|a\|_\nu = \sum_{n=-\infty}^{\infty} |a_n|\nu^{|n|}$ $\qquad \nu > 1$

The pair $(\ell^1_\nu, \|\cdot\|_\nu)$ is a Banach space.

- Let

$$\mathbb{S}_\rho = \{z \in \mathbb{Z} : \text{imag}(z) < \rho\}$$

- If $a \in \ell^1_\nu$, then the function $u(z)$ defined by

$$u(z) = \sum_{n\in\mathbb{Z}} a_n e^{inz}$$

is analytic on $\mathbb{S}_{\log(\nu)}$, continuous on its closure, and 2π periodic in the real part.

- $\|u\|_0 = \sup_{z\in\mathbb{S}_{\log\nu}} |u(z)| \leq \|a\|_\nu$ (giving up some strip width bounds derivatives)

- For $a, b \in \ell^1_\nu$ define their discrete convolution $a * b$ by

$$(a*b)_n = \sum_{k\in\mathbb{Z}} a_{n-k} b_k$$

Then $a * b \in \ell^1_\nu$ with

$$\|a*b\|_\nu \leq \|a\|_\nu \, \|b\|_\nu$$

That is, the triple $(\ell^1_\nu, \|\cdot\|_\nu, *)$ is a Banach Algebra.

- The terms of the discrete convolution are the Fourier coefficients of the pointwise product.
- Polynomials become iterated discrete convolutions.

Banach spaces of infinite sequences are convenient for computer assisted arguments.

Question of The Hour:

consider the Banach algebra "little ell nu one"

$$\ell^1_\nu = \{a = \{a_n\}_{n\in\mathbb{Z}} \, : \, a_n \in \mathbb{C} \text{ and } \|a\|_\nu < \infty\}$$

where $\|a\|_\nu = \sum_{n=-\infty}^{\infty} |a_n|\nu^{|n|}$ $\qquad \nu > 1$

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an analytic function.

We are interested in the function $F\colon \ell^1_\nu \to \ell^1_\nu$ defined by the rule

$$F(a) = b$$

if and only if

$$b = \{b_n\}_{n\in\mathbb{Z}}$$

are the Fourier coefficients of the function

$$f(u(z)) = \sum_{n\in\mathbb{Z}} b_n e^{inz}$$

That is, the sequence $F(a) = b = \{b_n\}_{n\in\mathbb{Z}}$ is defined by

$$\begin{aligned} b_n &= \frac{1}{2\pi i}\int_0^{2\pi} f(u(\theta))\, e^{-in\theta}\, d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} f\left(\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}\right) e^{-in\theta}\, d\theta \end{aligned}$$

Note: if f is polynomial then $F(a)$ is easily expressed in terms of discrete convolutions.

One can study analogous questions for Taylor series and Chebyshev series.

Question of The Hour:

Example problems in dynamical systems/nonlinear analysis:

- Initial value problem for an ODE:

  $x(t) = f(x(t))$ $\quad f\colon \mathbb{R}^d \to \mathbb{R}^d$

  $x(0) = x_0 \in \mathbb{R}^d$

  f real analytic

- System of scalar BVP on $[a, b] \subset \mathbb{R}$:

  $\mathcal{L}u(x) + f(u(x)) = 0$ $\quad f\colon \mathbb{R}^d \to \mathbb{R}^d$

  periodic BCs

  f real analytic

- Spectral sub-manifold – discrete time.

  $P(\lambda s) = f(P(s))$ $\quad f\colon \mathbb{R}^d \to \mathbb{R}^d$

  Impose that

  $\|P'(0)\|^2 > 0$

  f real analytic

Taylor

Chebyshev

Fourier

- Spectral sub-manifold - continuous time.

  $\lambda s P'(s) = f(P(s))$ $\quad f\colon \mathbb{R}^d \to \mathbb{R}^d$

  Impose that

  $\|P'(0)\|^2 > 0$

  f real analytic

- Invariant circle - discrete time

  $P(\theta + \omega) = f(P(\theta))$ $\quad f\colon \mathbb{R}^d \to \mathbb{R}^d$

  P periodic

  phase cond

  f real analytic

- Invariant torus - continuous time

  $$\omega_1 \frac{\partial}{\partial \theta_1} P(\theta_1, \theta_2) + \omega_2 \frac{\partial}{\partial \theta_2} P(\theta_1, \theta_2) = f(P(\theta_1, \theta_2))$$

  $f\colon \mathbb{R}^d \to \mathbb{R}^d$

  P periodic

  phase cond

Question of The Hour:

Example problems in dynamical systems/nonlinear analysis:

- $x(t) = f(x(t))$
- $\mathcal{L}u(x) + f(u(x)) = 0$
- $P(\lambda s) = f(P(s))$
- $\lambda s P'(s) = f(P(s))$
- $P(\theta + \omega) = f(P(\theta))$
- $\omega_1 \dfrac{\partial}{\partial \theta_1} P(\theta_1, \theta_2) + \omega_2 \dfrac{\partial}{\partial \theta_2} P(\theta_1, \theta_2) = f(P(\theta_1, \theta_2))$

Taylor

Chebyshev

Fourier

Conjugacy problems!

Computer assisted proofs for these problems (and many others) can be obtained using a common approach.

Step 1: Formulate as a zero finding problem on an appropriate Banach space.

Step 2: Project and solve numerically.

Step 3: Make a Newton-Kantorovich or Nash-Moser argument: prove a true solution nearby.

