Cusped waves and special functions

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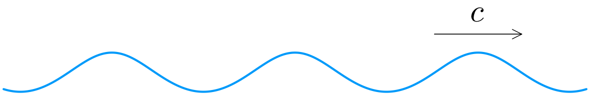
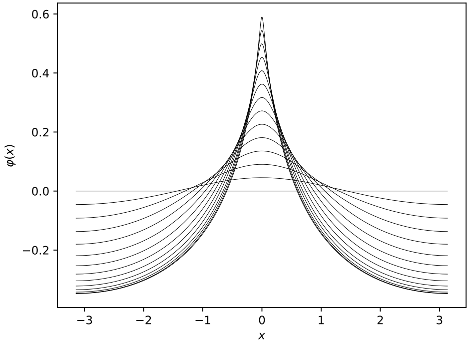
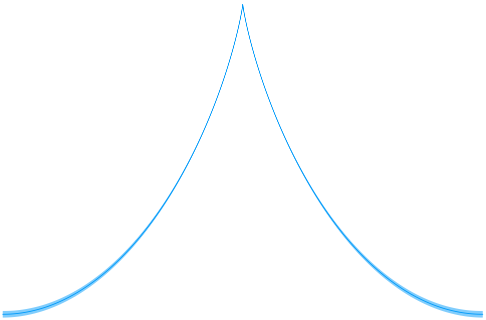
Cusped traveling waves







Fractional KdV: $\quad -c\varphi' + \varphi\varphi' = |D|^{\alpha}\varphi', \qquad \widehat{|D|^{\alpha} f}(\xi) = |\xi|^{\alpha}\widehat{f}(\xi)$

Software

Arb in Julia through Arblib.jl

```
julia> zeta(ArbSeries((0.5, 1)), Arb(0.3))
[0.0111527803099698 +/- 7.32e-17] + [-1.83287963677582 +/- 3.39e-15]⋅x + 𝒪(x^2)
```

Low-medium precision 30-100 bits

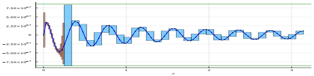
Short series degree < 10

Special functions Lots of them! Often around removable singularities.

Main methods

Integration

$$\int_0^{\pi} |I_\alpha(x, y)| w_\alpha(y)\, dy$$



Bounding functions

Taylor models

$$f(x) = \sum_{i=0}^{n} p_i (x - x_0)^i + \Delta (x - x_0)^{n+1}$$

Removable singularities

$$\frac{\sin x}{x}$$

Theorem

Theorem (D, Gómez-Serrano, 2022)

There is a 2π-periodic traveling wave φ of the Burgers-Hilbert equation, which behaves asymptotically at $x = 0$ as

$$\varphi(x) = c + \frac{1}{\pi}|x| \log |x| + \mathcal{O}(|x|\sqrt{\log |x|}).$$

Theorem (D, in preparation)

There is a 2π-periodic traveling wave φ of the fractional KdV equation for every $\alpha \in (-1, 0)$, which behaves asymptotically at $x = 0$ as

$$\varphi(x) = c - \nu_\alpha |x|^{-\alpha} + \mathcal{O}(|x|^p),$$

for a given $\nu_\alpha > 0$ and $-\alpha < p \leq 1$.

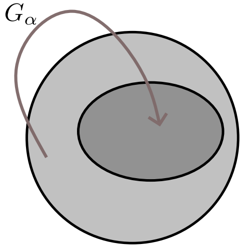
Reduction to fixed point problem

$$u(x) = u_\alpha(x) + w_\alpha(x)v(x)$$

$$v = G_\alpha[v], \quad G_\alpha[v] := (I - T_\alpha)^{-1}\left(-F_\alpha - N_\alpha v^2\right)$$

$$n_\alpha = \|N_\alpha\|_{L^\infty(\mathbb{T})}, \quad D_\alpha = \|T_\alpha\|, \quad \delta_\alpha = \|F_\alpha\|_{L^\infty(\mathbb{T})}$$

If $D_\alpha < 1$ then $\delta_\alpha < \dfrac{(1 - D_\alpha)^2}{4n_\alpha} \implies G$ is a contraction



Goal: Compute bounds for D_α, δ_α and n_α

n_α, D_α and δ_α

$$\delta_\alpha < \frac{(1-D_\alpha)^2}{4n_\alpha}$$

$$\begin{aligned}
n_\alpha &= \sup_{0<x<\pi} |N_\alpha(x)| = \sup_{0<x<\pi} \left| \frac{w_\alpha(x)}{2u_\alpha(x)} \right| \\
D_\alpha &= \sup_{0<x<\pi} \mathcal{T}_\alpha(x) = \sup_{0<x<\pi} \left| \frac{1}{\pi w_\alpha(x) u_\alpha(x)} \right| \int_0^\pi |I_\alpha(x,y) w_\alpha(y)| \, dy \\
\delta_\alpha &= \sup_{0<x<\pi} |F_\alpha(x)| = \sup_{0<x<\pi} \left| \frac{1}{w_\alpha(x) u_\alpha(x)} \left(\mathcal{H}^\alpha[u_\alpha](x) + \frac{1}{2} u_\alpha(x)^2 \right) \right|
\end{aligned}$$

Choosing u_α

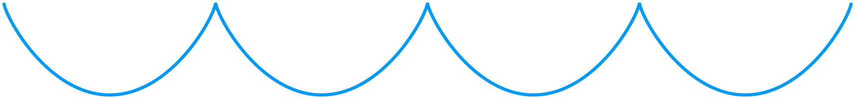
Requirements for u_α

- 2π-periodic
- $\mathcal{H}^\alpha[u_\alpha]$ easy to handle
- $u_\alpha(x) = \nu_\alpha |x|^{-\alpha} + \mathcal{O}(|x|^p)$

