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Some new tricks for formalising advanced mathematics

Manuel Eberl

Outline

1. A quick look at Isabelle/HOL

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2. Some recent work in formalising graduate-level maths

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2. Some recent work in formalising graduate-level maths
3. Some tools and tricks we had to develop along the way

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- Large library of mathematics (especially analysis)
- Archive of Formal Proofs: Growing collection of entries on Mathematics, Computer Science, Logic; 748 articles, 457 authors

Some Examples

A typical structure Isabelle proof: If one of a and b is even, then $a \cdot b$ is even.

```
lemma
  fixes a b :: int
  assumes "even a ∨ even b"
  shows   "even (a * b)"
proof
  from assms show "even (a * b)"
  proof
    assume "even a"
    then obtain a' where a': "a = 2 * a'"

    have "even (2 * (a' * b))"

    also have "2 * (a' * b) = a * b"
      using a'
    finally show "even (a * b)"
  next
    assume "even b"
    (* ... *)
```

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    assume "even a"
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      by (elim evenE)
    have "even (2 * (a' * b))"
      by simp
    also have "2 * (a' * b) = a * b"
      using a' by simp
    finally show "even (a * b)" .
  next
    assume "even b"
    (* ... *)
```

Some Examples

Dirichlet's Theorem: Given coprime integers *h* and *n*, there are infinitely many primes congruent *h* modulo *n*:

```
theorem Dirichlet:
  assumes "coprime h n"
  shows   "infinite {p. prime p ∧ [p = h] (mod n)}"
```

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A famous equality involving the Riemann ζ function:

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

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```
theorem zeta_even_nat:
  "zeta (2 * of_nat n) =
    of_real ((-1)^(n+1) * bernoulli (2*n) * (2*pi)^(2*n) / (2 * fact (2*n)))"
```

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Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

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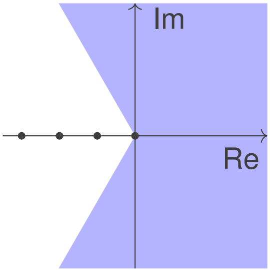
```
theorem fact_asymp_equiv:
  "fact ∼[at_top] (λn. sqrt (2*pi*n) * (n / exp 1) ^ n :: real)"
```

Some Examples

Stirling's formula for the complex Γ function:

$$\Gamma(s) \sim \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s \text{ uniformly for } |s| \to \infty \text{ with } |\mathrm{Arg}\, s| \leq \alpha < \pi$$



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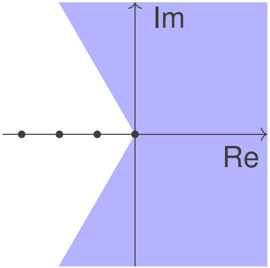


```
lemma Gamma_complex_asymp_equiv:
  fixes F and α
  assumes "α ∈ {0..<pi}"
  defines "F ≡ at_infinity ⊓ principal {z. ¦Arg z¦ ≤ α}"
  shows   "Gamma ∼[F] (λs. sqrt (2 * pi / s) powr (1 / 2) * (s / exp 1) powr s)"
```