Lanford, Eckman, Koch, Plum... school of a-posteriori analysis

A key step in studying any of these equations is studying the compositions with f

Theorem 5.1 (Newton Kantorovich (with smoothing approximate inverse))**.** *Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and that that $\mathcal{F}\colon \mathcal{X} \to \mathcal{Y}$ is a Fréchet differentiable map. Assume that $A^\dagger\colon \mathcal{X} \to \mathcal{Y}$, and $A\colon \mathcal{Y} \to \mathcal{X}$ are bounded linear operators with A one-to-one, and that $\bar{x} \in \mathcal{X}$, and $r_*, Y, Z_0, Z_1, Z_2 > 0$ have that*

- ***Approximate root:***
$$\|AF(\bar{x})\|_{\mathcal{X}} \leq Y,$$
- ***Approximate inverse:***
$$\|Id_{\mathcal{X}} - AA^\dagger\|_{B(\mathcal{X})} \leq Z_0,$$
- ***Approximate derivative:***
$$\left\| A \left[D\mathcal{F}(\bar{x}) - A^\dagger \right] \right\|_{B(\mathcal{X})} \leq Z_1,$$
- ***Local Lipschitz bound on the Fréchet derivative:***
$$\sup_{x, y \in B_{r_*(\bar{x})}} \|A\left[D\mathcal{F}(x) - D\mathcal{F}(y)\right]\|_{B(\mathcal{X})} \leq Z_2 \|x - y\|_{\mathcal{X}}.$$

Suppose that
$$Z_0 + Z_1 < 1, \quad \text{and that} \quad (1 - Z_0 - Z_1)^2 > 4 Z_2 Y.$$
Define
$$\beta = (1 - Z_0 - Z_1) + \sqrt{(1 - Z_0 - Z_1)^2 - 4 Z_2 Y},$$
$$C = \frac{\beta}{2 Z_2},$$
and
$$R = \min\left(r_*, C\right).$$

Then, for any $r > 0$ with
$$4\beta^{-1} Y \leq r \leq C,$$
there exists a unique
$$x_* \in B_r(\bar{x}),$$
so that
$$F(x_*) = 0.$$
Moreover, $AD\mathcal{F}(x_*)$ is a Banach space isomorphism.

Theorem 5.1 (Newton Kantorovich (with smoothing approximate inverse)). *Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and that that $\mathcal{F} \colon \mathcal{X} \to \mathcal{Y}$ is a Fréchet differentiable map. Assume that $A^\dagger \colon \mathcal{X} \to \mathcal{Y}$, and $A \colon \mathcal{Y} \to \mathcal{X}$ are bounded linear operators with A one-to-one, and that $\bar{x} \in \mathcal{X}$, and $r_*, Y, Z_0, Z_1, Z_2 > 0$ have that*

- ***Approximate root:***
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$$\sup_{x,y \in B_{r_*}(\bar{x})} \|A\left[D\mathcal{F}(x) - D\mathcal{F}(y)\right]\|_{B(\mathcal{X})} \leq Z_2 \|x - y\|_{\mathcal{X}}.$$

Suppose that
$$Z_0 + Z_1 < 1, \quad \text{and that} \quad (1 - Z_0 - Z_1)^2 > 4 Z_2 Y.$$
Define
$$\beta = (1 - Z_0 - Z_1) + \sqrt{(1 - Z_0 - Z_1)^2 - 4Z_2 Y},$$
$$C = \frac{\beta}{2Z_2},$$
and
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Then, for any $r > 0$ with
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$$x_* \in B_r(\bar{x}),$$
so that
$$F(x_*) = 0.$$
Moreover, $AD\mathcal{F}(x_*)$ is a Banach space isomorphism.

- The proof studies the Newton-like operator
$$T(x) = x - AF(x)$$
- The hypotheses of the theorem are exactly what is needed to show that T is a contraction on the complete metric space $\overline{B_r(\bar{x})}$.
- Another consequence of the theorem is that $ADF(x_*)$ is an isomorphism. This information can sometimes be parlayed into stability/transversality information.

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad\qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.
For example if $f(z) = e^z - 1$ this is a toy model for a suspension bridge.

Traveling waves on unbounded domains

MR2220064 Breuer, B.; Horák, J.; McKenna, P. J.; Plum, M.
A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam.
J. Differential Equations 224 (2006), no. 1, 60–97.

Stability

MR4017416 Nagatou, K.; Plum, M.; McKenna, P. J.
Orbital stability investigations for travelling waves in a nonlinearly supported beam.
J. Differential Equations 268 (2019), no. 1, 80–114.

Today's talk focuses on even periodic solutions.

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad\qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

We seek a solution of the form

$$u(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx} \qquad\qquad (\text{impose } a_{-n} = a_n)$$

where $a \in \ell_\nu^1$ with $\nu > 1$.

Plugging this ansatz into the BVP leads to the system of equations

$$(n^4 - \beta n^2) a_n + b_n(a) = 0, \qquad\qquad n \in \mathbb{Z}$$

where $\{b_n\}_{n \in \mathbb{Z}}$ are the Fourier series coefficients of

$$f(u(x)) = \sum_{n \in \mathbb{Z}} b_n e^{inx}$$

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

Note that

$$[DF(a)h]_n = (n^4 - \beta n^2) h_n + [b'(a) * h]_n$$

where $b'(a)$ is the map on coefficient space induced by the function $f'(z)$.

How to truncate?

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

How to truncate?

Imagine that $f(z) = z^2$.

Then

$$b(a)_n = \sum_{k \in \mathbb{Z}} a_{n-k} a_k$$

Truncation to N modes $(a_{-N}, \ldots, a_N)$ is given by

$$[b(a^N)]_n^N = \sum_{\substack{j+k=n \\ |j|,|k| \leq N}} a_j^N a_k^N$$

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

How to truncate?

Polynomial $f(z)$ is similar.

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

How to truncate?

More interested in the case where $f(z)$ is analytic but not polynomial.

$$f(z) = e^z - 1$$

$$f(z) = \cos(z)$$

$$f(z) = \frac{1}{(1+z^2)^{2/3}}$$

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

How to truncate?

