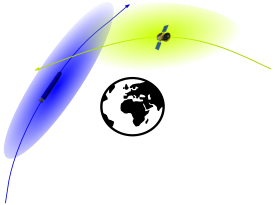
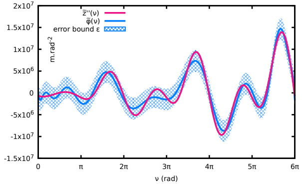
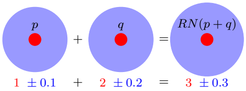
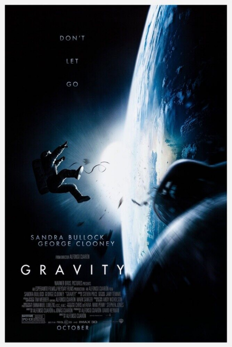
Mioara Joldes
http://homepages.laas.fr/mmjoldes/

Ten Years of Space Junk and related Symbolic-Numeric Algorithms









How it started? How it's going?

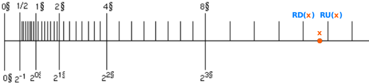
Joint works with D. Arzelier, F. Bréhard, M. Masson, J.-B. Lasserre, B. Salvy, R. Serra

Research Context

Can/Should we trust the numerics?

Numerics: floating-point arithmetic $\rightsquigarrow$ FAST

- Global optimization, systems of diff. equations, integration
- Usually, solutions lack certification of the output accuracy



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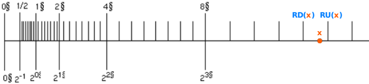

Computer Algebra Systems (eg. Maple) $\rightsquigarrow$ EXACT

Usually, pure symbolic methods are scarce or computationally expensive

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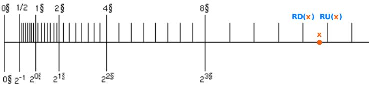

Need of **fast and certified** sols:

computer-aided proofs

safety-critical applications e.g., control units of aircraft, particle accelerators autonomous GNC of spacecraft

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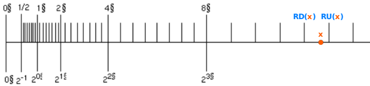

Validated numerics:

Speed & reliability via set-based computations

efficient numerics + bounds on roundoff, discretization, truncation errors

(Moore'62, Kulisch, Kaucher, Miranker '80s, Makino&Berz '90, Tucker'99, Lessard'12, Nakao&Plum'19...)

Difficult task for generic classes of numerical routines.

Need of **fast and certified** sols:

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safety-critical applications e.g., control units of aircraft, particle accelerators autonomous GNC of spacecraft

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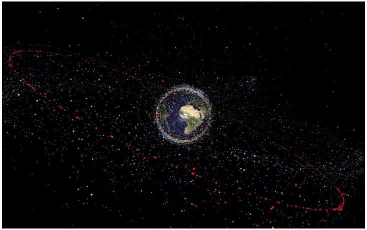
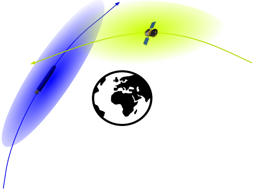


Space debris population model (source : ESA)

Cerise hit by a debris in 1996 (CNES/D. Ducros)

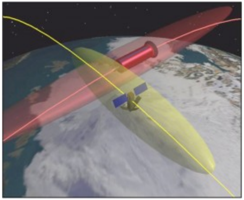


Conjunction illustration

Thomas Pesquet, when a debris whizzes past the ISS:**

"Climb into an escape shuttle, wait and hope. This happened four times"

** http://www.chron.com/news/science-environment/article/Thousands-of-tiny-satellites-are-about-to-go-into-11088984.php



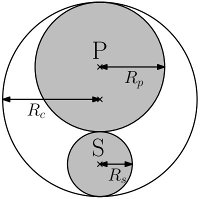
Two objects: primary P (operational satellite) and secondary S (space debris)

High relative velocity

Assumptions:

- Rectilinear relative motion
- No velocity uncertainty; Gaussian position uncertainty
- Infinite encounter time horizon

$\rightsquigarrow$ Probability of collision: 2-D integral over a disk.

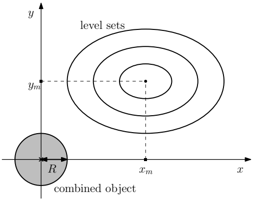


Formula

$$\mathcal{P} = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\mathcal{B}((0,0),R)} \exp\left(-\frac{(x-x_m)^2}{2\sigma_x{}^2} - \frac{(y-y_m)^2}{2\sigma_y{}^2}\right) \mathrm{d}x\mathrm{d}y,$$

where

- R: radius of combined object
- x_m, y_m: mean relative coordinates
- σ_x, σ_y: standard deviations of relative coordinates



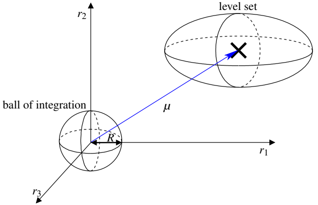
Short-term encounter model and probability of collision

3D generalization: instantaneouous collision probability

$$\mathcal{P}_{inst} = \frac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}} \int_{\mathcal{B}(0,R)} \exp\left(-\frac{1}{2}(r-\mu)^T\Sigma^{-1}(r-\mu)\right) \mathrm{d}r$$

Need for fast and reliable evaluation:

Simple, numerically stable, double-prec evaluation & effective error bounds.