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```
theorem sophomores_dream1:
  "integral {0..1} (λx. x powr (-x)) = (∑∞ k∈{(1::nat)..}. k powi (-k))"
```

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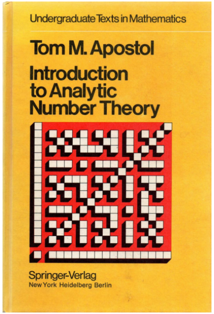
- 5 years ago, I formalised an entire textbook (for fun), with PNT as a part of it.
- Now we are tackling graduate-level maths (elliptic functions, modular forms).

A look back

(Almost) an entire undergraduate maths textbook formalised



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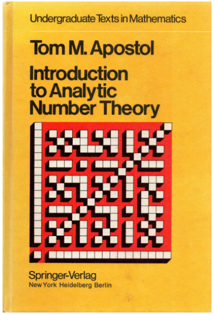
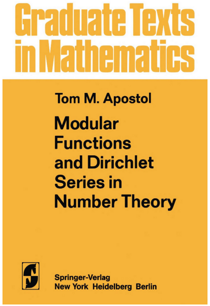


Contents: Dirichlet series, characters, Gauss sums, Riemann ζ function, L functions, lots of facts about prime numbers

Current work



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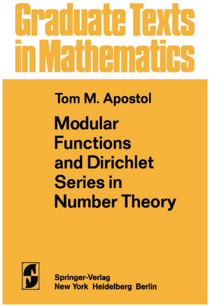

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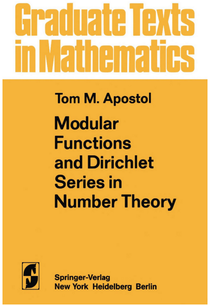

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Already done: Elliptic functions, Klein's j function, basic properties of modular forms, Dedekind's η function

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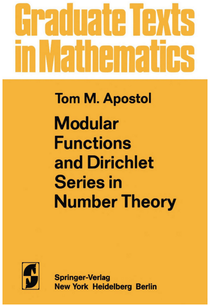

About 50 % formalised.

Already done: Elliptic functions, Klein's j function, basic properties of modular forms, Dedekind's η function

Up next: Rademacher's formula for the partition function, advanced results about modular forms

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But my impression is that things are getting better.

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Multiseries expansions

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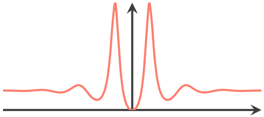

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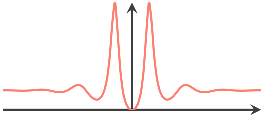

- For any $n \in \mathbb{N}$, the series

$$\sum_{k=1}^{\infty} \left(\frac{\log(k)^n}{k} - \frac{\log(k+1)^{n+1} - \log(k)^{n+1}}{n+1} \right)$$

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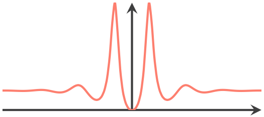

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converges due to the comparison test, since the summand is $\sim k^{-2} \log(k)^{n-1}$.

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How do computer algebra systems achieve this?

By computing Multiseries expansions (or something very similar)!

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Not competitive with Mathematica/Maple in scope or performance.

But good enough to help with all asymptotic problems I encountered in "real-world" formalisation.

Example

```
lemma
  fixes f :: "real ⇒ real" and y :: real
  assumes "y > 0"
  defines "f ≡ (λx. (cos (x/2) ^ 2 + sinh (x*y/2) ^ 2) / (cosh (x*y) - cos x))"
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  using ‹y > 0› unfolding f_def by real_asymp+
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Work in Progress: Laurent series expansions

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- A meromorphic function $f : \mathbb{C} \to \mathbb{C}$ can locally be expanded as $f(z) = \sum_{n \geq n_0} a_n (z - z_0)^n$ for some $n_0 \in \mathbb{Z}$

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The theory is all there, but a lot of code has to be written to make it nicely usable.

Application: Elliptic Functions

Definition

Let ω_1, ω_2 be two complex numbers. These span a lattice $\Lambda = \{a\omega_1 + b\omega_2 \mid a, b \in \mathbb{Z}\}$.

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- $\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$ is the “simplest” non-constant elliptic function.

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- It has the Laurent series expansion $\wp(z) = z^{-2} + \sum_{n=2}^{\infty} (n+1) G_{n+2} z^n$ at $z = 0$.

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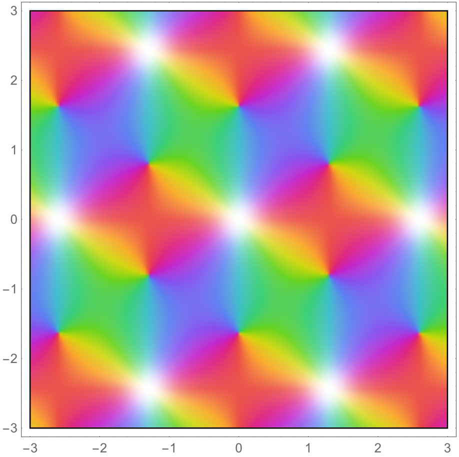