More interested in the case where $f(z)$ is analytic but not polynomial.

Let

$$u^N(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

Approximate the Fourier coefficients of

$$f\left(u^N(x)\right)$$

using the DFT/FFT.

... you make several kinds of mistakes here...

Example:

$$u^N(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

Approximate the Fourier coefficients of

$$f\left(u^N(x)\right)$$

Choose uniformly spaced $(x_{-N}, \ldots, x_N)$ in $[-\pi, \pi]$.

Evaluate

$$(f(u^N(x_{-N})), \ldots, f(u^N(x_N))) = (f_{-N}, \ldots, f_N)$$

Want $\{b_n\}_{n \in \mathbb{Z}}$ with

$$f(u^N(x)) = \sum_{n \in \mathbb{Z}} b_n e^{inx}$$

Instead, compute $\{\bar{b}_n\}_{n=-N}^{N}$ with

$$f_{-N} = \sum_{n=-N}^{N} \bar{b}_n e^{inx_{-N}} \quad \ldots \quad f_N = \sum_{n=-N}^{N} \bar{b}_n e^{inx_N}$$

That is: solve the linear system

$$\begin{bmatrix} e^{-iNx_{-N}} & \ldots & e^{iNx_{-N}} \\ \vdots & \ddots & \vdots \\ e^{-iNx_N} & \ldots & e^{iNx_N} \end{bmatrix} \begin{pmatrix} \bar{b}_{-N} \\ \vdots \\ \bar{b}_N \end{pmatrix} = \begin{pmatrix} f_{-N} \\ \vdots \\ f_N \end{pmatrix}$$

or

$$M\mathbf{b} = \mathbf{f}$$

One can work out $M^{-1}$by hand!

So

$$\mathbf{b} = M^{-1}\mathbf{f}$$

The FFT is just a fast algorithm for evaluating $u^N(x_j)$.

The IFFT is just a fast algorithm for evaluating the matrix vector product $M^{-1}\mathbf{f}$.

But!

The $\{\bar{b}_n\}_{n=-N}^{N}$ are only approximately the Fourier coefficients of $f\left(u^N(x)\right)$

... trig interpolation versus projection...

Aliasing error...

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad\qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad\qquad n \in \mathbb{Z}$$

How to truncate?

Given $\{a_n\}_{n=-N}^{N}$ let

$$\begin{aligned}
[b(a^N)]_n^N &= \bar{b}_n \qquad\qquad -N \leq n \leq N. \\
&\approx b_n \\
&= b(a^N)_n
\end{aligned}$$

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Define the map

$$F(a)_n = (n^4 - \beta n^2) a_n + b_n(a) \qquad n \in \mathbb{Z}$$

How to truncate?

Define $F^N : \mathbb{C}^{2N+1} \to \mathbb{C}^{2N+1}$ by

$$F^N\left(a^N\right) = (n^4 - \beta n^2) a_n^N + [b_n(a^N)]_n^N$$

Have

$$DF^N\left(a^N\right) h^N = (n^4 - \beta n^2) h_n^N + [c_n(a^N)]^N * h^N]_n^N$$

Where

$$[c_n(a^N)]^N$$

are the approximate Fourier coefficients of $f'(u^N(x))$ computed by DFT/FFT.

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.
Take $f(z) = e^z - 1$ and $\beta = 1.1$
Truncated map is

$$F^N\left(a^N\right) = (n^4 - \beta n^2) a_n^N + [b_n(a^N)]_n^N$$

Projecting onto $N = 30$ Fourier modes and applying Newton we find the following numerical approximate solution:

$$\bar{u}^N(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

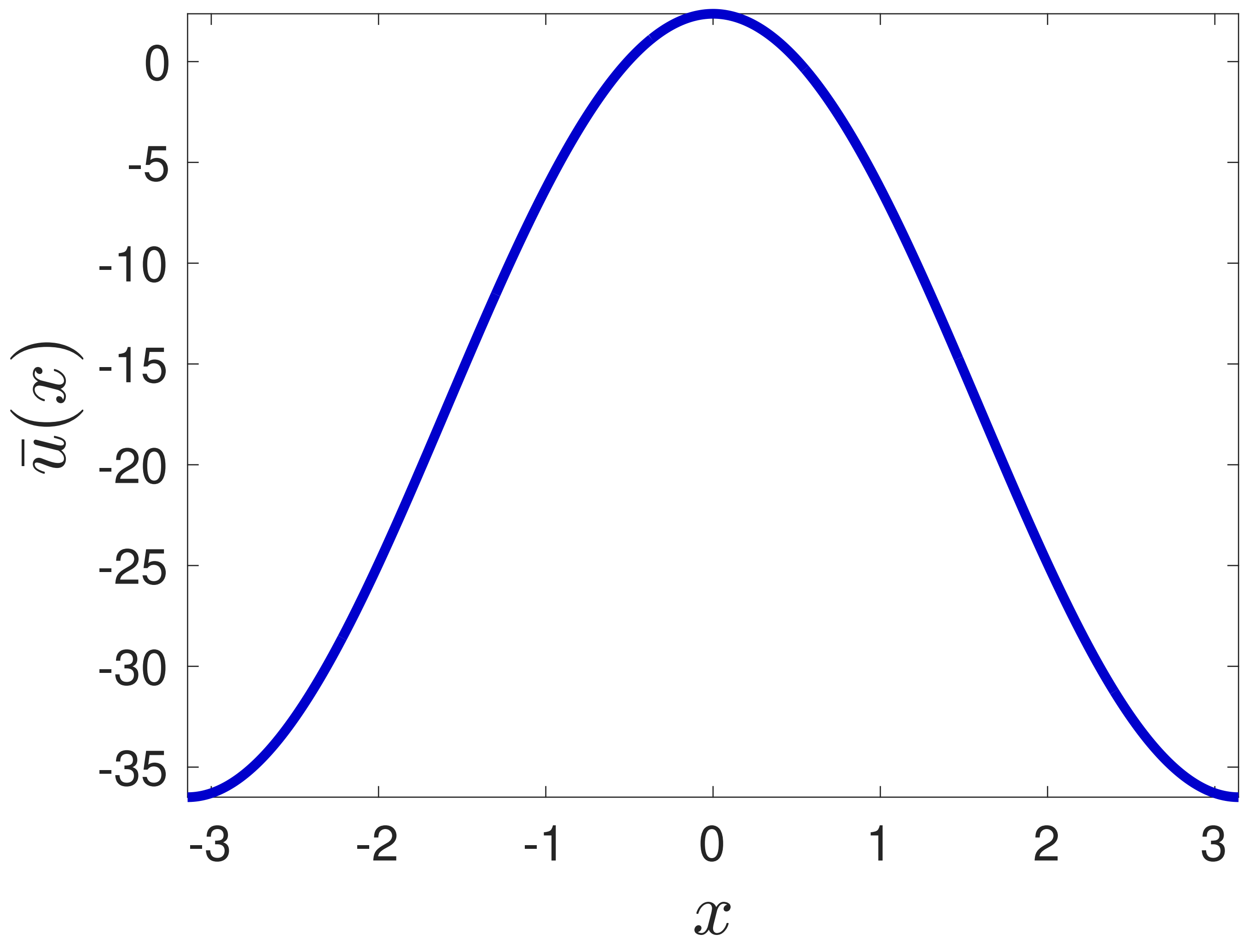


Figure 1: A periodic solution of the suspension bridge equation (14) at $\beta = 1.1$.

- $N = 30$ is sufficient to insure that a_n decay to machine precision.
- The nonlinearity and the Frechet derivative are evaluated via DFT/FFT.
- Based on the decay rate of the coefficients of $\bar{u}$ we guess an appropriate $\nu > 1$ in which to formulate the proof.

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad\qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.

Truncated map is

$$F^N\left(a^N\right) = (n^4 - \beta n^2) a_n^N + [b_n(a^N)]_n^N$$

and $\bar{a}^N = (\bar{a}_{-N}, \ldots, \bar{a}_N)$ has

$$F^N(\bar{a}^N) \approx 0$$

Let A^N be an approximate inverse (numerically computed) for $DF^N(\bar{a}^N)$.

Note that the derivative

$$[DF(a)h]_n = (n^4 - \beta n^2) h_n + [b'(a) * h]_n$$

has unbounded diagonal term

$$\mu_n = n^4 - \beta n^2$$

with explicit inverse

$$\mu_n^{-1} = \frac{1}{n^4 - \beta n^2}$$

Define the bounded linear operator $A \colon \ell_\nu^1 \to \ell_\nu^1$

$$[Ah]_n = \begin{cases} [A^N h^N]_n & \text{if } |n| \leq N \\ \mu_n^{-1} h_n & \text{if } |n| > N \end{cases}$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Define the map

$$F(a)_n = (n^4 - \beta n^2)a_n + b_n(a) \qquad n \in \mathbb{Z}$$

Truncated map is

$$F^N\left(a^N\right) = (n^4 - \beta n^2)a_n^N + [b_n(a^N)]_n^N$$

and $\bar{a}^N = (\bar{a}_{-N}, \ldots, \bar{a}_N)$ has

$$F^N(\bar{a}^N) \approx 0$$

Define the bounded linear operator $A \colon \ell_\nu^1 \to \ell_\nu^1$

$$[Ah]_n = \begin{cases} [A^N h^N]_n & \text{if } |n| \leq N \\ \mu_n^{-1} h_n & \text{if } |n| > N \end{cases}$$

Then

$$F(\bar{a}^N)_n = \begin{cases} (n^4 - \beta n^2)\bar{a}_n + \bar{b}_n + \epsilon_n & |n| \leq N \\ b_n & |n| \geq N \end{cases}$$

and

$$[AF(\bar{a}^N)]_n = \begin{cases} A^N F^N(\bar{a}^N)_n + [A^N \epsilon]_n & |n| \leq N \\ \frac{b_n}{n^4 - \beta n^2} & |n| \geq N \end{cases}$$

Then

$$\|AF(\bar{a}^N)\|_\nu = \sum_{n \in \mathbb{Z}} \left| [AF(\bar{a}^N)]_n \right| \nu^{|n|}$$

$$= \sum_{|n| \leq N} \left| [A^N F^N(\bar{a}^N)]_n \right| \nu^{|n|} + \sum_{|n| \leq N} \left| [A^N \epsilon]_n \right| \nu^{|n|} + \sum_{|n| > N} \frac{|b_n|}{n^4 - \beta n^2} \nu^{|n|}$$

Need bounds on

$$\epsilon_n = b_n - \bar{b}_n \qquad |n| \leq N$$

and

$$|b_n| \qquad |n| > N$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Need bounds on

$$\epsilon_n = b_n - \bar{b}_n \qquad |n| \leq N$$

and

$$|b_n| \qquad |n| > N$$

Strategy:

Choose an N_{tail} so that the Fourier coefficients of $f \circ \bar{u}$ decay to machine precision.

Choose an $N_{\text{FFT}} > 2N_{\text{tail}}$ (a power of 2) which will be used to control the aliasing.

