Existence of invariant tori in the large: from normally hyperbolic to lagrangian.

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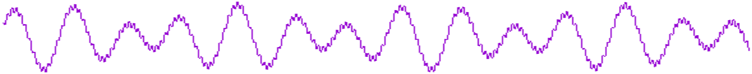

Certified and Symbolic Numeric Computation
ENS de Lyon

Goal(s) of the talk

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In this talk I plan to show some of the results we have been able to develop in the context of validating the existence of invariant tori in different contexts: from hyperbolic tori in skew product systems, to KAM (Kolmogorov-Arnold-Moser) tori in both twist maps and hamiltonian systems.

All our results are looking at being able to prove the existence of these objects for systems far from *the perturbative regime.*

Finally, if time permits, I want to share with you some thoughts that I collected from different discussions with different people around CAPs (Computer Assisted Proofs).

A prelude to a prelude

I am very aware of the fact that in this audience *ALL* of you are experts in the field and that know what a CAP is, but let me waste your time showing you some slides I am using nowadays when I talk about CAPs in front of a more common audience.

I also think that this exercise could contribute to some of the discussions we are here this week: which relation exists between CAPs and mathematicians and what do we want to achieve.

A mathematical prelude: CAP (Computer Assisted Proofs)

In this talk it is crucial that the audience understands how CAPs are performed in analysis and, for this, Interval Analysis plays a crucial role.
Short and simple:

Fundamental Principle of CAPs in Analysis

Given any function that is expressable as a finite combination $(+, -, \cdot, /, \circ)$ of standard functions (pol, trigo, logs, exps...) plus bounded unknown functions (reminders of expansions), and given a bounded interval, then a computer gives you back another interval containing the image of the former.

A mathematical prelude: CAP (Computer Assisted Proofs)

With the previous Fundamental Principle one can do a lot. For example, proving that a finite list of inequalities defined on compact domains in $\mathbb{R}^n$ is true when all sides are of the previous form.

A mathematical prelude: CAP (Computer Assisted Proofs)

Remark: if a problem suits the Fundamental Principle, then it can be proved via CAPs. But, if we don't see how to accomodate it, then CAPs are not the way.

Some people dismiss CAPs with sentences like: *But a CAP will never be able to prove this specific result while...*
Well, we are now at the level of *Nash-Moser schemes are useless because they don't help us proving Fermat's last theorem.*

Our approach in validating invariant tori

Focus on our results: CAP (Computer Assisted Proofs)

We produce theorems that suit to the Fundamental Principle. We write our theorems and, later on, check their hypotheses on specific problems with the help of CAPs.

We have in mind that the initial data that we have for checking the hypotheses are an approximation of the invariant torus and, of course, the dynamical system (*explicitly*).

Our methods fall in the cathegory of a-posteriori results.

Classical vs a-posteriori approach

The original approach is in the perturbative setting.

Like studying $f(x, \varepsilon) = 0$ for ε close to zero by using power series.

Or like performing an ε-close to identity transformation to obtain a solvable equation.

Our approach is to reconstruct the necessary statements with quantitative estimates that require only to have a good numerical guess of the solution.

Like studying solutions of $g(x) = f(x, \varepsilon_0) = 0$ by using Newton-Kantorovich.

Interlude: CAPs on zeros of functions

If I want to validate the existence and uniqueness of a zero for a differentiable function f on the interval $[0, 1]$ the easiest way is:

- Obtain enclosure of $f([0, 0])$ and check that zero is not there.
- Obtain enclosure of $f([1, 1])$ and check that zero is not there.
- Check that the previous 2 enclosures are of opposite sign.
- Obtain enclosure of $f'([0, 1])$ and check that zero is not there.

Compare this approach with: Given a function $g(x, y)$ such that $g(\frac{1}{2}, 0) = 0$ and $\partial_x g(\frac{1}{2}, 0) \neq 0$ then I know that $g(x, \varepsilon)$ has a zero with (unknown) small ε.