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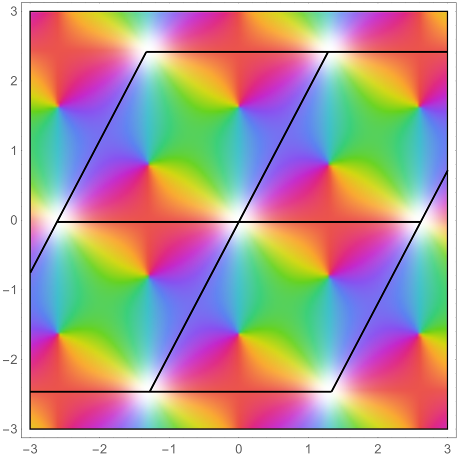

Application: Elliptic Functions

Theorem

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4\wp(z) - 140G_6$$

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Evaluating Winding Numbers

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We are mainly interested in simple closed counter-clock-wise curves.
i.e. all points have winding number 1 or 0

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Definitely *not* a fully automatic tool, but it does help immensely.

Deforming Integration Contours

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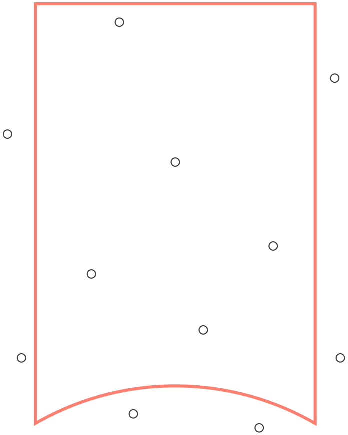
This concerns another common problem encountered when formalising a complex-analytic argument:

Deforming an integration contour in order to avoid “bad” points.

Counting Points



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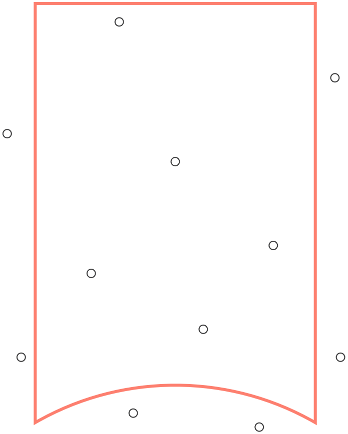

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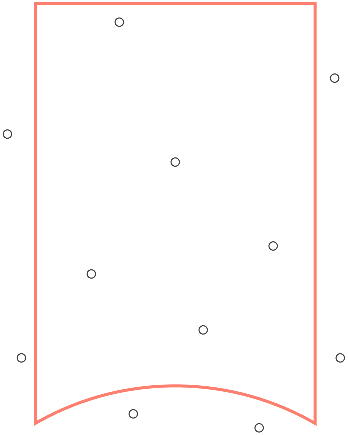

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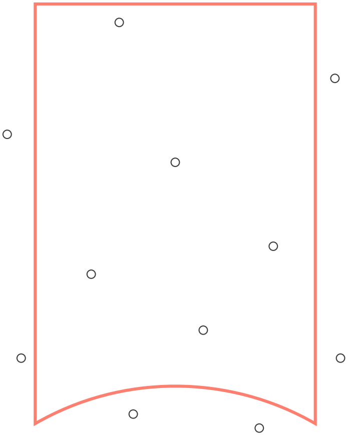

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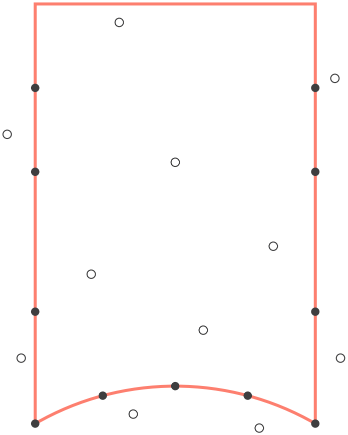