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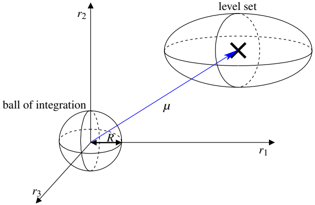

Previous works:
2D: Numerical integration (Foster '92, Patera '01, Alfano '05);
Power/Hermite series (Pelayo-Ayuso'16), with *trial and error* truncation,
or simplifying assumptions ($\sigma_x = \sigma_y$) (Chan '97);

3D: Equivalent volume –cuboids, approx distribution– (Chan '08, Zhang '20)

$$\begin{tikzcd}
\boxed{g(z) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\mathcal{B}((0,0),\sqrt{z})} \exp\left(-\frac{(x-x_m)^2}{2{\sigma_x}^2} - \frac{(y-y_m)^2}{2{\sigma_y}^2}\right) \mathrm{d}x\mathrm{d}y} \arrow[d, "\text{Laplace Transform}"] & \boxed{= \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}} \\
\boxed{\mathcal{L}(g)(\lambda) = \frac{\exp\left(-\lambda\left(\frac{{x_m}^2}{2\lambda{\sigma_x}^2+1} + \frac{{y_m}^2}{2\lambda{\sigma_y}^2+1}\right)\right)}{\lambda\sqrt{(2\lambda{\sigma_x}^2+1)(2\lambda{\sigma_y}^2+1)}}} \arrow[r, "\text{expansion at } \infty"] & \boxed{= \sum_{i=0}^{\infty} \ell_i \left(\frac{1}{\lambda}\right)^i} \arrow[u, "\text{Borel}"']
\end{tikzcd}$$

Lasserre, Zerron (2001)

$\mathcal{L}(g)$ is D-finite – solution of linear ODE with polynomial coefficients –

ℓ_i can be efficiently symbolically described and computed
–solution of linear recurrence with polynomial coefficients–

$\rightsquigarrow$Gfun Maple package (Salvy, Zimmermann 1994)

Orbital collision probability evaluation

Step 1: Symbolic representation

$$g(z) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{\mathcal{B}((0,0),\sqrt{z})} \exp\left(-\frac{(x-x_m)^2}{2\sigma_x{}^2} - \frac{(y-y_m)^2}{2\sigma_y{}^2}\right) \mathrm{d}x\mathrm{d}y$$

Laplace Transform

Lasserre, Zerron (2001)

$$\mathcal{L}(g)(\lambda) = \frac{\exp\left(-\lambda\left(\frac{x_m{}^2}{2\lambda\sigma_x{}^2+1} + \frac{y_m{}^2}{2\lambda\sigma_y{}^2+1}\right)\right)}{\lambda\sqrt{(2\lambda\sigma_x{}^2+1)(2\lambda\sigma_y{}^2+1)}}$$

expansion at ∞

$$= \sum_{i=0}^{\infty} \ell_i \left(\frac{1}{\lambda}\right)^i$$

Borel

$$\simeq \sum_{i=0}^{N} \frac{\ell_i}{(i+1)!} z^{i+1}$$

Numerical Issues

$\mathcal{L}(g)$ is D-finite – solution of linear ODE with polynomial coefficients –

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Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

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Cancellation in finite precision power series evaluation

Example: $\sigma_x = 115, \sigma_y = 1.41, x_m = 0.15, y_m = 3.88, \sqrt{z} = 15$

$$g(z) = \sum_{i=0}^{\infty} \frac{\ell_i}{(i+1)!} z^{i+1}$$

$$g(225) = 0.16 \cdot 10^{-1} + 1.5 + 16.1 - 250 \ldots + 2.2 \cdot 10^{19} - 2.6 \cdot 10^{19} - \ldots + 4.3 - 0.14 - 0.60 \ldots$$

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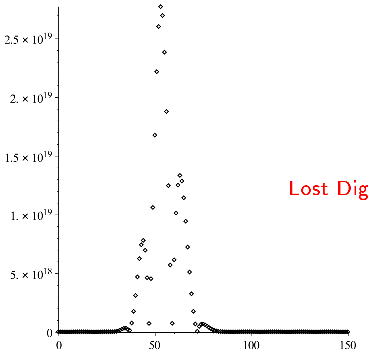
Values of $\left|\frac{\ell_i 225^{i+1}}{(i+1)!}\right|$, compared to $g(225) \simeq 0.1004$:



Lost Digits: $d_g(z) \simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

Cancellation in finite precision power series evaluation

Example: $\exp(-x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}$

$$\exp(-20) = 1 - 20 \ldots + 1.66 \cdot 10^7 - 1.23 \cdot 10^7 + \ldots + 1.19 \cdot 10^{-8} - 3.45 \cdot 10^{-9} \ldots$$

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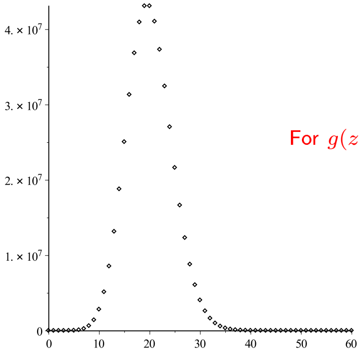
Values of $\left|\frac{(-1)^i 20^i}{i!}\right|$, compared to $\exp(-20) \simeq 2.06 \cdot 10^{-9}$:



For $g(z) = \sum_{i=0}^{\infty} g_i z^i$, lost digits $\simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

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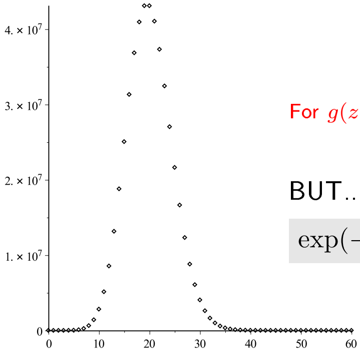

For $g(z) = \sum_{i=0}^{\infty} g_i z^i$, lost digits $\simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

BUT...

$$\exp(-x) = \frac{1}{\exp(x)}$$

No cancellation!

Cancellation in finite precision power series evaluation

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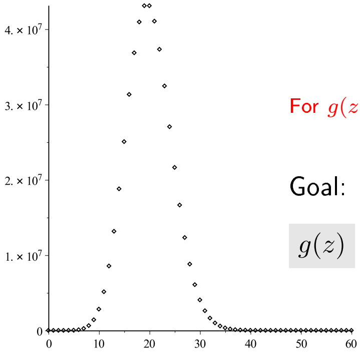


For $g(z) = \sum_{i=0}^{\infty} g_i z^i$, lost digits $\simeq \log \frac{\max_i |g_i z^i|}{|g(z)|}$

Goal:

$$g(z) = \frac{F(z)}{G(z)}$$

No cancellation!