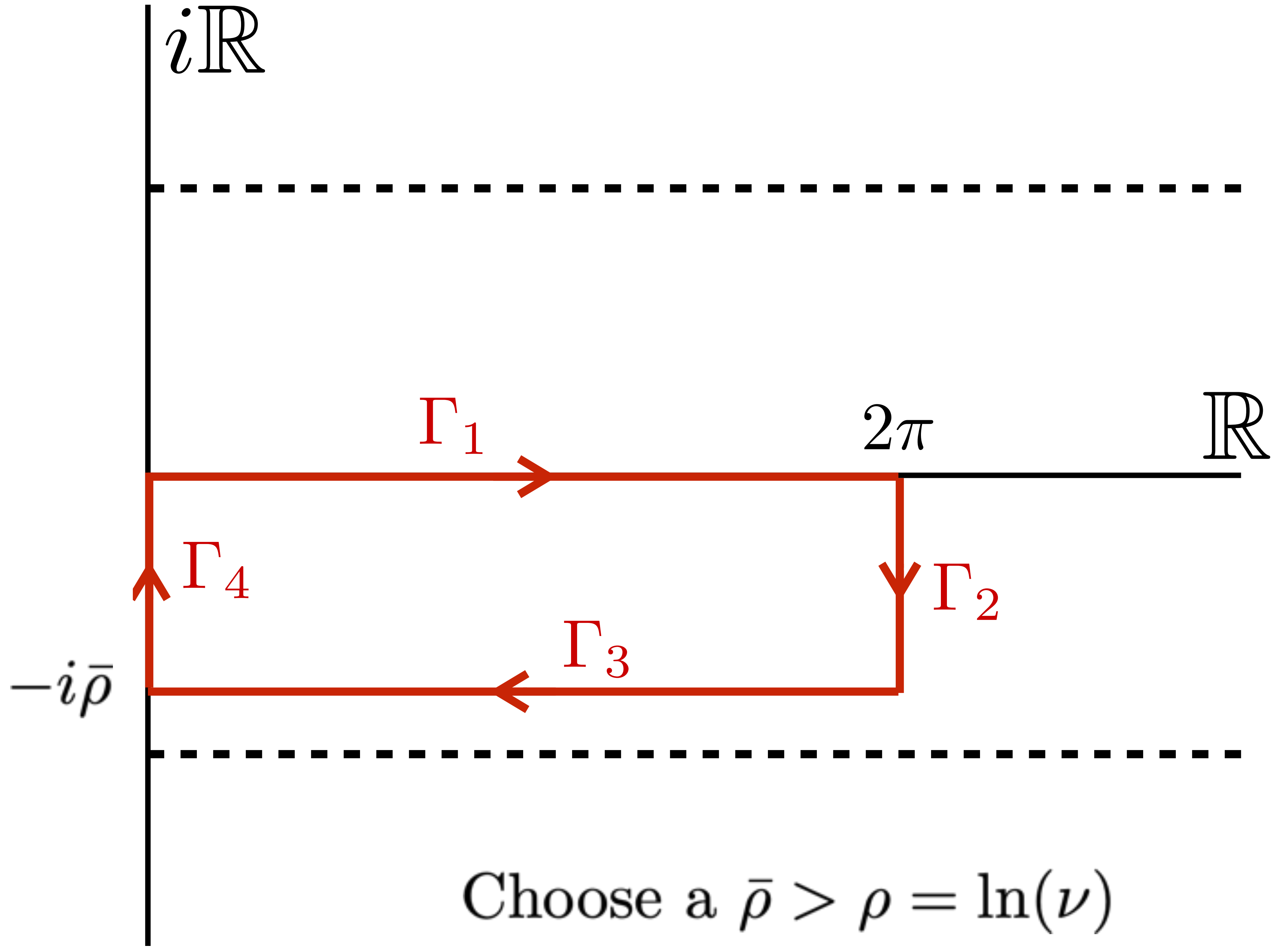
The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Need bounds on

$|b_n| \qquad |n| > N_{\text{tail}}$

For $n > 0$

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{2\pi} f(\bar{u}(x))e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{\Gamma_1} f(\bar{u}(z))e^{-inz}dz \\ &= -\frac{1}{2\pi}\int_{\Gamma_3} f(\bar{u}(z))e^{-inz}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\bar{u}(x-i\bar{\rho}))e^{-in(x-i\bar{\rho})}dx \\ &= \frac{1}{2\pi}e^{-n\bar{\rho}}\int_0^{2\pi} f(\bar{u}(x-i\bar{\rho}))e^{-inx}dx. \end{aligned}$$



Choose a $\bar{\rho} > \rho = \ln(\nu)$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Need bounds on

$|b_n| \qquad |n| > N_{\text{tail}}$

For $n > 0$

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{2\pi} f\left(\bar{u}(x)\right)e^{-inx}dx \\ &= \frac{1}{2\pi}\int_{\Gamma_1} f\left(\bar{u}(z)\right)e^{-inz}dz \\ &= -\frac{1}{2\pi}\int_{\Gamma_3} f\left(\bar{u}(z)\right)e^{-inz}dz \\ &= \frac{1}{2\pi}\int_0^{2\pi} f\left(\bar{u}(x - i\bar{\rho})\right)e^{-in(x - i\bar{\rho})}dx \\ &= \frac{1}{2\pi}e^{-n\bar{\rho}}\int_0^{2\pi} f\left(\bar{u}(x - i\bar{\rho})\right)e^{-inx}dx. \end{aligned}$$

Then

$$\begin{aligned} |b_n| &\leq \frac{1}{2\pi e^{n\bar{\rho}}}\int_0^{2\pi} \left|f\left(\bar{u}(x - i\bar{\rho})\right)\right| |e^{-inx}|dx \\ &\leq \frac{1}{\bar{\nu}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} \left| f(\bar{u}(x - i\bar{\rho}))\right| dx\right) \end{aligned}$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \le Y$

Need bounds on

$|b_n| \qquad |n| > N_{\text{tail}}$

$$\frac{1}{2\pi}\int_0^{2\pi} |\, f(\bar{u}(x - i\bar{\rho}))|\, dx \le C_{\bar{\rho}}.$$