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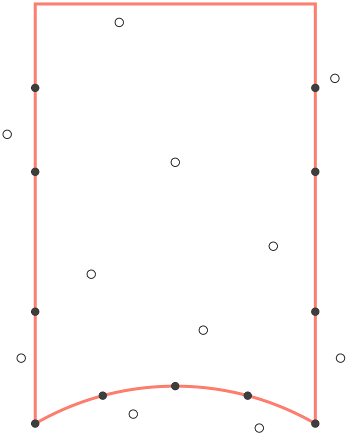

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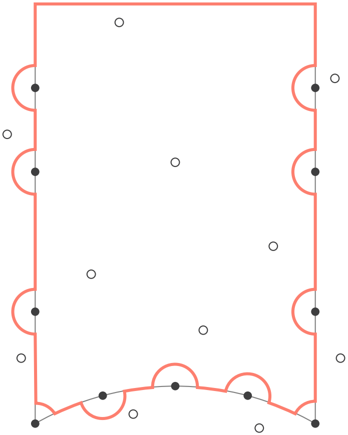
Solution: Deform γ to a contour γ_ε by adding small circular arcs of radius ε, then let $\varepsilon \to 0$.

A Complicated Integration Contour



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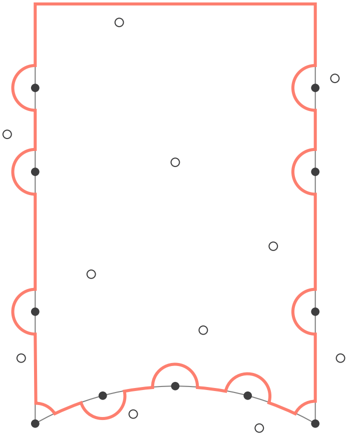

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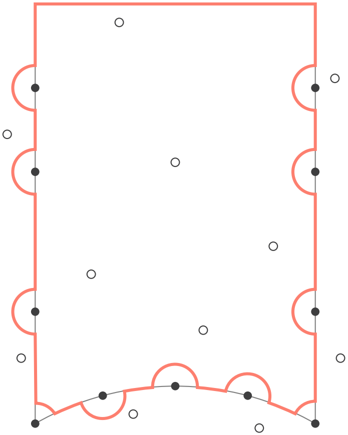

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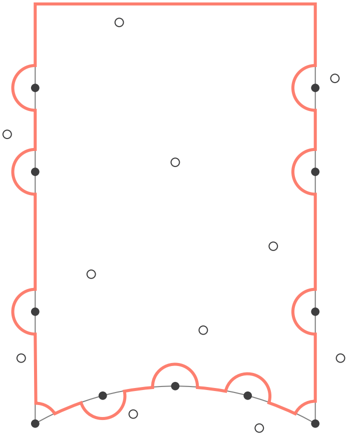

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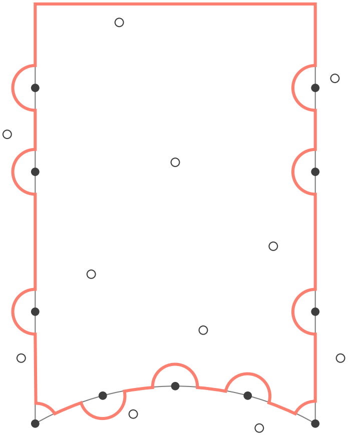

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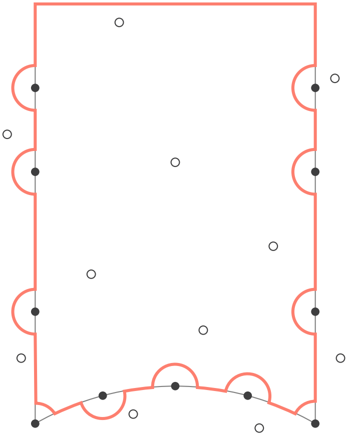

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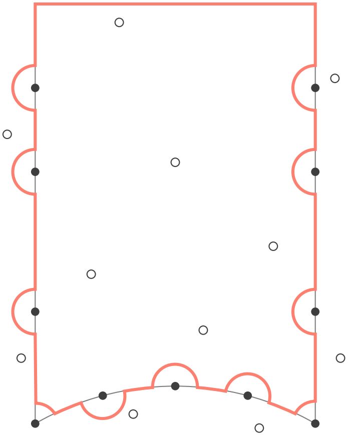

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So let’s try something smarter!