Orbital collision probability evaluation

Step 2: Reliable numerics

$$
\begin{tikzcd}
\underbrace{G(z)\,g(z)}_{F(z)} = G(z) \displaystyle\int \dots \arrow[d, "\text{Laplace Transform}"] & & \displaystyle = \sum_{i=0}^{\infty} \frac{f_i}{(i+1)!} z^{i+1} \\
\mathcal{L}(F)(\lambda) = \mathcal{L}(g)(\lambda - p) = \dots \arrow[rr, "\text{expansion at } \infty"] & & \displaystyle = \sum_{i=0}^{\infty} f_i \left(\frac{1}{\lambda}\right)^i \arrow[u, "\text{Borel}"']
\end{tikzcd}
$$

Gawronski, Müller, Reinhard (2007): choose $G(z) = \exp(p\,z)$, with $p \sim \dfrac{1}{2\sigma_y^2}$.

$\mathcal{L}(F)$ is D-finite

f_i can be efficiently symbolically described and computed
–solution of linear recurrence with polynomial coefficients–

$$\underbrace{G(z)\,g(z)}_{F(z)} = G(z)\int \dots$$

Laplace Transform

$$\mathcal{L}(F)(\lambda) = \mathcal{L}(g)(\lambda - p) = \dots$$

expansion at ∞

$$= \sum_{i=0}^{\infty} f_i \left(\frac{1}{\lambda}\right)^i$$

Borel

Reduced Cancellation

$$\simeq \sum_{i=0}^{N} \frac{f_i}{(i+1)!} z^{i+1}$$

Gawronski, Müller, Reinhard (2007): choose $G(z) = \exp(p\,z)$, with $p \sim \dfrac{1}{2\sigma_y^2}$.

$\mathcal{L}(F)$ is D-finite

f_i can be efficiently symbolically described and computed
–solution of linear recurrence with polynomial coefficients–

f_i are positive $\rightsquigarrow$ Reduced Cancellation

Positivity:

Consider $\bar{\varphi}(z) = \sum_{k=0}^{\infty} f_i z^i$, for $|z| \leq p^{-1}$, which is D-finite

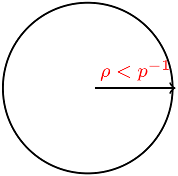
$$\bar{\varphi}'(z) = \underbrace{\frac{P(z)}{Q(z)}}_{\sum_{k=0}^{\infty} \beta_k z^k} \bar{\varphi}(z), \quad \bar{\varphi}(0) = C > 0,$$

with $\beta_k = p^{k+1} + \dots \qquad > 0$, hence

$$(i+1) f_{i+1} = \sum_{k=0}^{i} \beta_k f_{i-k} > 0.$$

$$\begin{tikzcd} \boxed{F(z) = \sum_{i=0}^{\infty} \frac{f_i}{(i+1)!} z^{i+1}} \\ \boxed{\bar{\varphi}(z) = \sum_{i=0}^{\infty} f_i z^i} \arrow[u, "\text{Borel}"] \end{tikzcd}$$

Closed-form; Conv. radius p^{-1}



Tail bounds:

$$\sum_{k=0}^{\infty} \frac{f_{k+n} z^{k+n}}{(k+n+1)!}$$

$$\leq \frac{(\frac{z}{\rho})^n}{(n+1)!} \bar{\varphi}(\rho)$$

for any $\rho < p^{-1}$ and $n \geq n_0$, with $n_0 = \left\lceil \frac{z}{\rho} \right\rceil$.

$$\boxed{F(z)=\sum_{i=0}^{\infty}\frac{f_i}{(i+1)!}z^{i+1}}$$

$$\Big\uparrow\ \text{Borel}$$

$$\boxed{\bar{\varphi}(z)=\sum_{i=0}^{\infty}f_i z^i}$$

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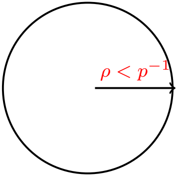

Tail bounds:

$$\sum_{k=0}^{\infty}\frac{f_{k+n}z^{k+n}}{(k+n+1)!}\leq\frac{(\frac{z}{\rho})^n}{(n+1)!}\sum_{k=0}^{\infty}f_{k+n}z^k\rho^n\underbrace{\frac{(n+1)!}{(k+n+1)!}}_{\leq 1/(n+1)^k}$$

$$\leq\frac{(\frac{z}{\rho})^n}{(n+1)!}\underbrace{\sum_{k=0}^{\infty}f_{k+n}\rho^{k+n}\underbrace{\frac{z^k}{\rho^k(n+1)^k}}_{\leq 1}}_{\bar{\varphi}(\rho)}$$

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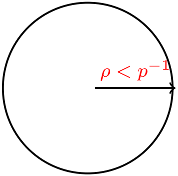

Tail bounds:

$$\sum_{k=0}^{\infty}\frac{f_{k+n}z^{k+n}}{(k+n+1)!} \le \frac{(\frac{z}{\rho})^n}{(n+1)!}\sum_{k=0}^{\infty} f_{k+n}z^k\rho^n \underbrace{\frac{(n+1)!}{(k+n+1)!}}_{\le 1/(n+1)^k}$$

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$$\le \frac{(\frac{z}{\rho})^n}{(n+1)!}\bar{\varphi}(\rho)$$

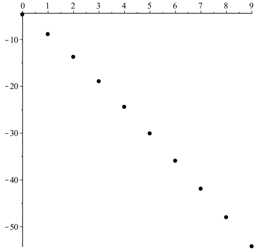
for any $\rho < p^{-1}$ and $n \ge n_0$, with $n_0 = \left\lceil \frac{z}{\rho} \right\rceil$.

In particular, set $\rho = p^{-1}/2$ and $z = R^2$.

