A Lindemann–Weierstrass theorem for E-functions

Effective Aspects in Diophantine Approximation

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Camille Jordan Institute, Lyon, France

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Structure of the talk

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1. Classical results and questions
2. Hermite–Lindemann and E-functions
3. Lindemann–Weierstrass and E-functions
4. Applications
5. Sketch of the proof
6. Linear dependence

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Completely effectively solved by Adamczewski–Rivoal (2018) with refinements by Bostan–Rivoal–Salvy (2022). Based on

- André's theory of E-operators (2000)
- Beukers' refinement of Siegel-Shidlovskii theorem (2006)

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If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are distinct, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are $\overline{\mathbb{Q}}$-linearly independent.

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Question 3

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ be $\mathbb{Q}$-linearly independent. Are $f(\alpha_1), \dots, f(\alpha_n)$ algebraically independent over $\mathbb{Q}$?

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Beukers (2006) : Refinement of the Siegel–Shidlovskii theorem

Let $Y = (f_1, \dots, f_n)^\top$ be a vector of E-functions satisfying $Y' = AY$ where A is an $n \times n$-matrix with entries in $\overline{\mathbb{Q}}(z)$. Denote the common denominator of the entries of A by $T(z)$.

Let $\alpha \in \overline{\mathbb{Q}}$ satisfying $\alpha T(\alpha) \neq 0$.

Then, for any homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ such that $P(f_1(\alpha), \dots, f_n(\alpha)) = 0$, there exists a polynomial $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the variables $X_1, \dots, X_n$, such that $Q(\alpha, X_1, \dots, X_n) = P(X_1, \dots, X_n)$ and $Q(z, f_1(z), \dots, f_n(z)) = 0$.

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- Bostan–Rivoal–Salvy (2022) : $|\mathrm{Exc}(f)| = d$

$$f(x) = \sum_{n=0}^{\infty} \binom{n+d}{d} \frac{1}{(a+d+1)_n} z^n, \quad (a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}).$$

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Every transcendental E-function f can be written in a unique way as $f = p + qg$ with $p, q \in \overline{\mathbb{Q}}[z]$, q monic, $q(0) \neq 0$, $\deg(p) < \deg(q)$ and g is a purely transcendental E-function.

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It takes an E-function f as input. It first says whether f is transcendental or not. Then it returns $\mathrm{Exc}(f)$ if f is transcendental.

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Remark : improved by Bostan–Rivoal–Salvy (2022). This yields an algorithm to compute canonical decompositions.

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On Beukers' lifting result

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Even more : Every such relation can be explicitely given by the system through a desingularization process by Beukers.

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Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers and consider $f_i(z) = e^{\alpha_i z}$.

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- Are $f(\alpha_1 z), \ldots, f(\alpha_n z)$ linearly independent over $\overline{\mathbb{Q}}(z)$?
- Is $T(1)$ non-zero ?

How to choose n and $\alpha_1, \ldots, \alpha_n$ to answer yes twice ?

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Examples

- $\psi(\exp)=1/(1-z)$ and $\mathfrak{S}(\exp)=\{1\}$.
- $\psi(J_0)=1/\sqrt{1+z^2}$ and $\mathfrak{S}(J_0)=\{-i,i\}$.

A Lindemann–Weierstrass theorem for E-functions

D. (2022)

Let f be an E-function and $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ non-zero such that

- for all i, $\alpha_i \notin \mathrm{Exc}(f)$,
- for all $i \neq j$ and all $\rho_1, \rho_2 \in \mathfrak{S}(f)$, $\alpha_i/\alpha_j \neq \rho_1/\rho_2$.

Then $1, f(\alpha_1), \ldots, f(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.

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exp is purely transcendental and $\mathfrak{S}(\exp) = \{1\}$. The second condition reads $\alpha_i \neq \alpha_j$: we retrieve the Lindemann–Weierstrass theorem.

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Bessel function

J_0 is purely transcendental and $\mathfrak{S}(J_0) = \{-i, i\}$. The second condition reads $\alpha_i^2 \neq \alpha_j^2$. We retrieve the linear part of Siegel's result.

Structure of the talk

Entire hypergeometric functions

We consider (entire) hypergeometric functions

$$F(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n}{(b_1)_n \cdots (b_s)_n} z^n,$$

where $s > r \geq 0$, $a_i, b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $(a)_n$ denotes the Pochhammer symbol defined by $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \geq 1$.

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Let F be a non-polynomial hypergeometric function with $s > r$ and rational parameters. Let $\alpha_1, \ldots, \alpha_n$ be pairwise non-zero distinct algebraic numbers which are not exceptional values for F. Then $1, F(\alpha_1), \ldots, F(\alpha_n)$ are $\overline{\mathbb{Q}}$-linearly independent.

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- $\psi(f) = H(kz^k)$ where H is a hypergeometric G-function.

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Hints : Write $k = s - r$ and consider the E-function $f(z) = F(z^k)$.