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But: For compositionality, we need to generalise to non-closed contours.

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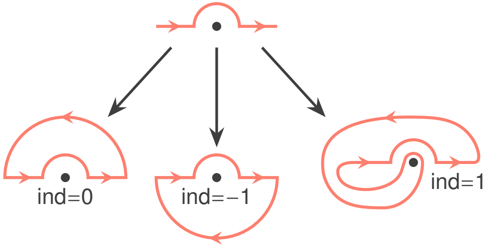

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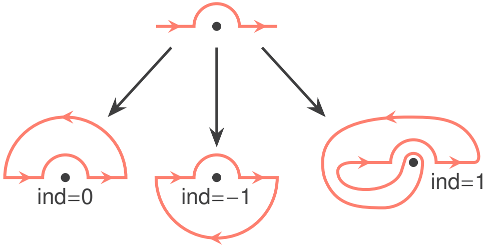

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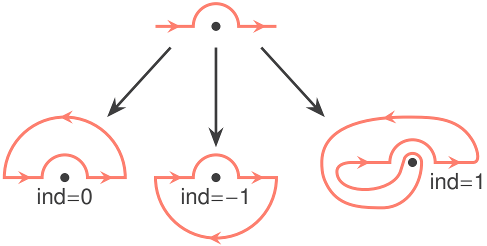

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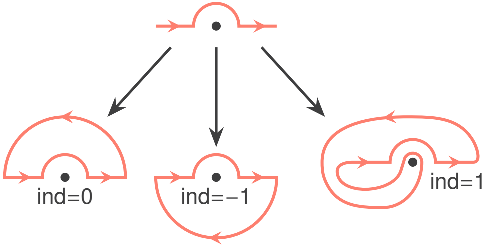

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If γ is a loop, we can easily recover the inside/outside information from this:
For a counter-clockwise loop, left = inside and right = outside.

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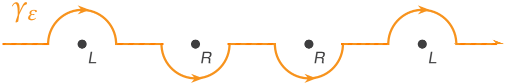

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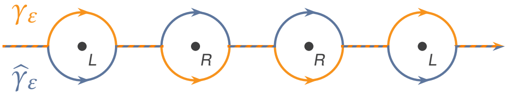

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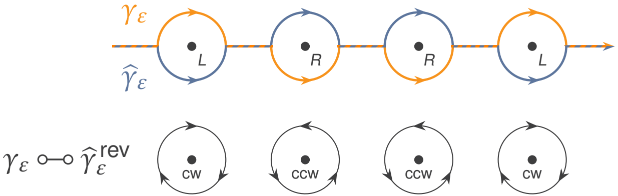

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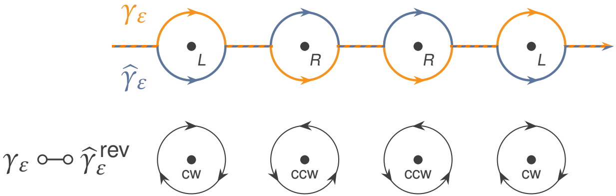

$$\mathrm{ind}_{\gamma_\varepsilon \circ\!-\!\circ \widehat{\gamma}_\varepsilon^{\mathrm{rev}}}(z) = \begin{cases} -1 & \text{if } z \in L \\ 1 & \text{if } z \in R \end{cases}$$

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Theorem

If $\gamma \underset{L/R}{\approx} \gamma_\varepsilon : \widehat{\gamma}_\varepsilon$ and γ is closed, then $\gamma \underset{L/R}{\approx} \gamma_\varepsilon$ if γ is clockwise and $\gamma \underset{L/R}{\approx} \widehat{\gamma}_\varepsilon$ if γ is counter-clockwise.

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This reduces proving $\approx$ for a concrete complicated path to proving it locally for its simple constituent paths!