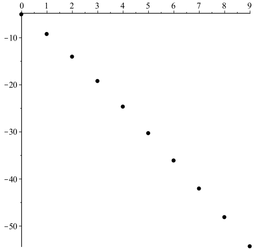
Case #	Input parameters (km)				
	σ_x	σ_y	R	x_m	y_m
1	0.05	0.025	0.005	0.01	0
2	0.05	0.025	0.005	0	0.01
3	0.075	0.025	0.005	0.01	0
4	0.075	0.025	0.005	0	0.01
5	3	1	0.01	1	0
6	3	1	0.01	0	1
7	3	1	0.01	10	0
8	3	1	0.01	0	10
9	10	1	0.01	10	0
10	10	1	0.01	0	10
11	3	1	0.05	5	0
12	3	1	0.05	0	5

Examples: accuracy η and $\log$ plot of terms in the series

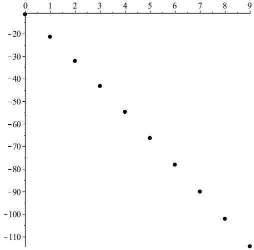
Case 1: $\eta = 23$



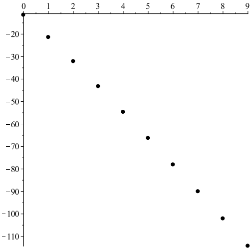
Case 2: $\eta = 22$



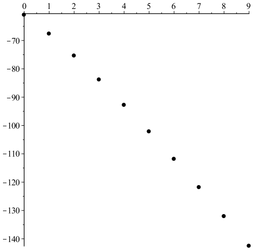
Case 4: $\eta = 22$



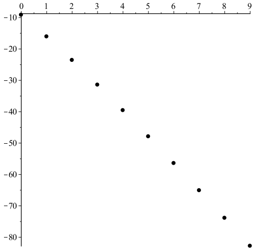
Case 6: $\eta = 47$



Case 8: $\eta = 33$



Case 11: $\eta = 34$



Efficient and reliable algorithm (2016)*

Error bounds

Linear complexity

Algorithm 12 Approximate evaluation of $\mathcal{P}_c$.

Input: Parameters: $\sigma_x, \sigma_y, x_m, y_m, R$; Number of terms: N.

Output: $\tilde{\mathcal{P}}_c$ – truncated series evaluation of $\mathcal{P}_c$.

$p = \frac{1}{2\sigma_y^2}$; $\varphi = 1 - \frac{\sigma_y^2}{\sigma_x^2}$; $\omega_x = \frac{x_m^2}{4\sigma_x^4}$; $\omega_y = \frac{y_m^2}{4\sigma_y^4}$; $\alpha_0 = \frac{1}{2\sigma_x\sigma_y}\exp\left(-\frac{1}{2}\left(\frac{x_m^2}{\sigma_x^2} + \frac{y_m^2}{\sigma_y^2}\right)\right)$;

$c_0 = \alpha_0 R^2$;

$c_1 = \frac{\alpha_0 R^4}{2}\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)$;

$c_2 = \frac{\alpha_0 R^6}{12}\left(\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)^2 + p^2\left(\frac{\varphi^2}{2}+1\right) + 2p\varphi\omega_x\right)$;

$$\begin{aligned} c_3 &= \frac{\alpha_0 R^8}{144}\left(\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)^3 + 3\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)\left(p^2\left(\frac{\varphi^2}{2}+1\right) + 2p\varphi\omega_x\right)\right. \\ &+ \left. 2\left(p^3\left(\frac{\varphi^3}{2}+1\right) + 3p^2\varphi^2\omega_x\right)\right); \end{aligned}$$

for $k \leftarrow 0$ to $N-5$ **do**

$$\begin{aligned} c_{k+4} &= -\frac{R^8 p^3 \varphi^2 \omega_y}{(k+2)(k+3)(k+4)^2(k+5)} c_k + \frac{R^6 p^2 \varphi\left(p\varphi\left(k+\frac{5}{2}\right) + 2\omega_y\left(\frac{\varphi}{2}+1\right)\right)}{(k+3)(k+4)^2(k+5)} c_{k+1} \\ &- \frac{R^4 p\left(p\varphi\left(\frac{\varphi}{2}+1\right)(2k+5) + \varphi\left(2\omega_y + \frac{3p}{2}\right) + \omega_x + \omega_y\right)}{(k+4)^2(k+5)} c_{k+2} \\ &+ \frac{R^2\left(p(2\varphi+1)(k+3) + p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)}{(k+4)(k+5)} c_{k+3} \end{aligned}$$

end for

$s \leftarrow 0$

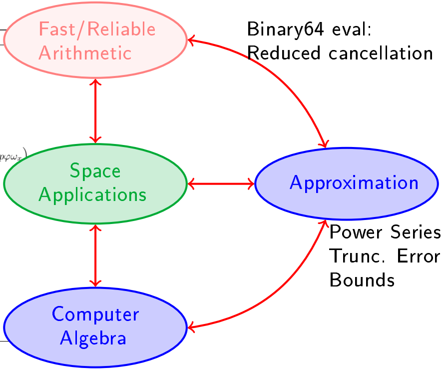
for $k \leftarrow 0$ to $N-1$ **do**

$s \leftarrow s + c_k$;

end for

$\tilde{\mathcal{P}}_c \leftarrow \exp\left(-pR^2\right) s$;

return $\tilde{\mathcal{P}}_c$.



Algorithms
D-finite functions
$\exp, \mathrm{erf}, \sin$, Gaussian

*JGCD, joint work with R. Serra, D. Arzelier, A. Rondepierre, J.-B. Lasserre, B. Salvy

Efficient and reliable algorithm (2016)*

Error bounds

Linear complexity

Algorithm 12 Approximate evaluation of $\mathcal{P}_c$.

Input: Parameters: $\sigma_x, \sigma_y, x_m, y_m, R$; Number of terms: N.

Output: $\tilde{\mathcal{P}}_c$ – truncated series evaluation of $\mathcal{P}_c$.