Consider a number $N_{\texttt{FFT}}$ (a power of 2) and a uniform mesh of $[0, 2\pi]$ of mesh size $\frac{2\pi}{N_{\texttt{FFT}}}$, that is

$$x_k = \frac{2k\pi}{N_{\texttt{FFT}}}, \qquad k = 0, \ldots, N_{\texttt{FFT}}.$$

$N_{\texttt{FFT}} \gg 2N_{\text{tail}}$

For each $k = 0, \ldots, N_{\texttt{FFT}}$, denote by $C_{\bar{\rho}}^k$ an upper bound satisfying

$$\sup_{x \in [x_k, x_{k+1}]} |f(\bar{u}(x - i\bar{\rho}))| \le C_{\bar{\rho}}^k.$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\bar{u}(x - i\bar{\rho}))|\, dx = \frac{1}{2\pi}\sum_{k=0}^{N_{\texttt{FFT}}-1}\int_{x_k}^{x_{k+1}} |f(\bar{u}(x - i\bar{\rho}))|\, dx \le \frac{1}{N_{\texttt{FFT}}}\sum_{k=0}^{N_{\texttt{FFT}}-1} C_{\bar{\rho}}^k.$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Need bounds on

$$|b_n| \qquad |n| > N_{\text{tail}}$$

To find $C^k_{\bar\rho}$ with

$$\sup_{x\in[x_k,x_{k+1}]} |f(\bar u(x-i\bar\rho))| \leq C^k_{\bar\rho}.$$

is an interval FFT:

First note that

$$\bar u(x-i\bar\rho) = \sum_{|n|\leq N} \bar a_n e^{in(x-i\bar\rho)}$$
$$= \sum_{|n|\leq N} (\bar a_n e^{n\bar\rho})e^{inx},$$

For a given $x \in [x_k, x_{k+1}]$, there exists $\delta \in \left[0, \frac{2\pi}{N_{\text{FFT}}}\right]$ such that $x = x_k + \delta$. Hence,

$$\begin{aligned} f(\bar u(x-i\bar\rho)) &= f\left(\sum_{|n|\leq N} (\bar a_n e^{n\bar\rho})e^{inx}\right) \\ &= f\left(\sum_{|n|\leq N} (\bar a_n e^{n(\bar\rho+i\delta)})e^{inx_k}\right) \\ &= f\left(\sum_{n=-\frac{N_{\text{FFT}}}{2}}^{\frac{N_{\text{FFT}}}{2}-1} \alpha_n e^{inx_k}\right), \end{aligned}$$

where

$$\alpha_n \stackrel{\text{def}}{=} \begin{cases} \bar a_n e^{n(\bar\rho+i\delta)}, & n = -N,\dots,N \\ 0, & n \in \{-\frac{N_{\text{FFT}}}{2},\dots,-N-1\} \cup \{N+1,\dots,\frac{N_{\text{FFT}}}{2}-1\}. \end{cases}$$

Now use interval arithmetic to compute an interval $\boldsymbol{\alpha}_n$ such that

$$\alpha_n = \bar a_n e^{n(\bar\rho+i\delta)} \in \boldsymbol{\alpha}_n, \quad \text{for all } \delta \in \left[0, \frac{2\pi}{N_{\text{FFT}}}\right].$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Need bounds on

$|b_n| \qquad |n| > N_{\text{tail}}$

To find $C_{\bar{\rho}}^k$ with

$$\sup_{x \in [x_k, x_{k+1}]} |f(\bar{u}(x - i\bar{\rho}))| \leq C_{\bar{\rho}}^k.$$

Denote the vector of intervals $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}_n\}_{n=-\frac{N_{\mathtt{FFT}}}{2}}^{\frac{N_{\mathtt{FFT}}}{2}-1}$, where $\boldsymbol{\alpha}_n = [0,0]$ for $|n| > N$. Then combining the FFT and interval arithmetic, compute

$$\sup_{x \in [x_k, x_{k+1}]} |f(\bar{u}(x - i\bar{\rho}))| \leq C_{\bar{\rho}}^k \stackrel{\text{def}}{=} \sup |f((\mathtt{FFT}(\boldsymbol{\alpha}))_k)|,$$

Finally, we apply the same approach to $-\bar{\rho}$ to get $C_{-\bar{\rho}}$ such that $|b_n| \leq \frac{C_{-\bar{\rho}}}{\bar{\nu}^{-n}}$ for all $n < 0$. Denoting

$$C \stackrel{\text{def}}{=} \max\{C_{-\bar{\rho}}, C_{\bar{\rho}}\}$$

we get that

$$|b_n| \leq \frac{C}{\nu^{|n|}}, \quad \text{for all } n \in \mathbb{Z}.$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

$$|b_n| \leq \frac{C}{\nu^{|n|}}, \quad \text{for all } n \in \mathbb{Z}.$$

Then

$$\|AF(\bar{a}^N)\|_\nu = \sum_{n\in\mathbb{Z}} \left|[AF(\bar{a}^N)]_n\right| \nu^{|n|}$$

$$= \sum_{|n|\leq N} \left|[A^N F^N(\bar{a}^N)]_n\right| \nu^{|n|} + \sum_{|n|\leq N} \left|[A^N \epsilon]_n\right| \nu^{|n|} + \sum_{|n|>N} \frac{|b_n|}{n^4 - \beta n^2} \nu^{|n|}$$

$$= \sum_{|n|\leq N} \left|[A^N F^N(\bar{a}^N)]_n\right| \nu^{|n|} + \sum_{|n|\leq N} \left|[A^N \epsilon]_n\right| \nu^{|n|} + \sum_{N<|n|< N_{\text{tail}}} \frac{|\bar{b}_n| + \epsilon_n}{n^4 - \beta n^2} \nu^{|n|} + \sum_{|n|> N_{\text{tail}}} \frac{|b_n|}{n^4 - \beta n^2} \nu^{|n|}$$