Basic Avoidance Patterns

Straight line: 

Basic Avoidance Patterns

Straight line: 

Circular arc: 

Basic Avoidance Patterns

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Straight line corner:

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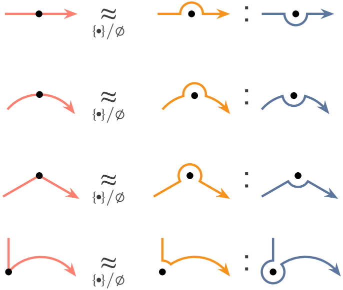

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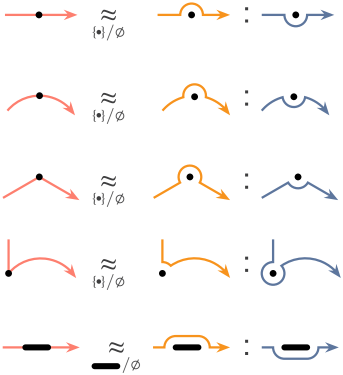

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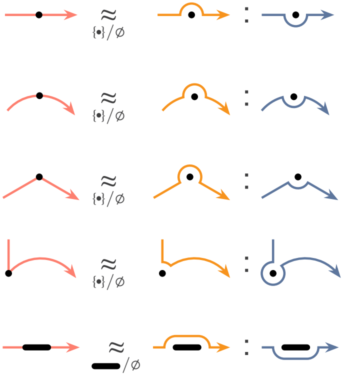
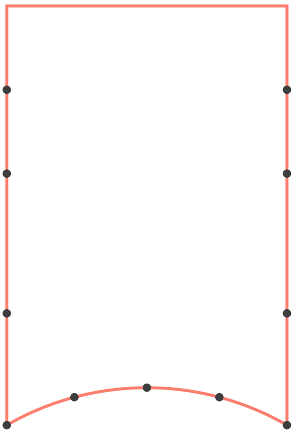


These rules have only trivial side conditions and are thus very easy to apply.

Back to our example



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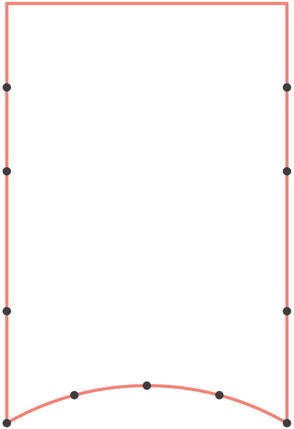

1. Prove that original contour γ is a simple counter-clockwise loop

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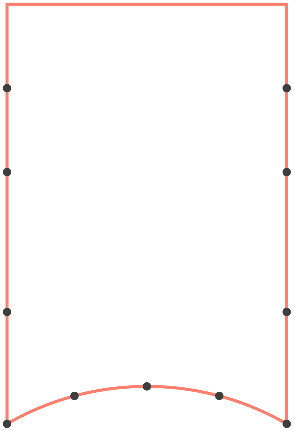

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2. Cut γ at arbitrary point between any pair of adjacent bad points

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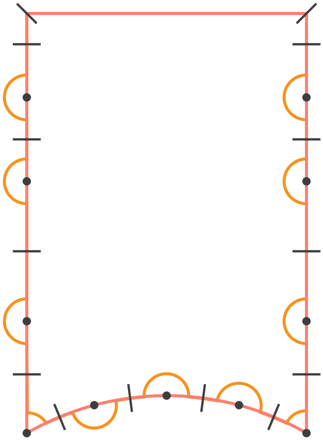

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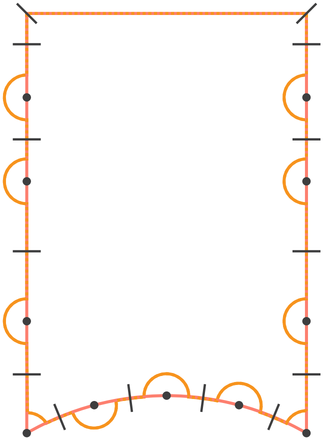

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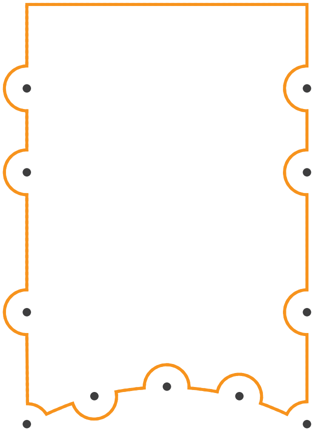