$p = \frac{1}{2\sigma_y^2};\ \varphi = 1 - \frac{\sigma_y^2}{\sigma_x^2};\ \omega_x = \frac{x_m^2}{4\sigma_x^4};\ \omega_y = \frac{y_m^2}{4\sigma_y^4};\ \alpha_0 = \frac{1}{2\sigma_x\sigma_y}\exp\left(-\frac{1}{2}\left(\frac{x_m^2}{\sigma_x^2} + \frac{y_m^2}{\sigma_y^2}\right)\right);$

$c_0 = \alpha_0 R^2;$

$c_1 = \frac{\alpha_0 R^4}{2}\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right);$

$c_2 = \frac{\alpha_0 R^6}{12}\left(\left(p\left(\frac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)^2 + p^2\left(\frac{\varphi^2}{2}+1\right) + 2p\varphi\omega_x\right);$

$$\begin{aligned} c_3 &= \tfrac{\alpha_0 R^8}{144}\left(\left(p\left(\tfrac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)^3 + 3\left(p\left(\tfrac{\varphi}{2}+1\right) + \omega_x + \omega_y\right)\left(p^2\left(\tfrac{\varphi^2}{2}+1\right) + 2p\varphi\omega_x\right)\right. \\ &+ \left.2\left(p^3\left(\tfrac{\varphi^3}{2}+1\right) + 3p^2\varphi^2\omega_x\right)\right); \end{aligned}$$

for $k \leftarrow 0$ to $N-5$ **do**

$$\begin{aligned} c_{k+4} &= -\frac{R^8p^3\varphi^2\omega_y}{(k+2)(k+3)(k+4)^2(k+5)}c_k + \frac{R^6p^2\varphi\left(p\varphi\left(k+\frac{5}{2}\right)+2\omega_y\left(\frac{\varphi}{2}+1\right)\right)}{(k+3)(k+4)^2(k+5)}c_{k+1} \\ &- \frac{R^4p\left(p\varphi\left(\frac{\varphi}{2}+1\right)(2k+5)+\varphi\left(2\omega_y+\frac{3p}{2}\right)+\omega_x+\omega_y\right)}{(k+4)^2(k+5)}c_{k+2} \\ &+ \frac{R^2\left(p(2\varphi+1)(k+3)+p\left(\frac{\varphi}{2}+1\right)+\omega_x+\omega_y\right)}{(k+4)(k+5)}c_{k+3} \end{aligned}$$

end for

$s \leftarrow 0$

for $k \leftarrow 0$ to $N-1$ **do**

$s \leftarrow s + c_k;$

end for

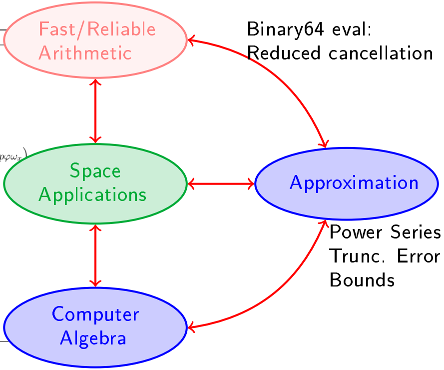
$\tilde{\mathcal{P}}_c \leftarrow \exp\left(-pR^2\right)s;$

return $\tilde{\mathcal{P}}_c$.

2017: CNES impl+tests

2021: CNES on-board impl**

Long-term collaboration
ADS, CNES, Thales, TAS



Algorithms
D-finite functions
$\exp, \mathrm{erf}, \sin$, Gaussian

*JGCD, joint work with R. Serra, D. Arzelier, A. Rondepierre, J.-B. Lasserre, B. Salvy

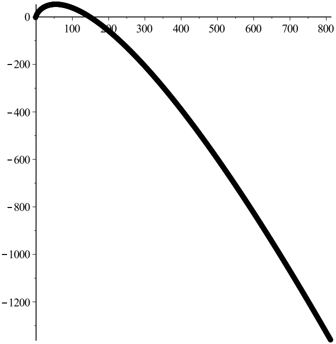
** 01/2023 https://lejournal.cnrs.fr/articles/un-algorithme-pour-eviter-les-debris-spatiaux

Trickier examples
(from [Alfano 2009])

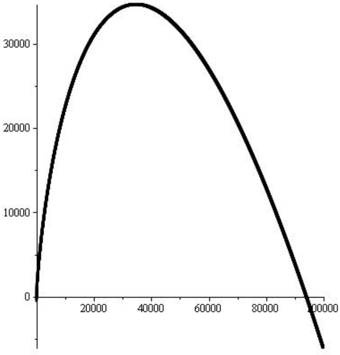
Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

Alfano's test case number	Probability of collision (-)		
	Alfano	New method	Reference (MC)
3	0.10038	0.10038	0.10085
5	0.044712	0.045509	0.044499

Case 3 Alfano: $\eta_{800} = 30$



Case 5 Alfano: $\eta_{121000} = 20$



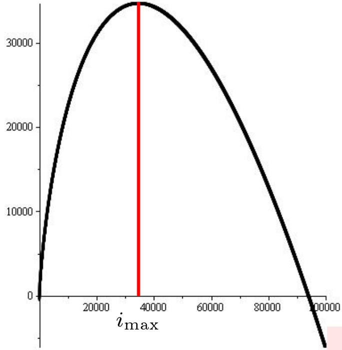
Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
3	114.25	1.41	15	0.15	3.88
5	177.8	0.038	10	2.12	-1.22

(from [Alfano 2009])

An annoying bump...

$$\log\left(\frac{f_i R^{2i}}{(i+1)!}\right)$$

$$i_{\max} \sim O\left(\frac{R^2}{2\sigma_y^2}\right)$$



Term index i

Case #	Input parameters (m)				
	σ_x	σ_y	R	x_m	y_m
5	177.8	0.038	10	2.12	-1.22

$$i_{max} \simeq 34626$$

An annoying bump...