$$= \sum_{|n|\leq N} \left|[A^N F^N(\bar{a}^N)]_n\right| \nu^{|n|} + \sum_{|n|\leq N} \left|[A^N \epsilon]_n\right| \nu^{|n|} + \sum_{N<|n|< N_{\text{tail}}} \frac{|\bar{b}_n| + \epsilon_n}{n^4 - \beta n^2} \nu^{|n|} + \frac{2C}{N_{\text{tail}}^4 - \beta\, N_{\text{tail}}^2} \sum_{n= N_{\text{tail}}}^{\infty} \left(\frac{\nu}{\bar{\nu}}\right)^{|n|}$$

$$= \sum_{|n|\leq N} \left|[A^N F^N(\bar{a}^N)]_n\right| \nu^{|n|} + \sum_{|n|\leq N} \left|[A^N \epsilon]_n\right| \nu^{|n|} + \sum_{N<|n|< N_{\text{tail}}} \frac{|\bar{b}_n| + \epsilon_n}{n^4 - \beta n^2} \nu^{|n|} + \frac{2C}{N_{\text{tail}}^4 - \beta\, N_{\text{tail}}^2} \frac{\left(\frac{\nu}{\bar{\nu}}\right)^{N_{\text{tail}}+1}}{1 - \frac{\nu}{\bar{\nu}}}$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Just need the ϵ_n, for $|n| \leq N_{\text{FFT}}$

$$|b_n| \leq \frac{C}{\nu^{|n|}}, \quad \text{for all } n \in \mathbb{Z}.$$

Recall the so called "discrete Poisson summation formula":

$$\epsilon_n = b_n - \bar{b}_n = -\sum_{j=1}^{\infty} (b_{n+jN_{FFT}} + b_{n-jN_{FFT}})$$

See for example Ch 6 of *The DFT: An Owner's Manual for the Discrete Fourier Transform* by Briggs and Henson.

$$\begin{aligned}
|\varepsilon_n| &\leq \sum_{j=1}^{\infty} |b_{n+jN_{\text{FFT}}}| + |b_{n-jN_{\text{FFT}}}| \\
&\leq C \sum_{j=1}^{\infty} \left(\frac{1}{\bar{\nu}^{n+jN_{\text{FFT}}}} + \frac{1}{\bar{\nu}^{jN_{\text{FFT}}-n}} \right) \\
&= C \left(\frac{1}{\bar{\nu}^n} + \bar{\nu}^n \right) \sum_{j=1}^{\infty} \left(\frac{1}{\bar{\nu}^{jN_{\text{FFT}}}} + \frac{1}{\bar{\nu}^{jN_{\text{FFT}}}} \right) \\
&= 2C \left(\frac{1}{\bar{\nu}^n} + \bar{\nu}^n \right) \sum_{j=1}^{\infty} \left(\frac{1}{\bar{\nu}^{N_{\text{FFT}}}} \right)^j \\
&= 2C \left(\frac{1}{\bar{\nu}^n} + \bar{\nu}^n \right) \frac{1}{\bar{\nu}^{N_{\text{FFT}}} - 1}.
\end{aligned}$$

The Defect Bound: $\|AF(\bar{a})\|_\nu \leq Y$

Just need the ϵ_n, for $|n| \leq N_{\text{FFT}}$

$$\|AF(\bar{a}^N)\|_\nu = \sum_{n\in\mathbb{Z}} \left|[AF(\bar{a}^N)]_n\right| \nu^{|n|}$$

$$= \sum_{|n|\leq N} \left|[A^N F^N(\bar{a}^N)]_n\right| \nu^{|n|} + \sum_{|n|\leq N} \left|[A^N \epsilon]_n\right| \nu^{|n|} + \sum_{N<|n|< N_{\mathtt{tail}}} \frac{|\bar{b}_n| + \epsilon_n}{n^4 - \beta n^2}\nu^{|n|} + \frac{2C}{N_{\mathtt{tail}}^4 - \beta\, N_{\mathtt{tail}}^2} \frac{\left(\frac{\nu}{\bar{\nu}}\right)^{N_{\mathtt{tail}}+1}}{1 - \frac{\nu}{\bar{\nu}}}$$

where

$$|\varepsilon_n| \leq 2C\left(\frac{1}{\bar{\nu}^n} + \bar{\nu}^n\right)\frac{1}{\bar{\nu}^{N_{\text{FFT}}} - 1}$$

Approximate derivative:

$$\|A[DF(\bar{a}) - A^{\dagger}]\|_{B(X)} \leq Z_1$$

Have to perform a similar analysis for

$$(b' * h)_n = \sum_{n_1+n_2=n} b'_{|n_1|} h_{|n_2|},$$

Where b' are the Fourier coefficients of $f' \circ \bar{u}$.

Use the fact that this is bounded and linear in h, hence in the dual space.
(Du8al of ℓ^1_ν is a weighted little ℓ infinity space)
Get explicit finite formulas that only require computing FFT for $f'(z)$.