Reducing the existence to a functional equation

We reduce the problem of invariance of a torus to finding a zero of a functional equation

$$\mathcal{F}[f](x) = 0\,.$$

The particular form of this equation is not important now. What matters is that we have a guess

$$\mathcal{F}[f_0](x) = e(x)\,.$$

The Newton method consists in obtaining a better solution

$$\mathcal{F}[f_0 + h_0] = \mathcal{F}[f_0] + D\mathcal{F}[f_0]\ h_0 + O_2(h_0) = 0$$

Linearized equation

$$D\mathcal{F}[f_0]\ h_0 = -e\,, \qquad f_k = f_{k-1} + h_{k-1}$$

Reducing the existence to a functional equation

Then we need to prove that the above procedure converges. In the case of **uniformly hyperbolic invariant tori** we set a suitable Banach space X, with $\mathcal{F} : X \to X$, and use that $f_k \in X$. Then the existence is derived by Banach Fixed Point Theorem.

Warning

However, in the context of **KAM solutions** the previous does not work since the map

$$D\mathcal{F}^{-1} : X \to X \qquad \text{is } \textbf{unbounded}$$

Due to the effect of **small divisors** we need to consider a scale of Banach spaces $X_1 \subset X_\rho \subset X_{\rho-\delta} \subset X_0$ and work with estimates

$$\left\| D\mathcal{F}[f] \circ \mathcal{L} - \mathrm{id} \right\|_{\rho-\delta} \leq \left\| \mathcal{F}[f] \right\|_{\rho}$$

This is a Nash-Moser scheme (Generalized IFT).

Flavour of our theorems

Our theorems require the following ingredients:

- Good candidates $\omega \in \mathbb{R}^{2n}$ and $K : \mathbb{T}^n \to \mathbb{R}^{2n}$ for solutions (Given by numerical computations).
- A quantitative theorem that asks bounds on the given candidate that can be computed rigorously with a computer in finite time.
    - ω is non-resonant: $|k \cdot \omega| \geq \gamma / |k|^\tau$. (only for KAM tori).
    - K is an embedding: $\|(K^* g)^{-1}\|_\rho$.
    - Error of invariance small enough: $\|\mathcal{F}[K]\|_\rho$.
    - Twist condition. (only for KAM tori).

Normally Hyperbolic tori

The system

We are interested in looking for invariant tori for the skew-product system $F : \mathbb{T}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}^d \times \mathbb{R}^n$ of the form

$$F(\theta, x) = \begin{cases} \theta + \omega \\ f(\theta, x) \end{cases}$$

Invariance equations

The invariant torus is parameterized by $K : \mathbb{T}^d \longrightarrow \mathbb{R}^n$ and satisfies the invariance equation

$$f(\theta, K(\theta)) = K(\theta + \omega).$$

Since we are interested in looking for hyperbolic invariant tori, they must also satisfy there exists maps $P : \mathbb{T}^d \longrightarrow GL(\mathbb{R}^n)$, and $\Lambda_u : \mathbb{T}^d \longrightarrow GL(\mathbb{R}^{n_u})$ and $\Lambda_s : \mathbb{T}^d \longrightarrow GL(\mathbb{R}^{n_s})$, $n_u + n_s = n$ such that

$$P(\theta + \omega)^{-1} D_x f(\theta, K(\theta)) P(\theta) = \begin{pmatrix} \Lambda_u(\theta) & 0 \\ 0 & \Lambda_s(\theta) \end{pmatrix}$$

with these Λ_i being hyperbolically expanding or contracting.

With these two equations and unknowns (K and P), and a Newton-Kantorovich type argument we can build a theorem suitable for validation.

A Validation Theorem

Theorem 3.7. *Let* $(f,F)\colon \mathscr{A} \subset \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{T}^d \times \mathbb{R}^n$ *be a skew-product on the annulus* $\mathscr{A}$*, with* F *being of class* $C^{1+\mathrm{Lip}}$ *with respect to the fiber variable* y*. Assume we are given:*

1.1) *an approximately invariant torus* $\mathscr{K}_0 = \mathrm{graph}(K_0)$ *with* $(\mathrm{id}, K_0) \in \Gamma(\mathscr{A})$*;*
1.2) *a Finsler norm* $|\cdot| : \mathbb{T}^d \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$*;*
1.3) *a closed annulus around* $\mathscr{K}_0$ *of radius* R *inside* $\mathscr{A}$*:*

$$\bar{\mathscr{A}}(K_0,R) := \{(\theta,y) \in \mathbb{T}^d \times \mathbb{R}^n \mid \text{for all } \theta \in \mathbb{T}^d,\ |K(\theta) - y|_\theta \le R\} \subset \mathscr{A}.$$