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- Elements of $\mathfrak{S}(f)$ are of the form ρ/k where ρ is a k-th root of unity.

Entire hypergeometric functions

We consider (entire) hypergeometric functions

$$F(z)=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}z^n,$$

where $s>r\geq 0$, $a_i,b_j\in\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$ and $(a)_n$ denotes the Pochhammer symbol defined by $(a)_0=1$ and $(a)_n=a(a+1)\cdots(a+n-1)$ for $n\geq 1$.

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- $\psi(f)=H(kz^k)$ where H is a hypergeometric G-function.
- Elements of $\mathfrak{S}(f)$ are of the form ρ/k where ρ is a k-th root of unity.
- if $\alpha_i=\beta_i^k$, then $\alpha_i\neq\alpha_j$ implies $\beta_i/\beta_j\neq\rho_1/\rho_2$.

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$$f(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor 2n/3 \rfloor} \frac{(1/4)_{n-m}}{(2n-3m)!(2m)!} z^n.$$

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Structure of the talk

General statement

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Let $f_1, \dots, f_n$ be E-functions with pairwise disjoint sets $\mathfrak{S}(f_i)$. Let $\alpha \in \overline{\mathbb{Q}}$ non-zero be such that $\alpha \notin \mathrm{Exc}(f_i)$ for all i. Then $1, f_1(\alpha), \dots, f_n(\alpha)$ are $\overline{\mathbb{Q}}$-linearly independent.

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- Laplace transform : $\psi(\mathcal{L}_0 f_0) + \cdots + \psi(\mathcal{L}_n f_n) = 0$.

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- for all i, $\lambda_i f_i(\alpha) = (\mathcal{L}_i f_i)(\alpha) \in \overline{\mathbb{Q}}$.
- $\lambda_i \neq 0$ for at least one i : $f_i(\alpha) \in \overline{\mathbb{Q}}$, a contradiction.

□

Structure of the talk

What about linear dependence ?

Question

How to determine all $\overline{\mathbb{Q}}$-linear relations between values of 1, exp, cos and sin at algebraic points ?

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How to determine all $\overline{\mathbb{Q}}$-linear relations between values of 1, exp, cos and sin at algebraic points ?

We shall show that they all come from specializations of

$$\sin(-z) = -\sin(z)$$

$$\cos(-z) = \cos(z)$$

$$e^{iz} = \cos(z) + i\sin(z).$$

Restrictions on linear relations

Let $f_1, \dots, f_n$ be E-functions and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ such that $f_i(\alpha_i)$ is transcendental and $\beta_1 f_1(\alpha_1) + \cdots + \beta_n f_n(\alpha_n) \in \overline{\mathbb{Q}}$, for algebraic numbers $\beta_1, \dots, \beta_n$ not all zero.

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Consequence

For every k, we have $F_k(1) \in \overline{\mathbb{Q}}$, that is

$$\textstyle\sum_{i \in C_k} \beta_i f_i(\alpha_i) \in \overline{\mathbb{Q}}.$$

Restrictions on linear relations

By Beukers' lifting result and André's theory of E-operators, a $\overline{\mathbb{Q}}$-linear relation between transcendental values of $1, f_1, \ldots, f_n$ at algebraic points $\alpha_1, \ldots, \alpha_n$ are spanned by specializations of $\overline{\mathbb{Q}}(z)$-linear relations between 1 and functions

$$f_i^{(k)}(gz)$$

with $1 \leq i \leq n$, $1 \leq k \leq r_i - 1$ and $g \in G$, where r_i is the differential order of f_i.

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- Consider $\exp(iz)$, $\cos(z)$ and $\sin(z)$.
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- Relations come from specializations at $\alpha \in \overline{\mathbb{Q}}^*$ of $\overline{\mathbb{Q}}(z)$-linear relations between $1, \exp(iz), \exp(-iz), \cos(-z), \cos(z), \sin(-z), \sin(z)$.

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- Modulo $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$, we search for a relation

$$P_1(z) + P_2(z)\cos(z) + P_3(z)\sin(z) = 0,$$

with $P_i(z)$ in $\overline{\mathbb{Q}}[z]$. It yields $P_1 = P_2 = P_3 = 0$.

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  with $P_i(z)$ in $\overline{\mathbb{Q}}[z]$. It yields $P_1 = P_2 = P_3 = 0$.
- All relations come from specializations at $\alpha \in \overline{\mathbb{Q}}^*$ of $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$.

What about the second formulation ?

Question 3

Let f be an E-function and $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ linearly independent over $\mathbb{Q}$.
Are $f(\alpha_1), \dots, f(\alpha_n)$ algebraically independent over $\mathbb{Q}$?

In the case of $f = \exp$, the Siegel–Shidlovskii theorem is sufficient because $\exp(\alpha_1 z), \dots, \exp(\alpha_n z)$ are algebraically independent over $\overline{\mathbb{Q}}[z]$.

Thank you for your attention !