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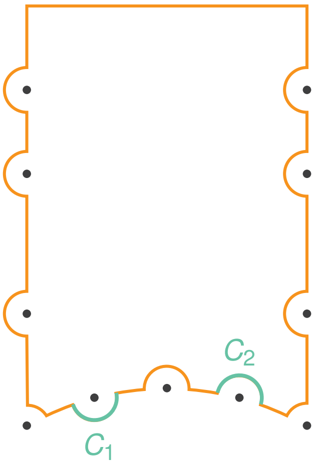
One last complication



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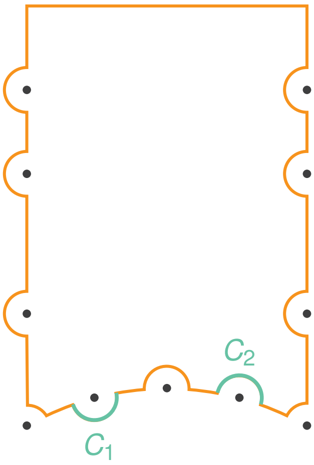

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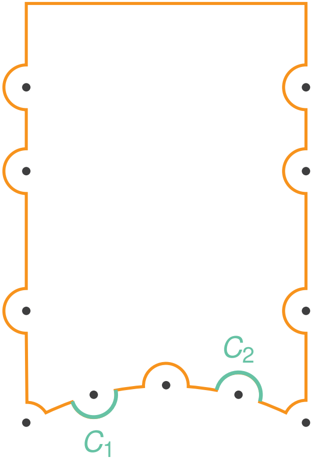

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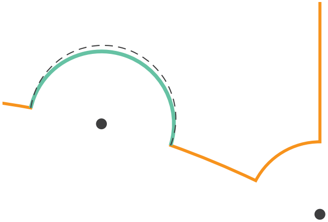

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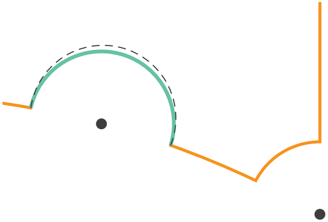

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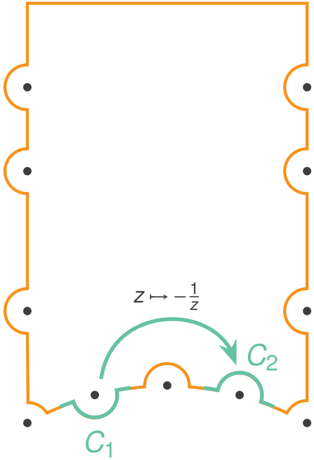

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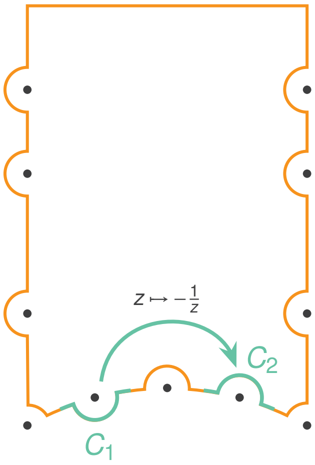

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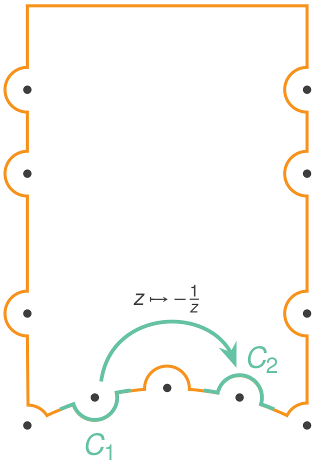

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Yes, because $\approx$ is closed under nice functions.

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In particular, this works for all affine injective functions and all holomorphic injective functions.

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In any case, thanks to our tools and libraries we are now at a point where a small team can routinely tackle graduate-level maths "for fun".