An alternative via Saddle-Point

Recall that

$$\mathcal{L}g(\lambda) = \frac{\exp\left(-\frac{{\sigma_x}^2{y_m}^2+{\sigma_y}^2{x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2\lambda{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2\lambda{\sigma_x}^2+1)}\right)}{\lambda\sqrt{(2\lambda{\sigma_x}^2+1)(2\lambda{\sigma_y}^2+1)}}$$

Inverse Laplace Transform

$$\mathcal{I} = \frac{1}{2i\pi}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty}\underbrace{e^{\xi\lambda}\mathcal{L}g(\lambda)}_{e^{\varphi(\lambda)}}\mathrm{d}\lambda$$

Integrand $e^{\varphi(\lambda_0+it)}$:

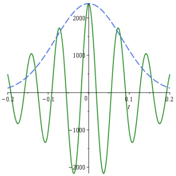
Recall that

$$\mathcal{L}g(\lambda) = \frac{\exp\left(-\frac{{\sigma_x}^2{y_m}^2+{\sigma_y}^2{x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2\lambda{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2\lambda{\sigma_x}^2+1)}\right)}{\lambda\sqrt{(2\lambda{\sigma_x}^2+1)(2\lambda{\sigma_y}^2+1)}}$$

Inverse Laplace Transform

$$\mathcal{I} = \frac{1}{2i\pi}\int_{\lambda_0-i\infty}^{\lambda_0+i\infty} \underbrace{e^{\xi\lambda}\mathcal{L}g(\lambda)}_{e^{\varphi(\lambda)}}\mathrm{d}\lambda$$

Integrand $e^{\varphi(\lambda_0+it)}$:



generic λ_0

An alternative via Saddle-Point

Recall that

$$\mathcal{L}g(\lambda) = \frac{\exp\left(-\frac{{\sigma_x}^2 {y_m}^2+{\sigma_y}^2 {x_m}^2}{2{\sigma_x}^2{\sigma_y}^2} + \frac{{y_m}^2}{2{\sigma_y}^2(2\lambda{\sigma_y}^2+1)} + \frac{{x_m}^2}{2{\sigma_x}^2(2\lambda{\sigma_x}^2+1)}\right)}{\lambda\sqrt{(2\lambda{\sigma_x}^2+1)(2\lambda{\sigma_y}^2+1)}}$$

Inverse Laplace Transform

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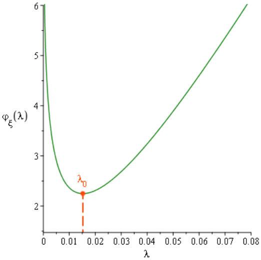
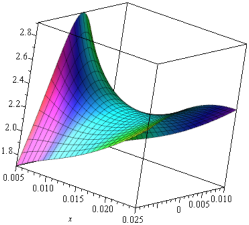
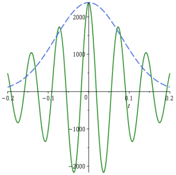


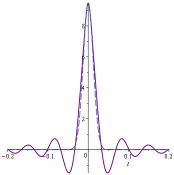


Saddle-point $\lambda_0 > 0$ solution of $\varphi'(\lambda_0) = 0$

Integrand $e^{\varphi(\lambda_0+it)}$:



generic λ_0



λ_0 is a saddle

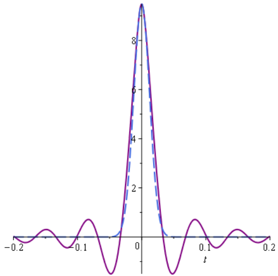
Laplace method in a nutshell

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

2. Central approximation

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(\lambda - \lambda_0)^2} \mathrm{d}\lambda$$



Integrand $e^{\varphi(\lambda_0 + it)}$

3. Complete the tails

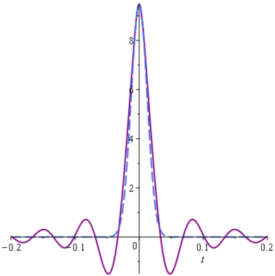
$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} e^{\varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(\lambda - \lambda_0)^2} \mathrm{d}\lambda$$

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi}\int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)}\mathrm{d}\lambda$$

2. Central approximation

$$\mathcal{I} \simeq \frac{1}{2i\pi}\int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(\lambda-\lambda_0)^2}\mathrm{d}\lambda$$



Integrand $e^{\varphi(\lambda_0 + it)}$

3. Complete the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi}\int_{\lambda_0 - i\infty}^{\lambda_0 + i\infty} e^{\varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(\lambda-\lambda_0)^2}\mathrm{d}\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2}t^2}\mathrm{d}t$$

$$\mathcal{I} \simeq \frac{e^{\varphi(\lambda_0)}}{2\sqrt{\varphi''(\lambda_0)\pi}}$$

Not sufficiently accurate

Can we do better?

Asymptotic expansions of Laplace type integrals

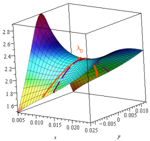
1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

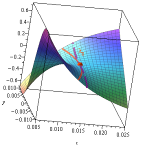
2. Change of variables in the neighborhood of λ_0:

$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

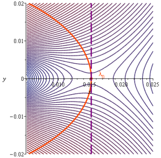
+ Local deformation of the integration path to match the constant phase contour passing through λ_0



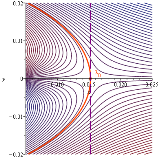
Surface $(x, y) \mapsto u_{\lambda_0}(x + iy)$



Surface $(x, y) \mapsto v_{\lambda_0}(x + iy)$



Level set u_{λ_0}



Level set v_{λ_0}.

Asymptotic expansions of Laplace type integrals

1. Neglect the tails

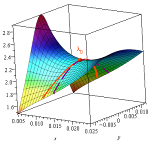
$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

2. Change of variables in the neighborhood of λ_0:

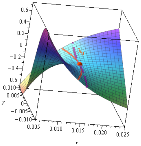
$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

3. Complete the tails

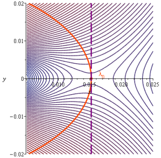
$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2} w^2} \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w} \mathrm{d}w$$



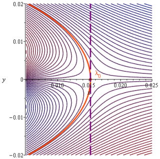
Surface $(x, y) \mapsto u_{\lambda_0}(x + iy)$



Surface $(x, y) \mapsto v_{\lambda_0}(x + iy)$



Level set u_{λ_0}



Level set v_{λ_0}.