Let ρ *be an* error bound *of the invariance equation for* K_0*,* c_H *be a* hyperbolicity bound *of the transfer operator* $\mathscr{M}_0$ *associated to the linear skew-product* (f, M_0) *given by the transfer matrix* $M_0(\theta) = \mathrm{D}_y F(\theta, K_0(\theta))$*, and* b *be the Lipschitz constant of the differential of the skew-product with respect to* y*. Assume that:*

2.1) *for each* $\theta \in \mathbb{T}^d$*,* $|F(\theta, K_0(\theta)) - K_0(f(\theta))|_{f(\theta)} \le \rho$*;*
2.2) *for each* $z \in \mathbb{C}$ *with* $|z| = 1$*,* $\|(\mathscr{M}_0 - z\mathrm{Id})^{-1}\| \le c_H$*;*
2.3) *for each* $(\theta, y_1), (\theta, y_2) \in \bar{\mathscr{A}}(K_0, R)$*,* $v \in \mathbb{R}^n$*,*

$$|(\mathrm{D}_y F(\theta, y_1) - \mathrm{D}_y F(\theta, y_2))v|_{f(\theta)} \le b|y_1 - y_2|_\theta |v|_\theta.$$

Assume that the convergence bound h *and the radii* r_0*,* r_1 *satisfy*

3.1) $c_H^2 b\rho \le h < \frac{1}{2}$*;*
3.2) $(1 - \sqrt{1-2h})(c_H b)^{-1} \le r_0 \le r_1 < \min\left((1 + \sqrt{1-2h})(c_H b)^{-1},\ R\right)$.

Then, there exists a unique torus $\mathscr{K}_* = \mathrm{graph}(K_*)$ *with* $(\mathrm{id}, K_*) \in \Gamma(\mathscr{A})$ *such that:*

a.1) *for each* $\theta \in \mathbb{T}^d$*,* $F(\theta, K_*(\theta)) - K_*(f(\theta)) = 0$*;*
a.2) *for each* $\theta \in \mathbb{T}^d$*,* $|K_*(\theta) - K_0(\theta)|_\theta \le r_1$.

Moreover:

a.3) *for each* $\theta \in \mathbb{T}^d$*,* $|K_*(\theta) - K_0(\theta)|_\theta \le r_0$.
a.4) $\mathscr{K}_*$ *is a FHIT.*

Figure: (from book *The parameterization method for Invariant Manifolds*, Haro et altri.)

KAM for twist maps

Twist maps

Twist maps are maps on $\mathbb{T} \times \mathbb{R}$ such that $|\partial_y F_1| \geq c > 0$ (as you increase the height the map sends you further in th x direction).

A paradigm of them is the Chirikov Standard Map.

$$\begin{aligned} F : \mathbb{T} \times \mathbb{R} &\longrightarrow \mathbb{T} \times \mathbb{R} \\ (x, y) &\longmapsto (x + y - \frac{\varepsilon}{2\pi} \sin(2\pi x), y - \frac{\varepsilon}{2\pi} \sin(2\pi x)). \end{aligned}$$

Invariant Tori for Twist Maps

An invariant torus for twists maps is the pair (ω, K), $K : \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{R}$ satisfying

$$F(K(\theta)) = K(\theta + \omega).$$

In this case the linearized dynamics around the torus is not hyperbolic, but conjugated to

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$a \neq 0$. This makes all the business more difficult, since the linearized operator is noninvertible!

One step correction

The idea is to correct a quasi-invariant torus K_0 with Δ so

$$F(K_0(\theta) + \Delta(\theta)) - K_0(\theta + \omega) - \Delta(\theta + \omega) = 0.$$

Linealizing this last equation and discarding h.o.t. we obtain

$$F(K_0(\theta)) - K_0(\theta + \omega) + DF(K_0(\theta))\Delta(\theta) - \Delta(\theta + \omega) = 0.$$

For solving this last linear equation we perform a change of variables $P = (DK_0||N)$ ($\Delta = P\xi$) and obtain the (almost) constant linear equation

$$\begin{pmatrix} 1 & T(\theta) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_L(\theta) \\ \xi_N(\theta) \end{pmatrix} - \begin{pmatrix} \xi_L(\theta + \omega) \\ \xi_N(\theta + \omega) \end{pmatrix} = 0.$$

These are the famous small divisors.