Local estimate:

$$\|A[DF(c) - DF(\bar{a})]\|_{B(X)} \leq Z_2(r)r, \quad \textit{for all } c \in \overline{B_r(\bar{a})} \textit{ and all } r > 0.$$

$$\begin{aligned}
\|[DF(c) - DF(\bar{a})]h_2\|_{1,\omega} &= \|[D\,f(\bar{a} + rh_1) - D\,f\,(\bar{a})]h_2\|_1 \\
&\leq r\|[D^2\,f\,(\bar{a} + rh)](h_1, h_2)\|_1 \\
&\leq r\|[D^2\,f\,(\bar{a} + r^*h)](h_1, h_2)\|_1 \\
&\leq r\|e^{\bar{a}+r^*h}\|_{1,\omega}\|h_1\|_1 \quad \|h_2\|_1 \\
&\leq r\|e^{\bar{a}+r^*h}\|_1 \\
&\leq r\|e^{\bar{a}}\|_{1,\omega}\|\|e^{r^*h}\|_1 \\
&\leq re^{r^*}\|e^{\bar{a}}\|_1
\end{aligned}$$

for any $r^* \geq r$, where the last inequality follows from using the Banach algebra. Hence, we set

$$Z_2 \stackrel{\text{def}}{=} \|A\|_{B(\ell^1)}e^{r^*}\|e^{\bar{a}}\|_1$$

Example: consider the scalar boundary value problem

$$u'''' + \beta u'' + f(u) = 0 \qquad\qquad x \in [-\pi, \pi]$$

with f an analytic function, and Neumann boundary conditions.
Take $f(z) = e^z - 1$ and $\beta = 1.1$

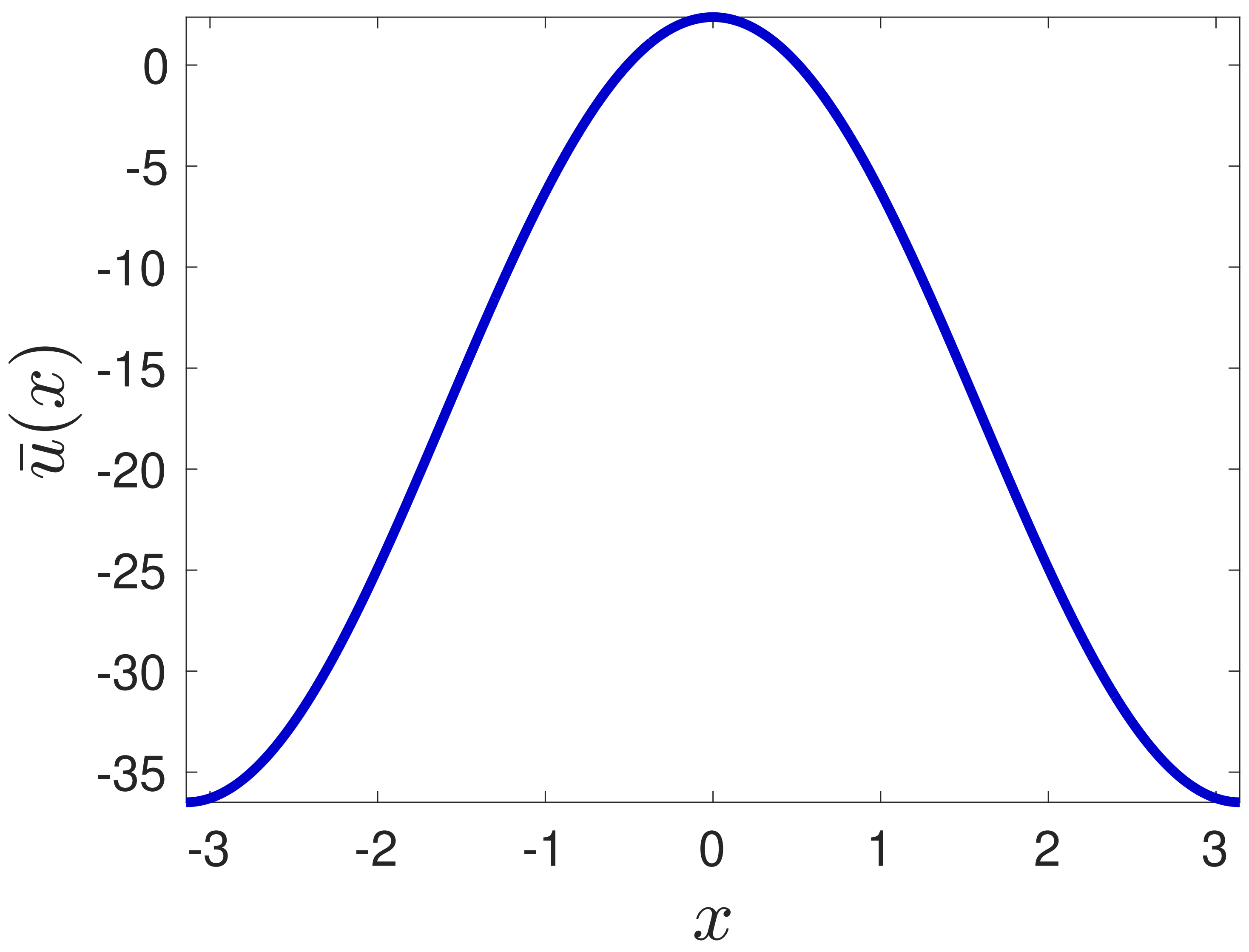


Figure 1: A periodic solution of the suspension bridge equation (14) at $\beta = 1.1$.

$N = 35$

$N_{\text{tail}} = 60$

$N_{\text{FFT}} = 512$

$\nu = 1.3$ $\qquad \ln(\nu) \approx 0.262$

$r_* = 10^{-5}$

$\vdots$

$Y_0 \approx 3.59705 \times 10^{-12}$

$r \approx 3.59714 \times 10^{-12}$

Proof takes about 1.1 seconds.

–The estimates just require that we can evaluate f and f' on interval data.

–Also need a local bound on $\|f''(u)\|$.

So changing the nonlinearity just means changing these formulas.

We currently have this implemented in MatLab/IntLab, but a much more general implementation is possible.

Extension to Taylor and Chebyshev is similar.

Multivariable extensions are also possible (as Jordi described)

Thank you to the organizers for the invitation!

Thank you for listening!