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

2. Change of variables in the neighborhood of λ_0:

$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

3. Complete the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2} w^2} \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w} \mathrm{d}w$$

1. Find Saddle-point $\lambda_0 > 0$, numerical solution of:

$$\varphi'(\lambda) = 0$$

$\downarrow$

2. Series Inversion:

$$\lambda(w) = \lambda_0 + iw + b_2(iw)^2 + b_3(iw)^3 + \dots$$

Expand in w, identify coeffs and solve for b_n.

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

2. Change of variables in the neighborhood of λ_0:

$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

3. Complete the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2} w^2} \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w} \mathrm{d}w$$

1. Find Saddle-point $\lambda_0 > 0$, numerical solution of:

$$\varphi'(\lambda) = 0$$

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$$\lambda(w) = \lambda_0 + iw + b_2(iw)^2 + b_3(iw)^3 + \dots$$

Expand in w, identify coeffs and solve for b_n.

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

2. Change of variables in the neighborhood of λ_0:

$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

3. Complete the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2} w^2} \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w} \mathrm{d}w$$

$$\boxed{\sum_{n=0}^{\infty} (n+1) b_{n+1} (iw)^n}$$

1. Find Saddle-point $\lambda_0 > 0$, numerical solution of:

$$\varphi'(\lambda) = 0$$

2. Series Inversion:

$$\lambda(w) = \lambda_0 + iw + b_2(iw)^2 + b_3(iw)^3 + \dots$$

Expand in w, identify coeffs and solve for b_n.

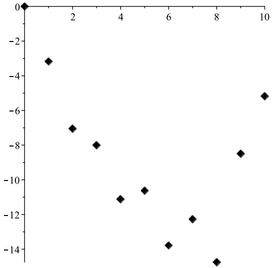
3. Term-by-term integration:

$$\mathcal{I} \simeq \frac{e^{\varphi(\lambda_0)}}{2\sqrt{\varphi''(\lambda_0)\pi}} \sum_{n=0}^{N} c_n,$$

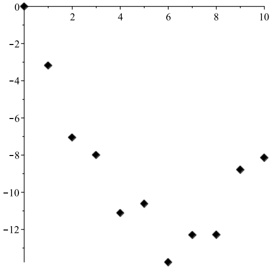
with $c_n = (-1)^n \dfrac{(2n+1)!!}{(\varphi''(\lambda_0))^n} b_{2n+1}$

log plot of c_n coefficients

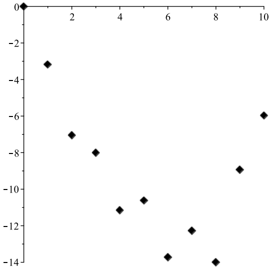
Case 2:



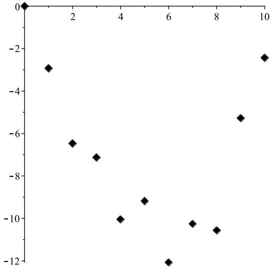
Case 3:



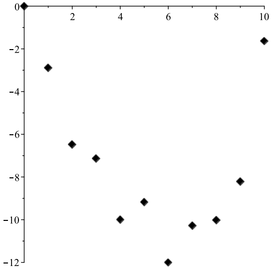
Case 4:



Case Alfano(3):



Case Alfano(5)



Accuracy with 5 terms:

Case2:	3 digits
Case3:	2 digits
Case6:	4 digits
CaseA3:	3 digits
CaseA5:	3 digits

Instantaneous (3D) probability by orbit propagation from $t_0 = 0$ to $t_f = 13200s$ on a grid 0f 301 points; accuracy threshold $\delta = 10^{-15}$.
Convergent series used if no. terms $\leq N_{max}$, otherwise saddle-point alternative.

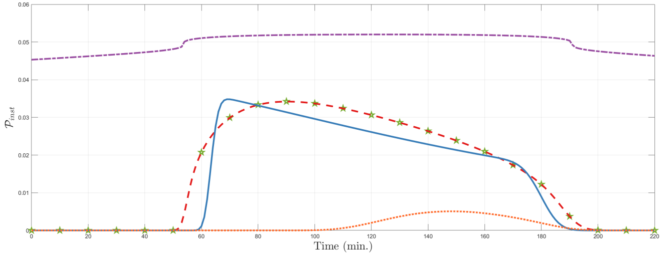


Figure: Alfano'09 test case 4: $\mathcal{P}_{inst}$ - Monte Carlo simulations (green stars), our algorithm (red dashed line), alternative methods AD (dashed dot purple line), EV (dotted orange line), EVC (solid blue line).

Two Satellites in Geosynchronus Equatorial Orbits (GEO) –continued–

Instantaneous (3D) probability by orbit propagation from $t_0 = 0$ to $t_f = 13200s$ on a grid 0f 301 points; accuracy threshold $\delta = 10^{-15}$.
Convergent series used if no. terms $\leq N_{max}$, otherwise saddle-point alternative.

N_{max}	Mean/median timings (s)	% calls of divergent series
10	0.2153/0.2509	100
400	0.2350/0.2705	52
1000	0.2495/0.2945	32.1
4000	0.3804/0.4503	0

Table: Mean/median timings (s) over 30 runs and percentages of calls of the divergent series.

Saddle-point method is faster, but currently lacks error bounds...