Interlude about small divisors.

Let me stop here with the hope that we get the flavour of why KAM schemes are a level higher in difficulty (needing to combine diophantine conditions on ω with reducing analycity at each step).

The fundamental linear equation one solves is

$$f(\theta + \omega) - f(\theta) = g(\theta).$$

In Fourier this is

$$\hat{f}_k = \frac{\hat{g}_k}{e^{2\pi i k \omega} - 1}.$$

To have a formal solution we need that ω is irrational, and if we want that $\hat{f}_k$ decays almost as fast as $\hat{g}_k$ we need that the denominator does not approach to 0 to fast (so ω Diophantine).

Theorem suitable for validation

I don't dare to write down the theorem that assures the existence of invariance of a torus given an approximate one but, the main ingredients are:

- Control of the error in a complex band $e = \|F(K(\theta)) - K(\theta + \omega)\|_\rho$.
- Control on the diophantine constants of ω.
- Control on the map around the torus (In particular of T, the torsion around the torus).

Then we obtain that, if

$$\frac{\mathcal{C}_1 e}{\gamma^4 \delta^{4\tau}} \leq 1$$

then the torus exists. The constant C_1 depends on a looooooot of intermediate computations (norm of the map, its derivative, torsion estimates...)

More details of the Computer Assisted Proof part

An important tool needed in our proofs is the use of Computer Assisted Analysis: How to encode Banach spaces (of periodic functions) and operate with them.

In particular, we are interested with the following questions:

- How to encode analytic periodic functions.
- How to bound the error of operating with them (addition, multiplication...)
- How to bound the error of compositions, e.g. $\sin(f(x))$.
- How to control analytic norms.

Encoding Banach spaces in a computer

Let $\mathcal{A}_\rho$ be the space of analytic periodic functions with radius of analyticity ρ. This is a Banach space.

Theorem

Let $f : \mathbb{T}_{\hat{\rho}} \to \mathbb{C}$ be analytic in the complex strip $\mathbb{T}_{\hat{\rho}}$ of size $\hat{\rho} > 0$. Let us consider a regular grid of $\mathbb{T}$ of N points. Then, for $0 \leq \rho < \hat{\rho}$:

$$\|f - \mathrm{FFT}(f)\|_\rho \leq C_N(\rho, \hat{\rho})\|f\|_{\hat{\rho}}.$$

where $C_N(\rho, \hat{\rho})$ is exponentially small.

FFT is computed using interval arithmetics (MPFI).

I will not get more into details here because *Jason Mireles James* promised me he will talk about it tomorrow at 9:30. Don't miss the talk!

An example

Let us consider the standard map

$$\begin{array}{rcl} F : \mathbb{T}\times\mathbb{R} & \longrightarrow & \mathbb{T}\times\mathbb{R} \\ (x,y) & \longmapsto & (x+y-\frac{\varepsilon}{2\pi}\sin(2\pi x), y-\frac{\varepsilon}{2\pi}\sin(2\pi x)). \end{array}$$

For $\varepsilon = 0$ we have invariant tori parametrized by

$$K(\theta) = \begin{pmatrix} \theta \\ \omega \end{pmatrix},$$

For $\varepsilon > 0$ (sufficiently small) KAM theory concludes that most of these curves persist, although they are slightly deformed. These curves are successively destroyed as ε is increased.

From now on, we pay attention to $\omega = \frac{\sqrt{5}-1}{2}$.

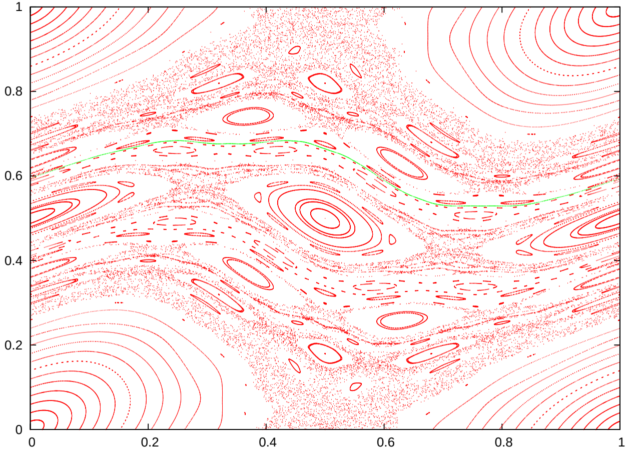

Figure: Phase-space of the standard map for $\varepsilon = 0.97$.