Asymptotic expansions of Laplace type integrals: error bounds

1. Neglect the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{\lambda_0 - is}^{\lambda_0 + is} e^{\varphi(\lambda)} \mathrm{d}\lambda$$

1. Bound tail

2. Change of variables in the neighborhood of λ_0:

$$\varphi(\lambda(w)) = \varphi(\lambda_0) + \frac{\varphi''(\lambda_0)}{2}(iw)^2$$

2. Series Inversion:
Bound truncation error

3. Complete the tails

$$\mathcal{I} \simeq \frac{1}{2i\pi} \int_{-\infty}^{+\infty} e^{\varphi(\lambda_0) - \frac{\varphi''(\lambda_0)}{2} w^2} \frac{\mathrm{d}\lambda(w)}{\mathrm{d}w} \mathrm{d}w$$

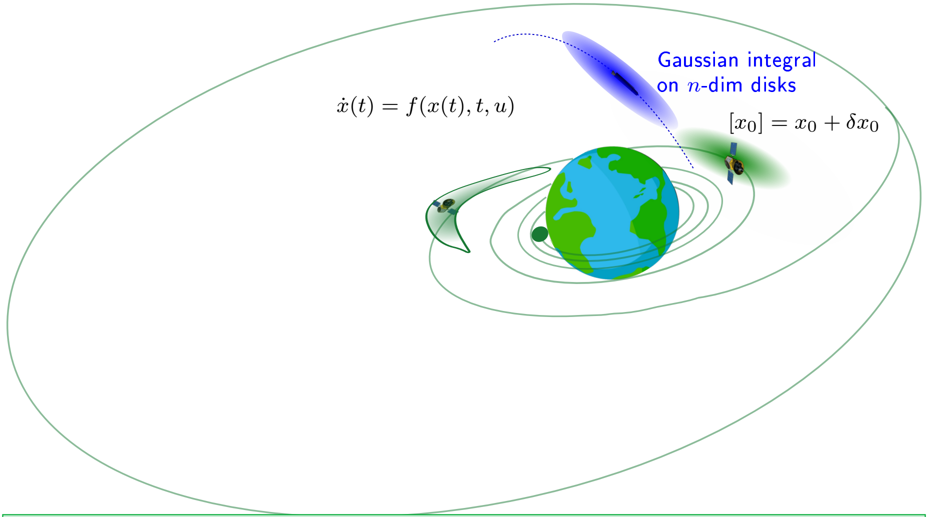
3. Term-by-term integration:
Bound extra added tails

Note: see works by e.g., Olver('60s), Wong ('80s) for some special functions

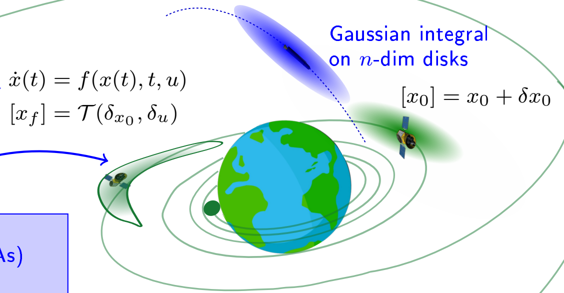
Long-term encounters

Multiple encounters

The issue of mega-constellations



Applications: Validated Autonomous Electric Orbit Raising
Uncertainty Propagation
Collision Risk Assessment/Mitigation, Mega-constellations



Sym-Num Algorithms based on
Rigorous Polynomial Approx (RPAs)

Generalized Fourier Series
Multivariate RPAs
Polynomial Chaos Expansions
RPAs for non-linear ODEs

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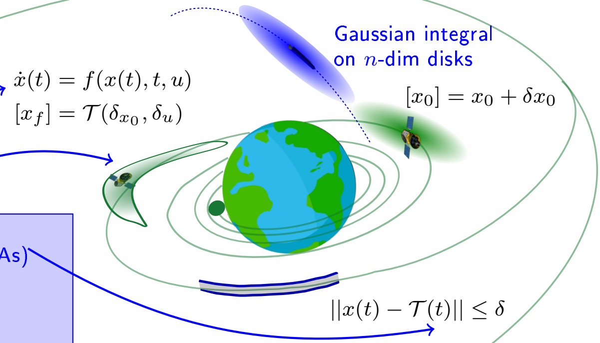
Other challenges:

Fast and Reliable Computations for Space – Exemplified Roadmap



Sym-Num Algorithms based on Rigorous Polynomial Approx (RPAs)

- Generalized Fourier Series
- Multivariate RPAs
- Polynomial Chaos Expansions
- RPAs for non-linear ODEs

Complexity/Accuracy?

✓ Recurrence prop. hold $\rightsquigarrow$ Sparsity

✓ Newton-like methods for validation

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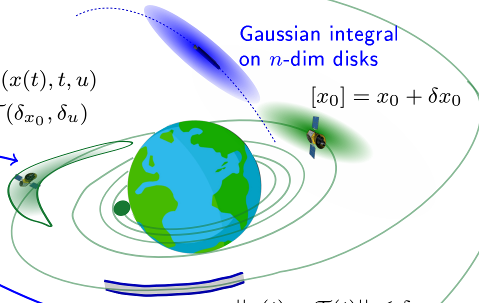
Arith:
Finite prec evaluation of RPAs

$$\dot{x}(t) = f(x(t), t, u)$$
$$[x_f] = \mathcal{T}(\delta_{x_0}, \delta_u)$$

Gaussian integral on n-dim disks

$[x_0] = x_0 + \delta x_0$



$||x(t) - \mathcal{T}(t)|| \leq \delta$

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Thank you for your attention!

a D-finite function.

Asymptotic expansions of Laplace type integrals

Watson's Lemma

Let f be a function of positive real variable s.t.
as $x \to 0^+$,

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^{\frac{s+k}{\mu}-1},$$

with $s, \mu > 0$.
Then

$$\int_0^{\infty} e^{-\lambda x} f(x) \mathrm{d}x \sim \sum_{k=0}^{\infty} \Gamma\left(\frac{s+k}{\mu}\right) \frac{a_k}{\lambda^{\frac{s+k}{\mu}}},$$

as $\lambda \to \infty$,
provided that the integral converges throughout its range for sufficiently large λ.

Note: $f(x) = \sum_{k=0}^{N-1} a_k x^{\frac{s+k}{\mu}-1} + \mathcal{O}\left(x^{\frac{s+N}{\mu}-1}\right).$