We proof that the golden curve persists for $\varepsilon \leq 0.9716$.

- It is known that for $\varepsilon = 0.9718$ the curve does not exist.
- Numerical computations suggest that the breakdown occurs at $\varepsilon_c \simeq 0.97163540$.
- We require to do 2^{22} evaluations of the map, and we obtain that
$$\|K_{\mathrm{true}} - K_{\mathrm{approx}}\|_{\rho_*} \leq 4 \cdot 10^{-22}\,.$$
- To numerically observe that there is no curve for $\varepsilon = 0.971636$, one needs to compute $N = 2^{26}$ iterates

Further advances

One important problem in Hamiltonian Mechanics, and Dynamical Systems in general, is to identify stability (and instability) regions in phase and parameter space.

> *Given a particular system with non-perturbative parameters, and given a particular region of interest in phase/parameter space, what is the abundance of quasiperiodic smooth solutions in that region?*

The only results in this direction were expressed as asymptotic bounds with respect to the size of the perturbation.

Given any family $\alpha \in A \mapsto f_\alpha$ of analytic circle diffeomorphisms, we answer the following problem:

> *Obtain (almost optimal) lower bounds for the Lebesgue measure of parameters $\alpha \in A$ such that the map f_α is analytically conjugated to a rigid rotation.*

This enlarges our vision about how stability can be effectively measured.

We have also produced the Theorem for Hamiltonian systems with sharp constants γ^2 instead of γ^4 in the denominator. Moreover, this result also deals for systems with first integrals without the need of removing them or the need of dealing with the full lagrangian torus, only with the dimension of the nonintegrable part.

Where do we find all these

- F., Haro, *Reliable Computation of Robust Response Tori on the Verge of Breakdown*, SIADS 2012.
- Haro. et altri *The parameterization Method for Invariant Tori*, Springer, 2016.
- F., Haro, Luque, *Rigorous Computer-Assisted Application of KAM Theory: A Modern Approach*, FOCM 2016.
- Haro, Luque, *A-posteriori KAM theory with optimal estimates for partially integrable systems*, JDE 2019.
- F., Haro, Luque, *Effective bounds for the measure of rotations*, Nonlinearity 2020.
- F., Haro, *A Modified Parameterization Method for Invariant Lagrangian Tori for Partially Integrable Hamiltonian Systems*, (submitted).

Some reflections about CAPs

Some points that make KAM heavy in terms of computer power

- In real (but low dimensional) applications the dimension of the tori are 2, 3, or more. This implies that we have $1024^3 = 1.073.741.824$ Fourier modes. This is memory expensive (4 teras) and computational expensive. (Evaluating the error can take 64 days).
- The previous point rises the following questions: How do you send a file that big to a journal? Who will referee something like this? Also, there is the need of big computers for doing these computations.
- Related to this: Don't we need to push for some kind of standards so the referee process is smoother, helps to guide the editors (who are not always knowledgeable about CAPs)?

A personal experience refereeing papers

The more papers I referee the longer and convoluted are the computations they involve. Some of them require that I run them a week under 120 parallel processes and the installation is not trivial. So, I have a question?

Shouldn't we standarize all these?

Sometimes it seems that we are still in an experimental phase where non-standarization is allowed.

Do we need to submit code? How long will survive the code? In 30 years, will it be possible to run these codes?

The CAPs Manifesto (even better: The Lyon Manifesto)

It will be nice to see that experts like all of you gather around a document and write some basic guidelines about how to submit a CAP.

> *A spectre is haunting Mathematics – the spectre of Computer Assisted Proofs.*

Some few suggestions (very debatable):

- There must be a standard one-to-one between statements on the paper and their computer proofs on the code. Easy to locate and easy to distinguish between pencil-and-paper proofs and CAPS; and code-proofs and auxiliar code.
- A suggested structure of the files and folders will be desirable, easy to navigate and know what is what.
- A demand to journals to keep the file codes.
- The Manifesto should propose a minimum number of times and different machines to run the code.

The End

Thank you very much!

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