Clausen functions

- $\tilde{C}_s(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)-1}{n^s} = \mathrm{Re}\left(\mathrm{Li}_s(e^{ix})\right) - \zeta(s)$
- $\mathcal{H}^\alpha[\tilde{C}_s(x)] = -\tilde{C}_{s-\alpha}(x)$
- $\tilde{C}_s(x) = \Gamma(1-s)\sin\left(\frac{\pi}{2}s\right)|x|^{s-1} + \mathcal{O}(x^2)$

$$u_\alpha(x) = a_{\alpha,0}\tilde{C}_{1-\alpha}(x) + \cdots$$



Choosing u_α

$$u_\alpha(x) = \sum_{j=0}^{N_{\alpha,0}} a_{\alpha,j}\,\tilde{C}_{1-\alpha+jp_\alpha}(x) + \sum_{n=0}^{N_{\alpha,1}} b_{\alpha,n}(\cos(nx) - 1)$$

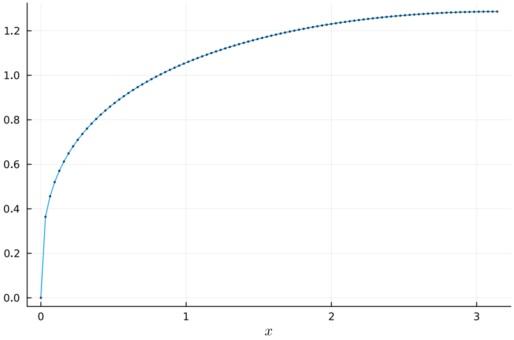
In reality: Asymptotic analysis and numerical optimization

Now: "God given"

Case study: $\alpha = -0.33$

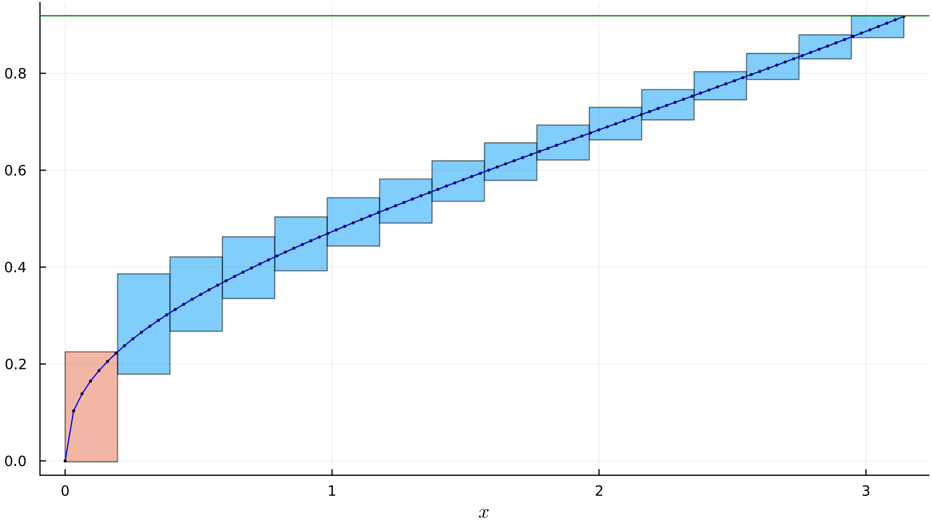
$$u_\alpha(x) = \sum_{j=0}^{3} a_{\alpha,j}\,\tilde{C}_{1-\alpha+jp_\alpha}(x) + \sum_{n=0}^{2} b_{\alpha,n}(\cos(nx) - 1)$$

$$w_\alpha(x) = |x|^{0.75}$$



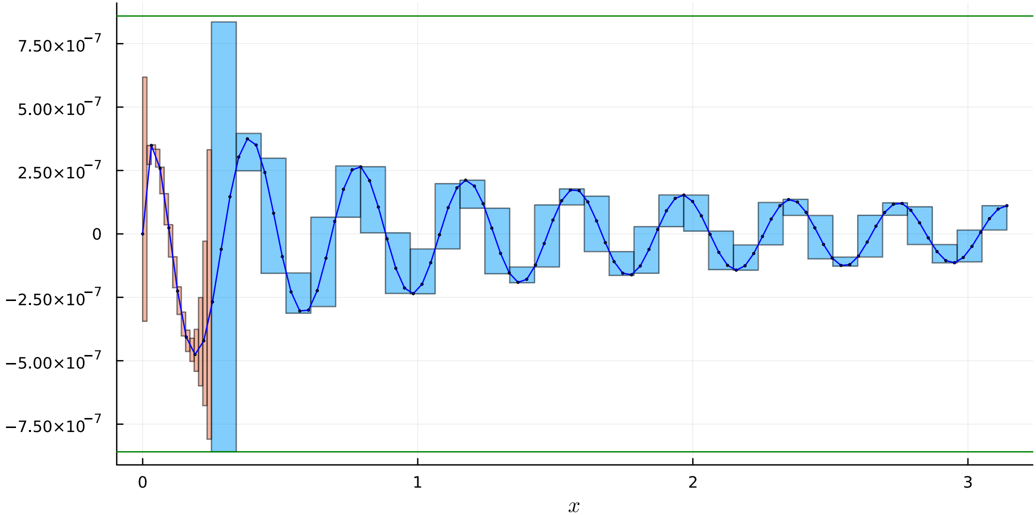
Case study: $\alpha = -0.33$ - n_α

$$n_\alpha = \sup_{0<x<\pi} |N_\alpha(x)| \leq 0.918755573220551$$



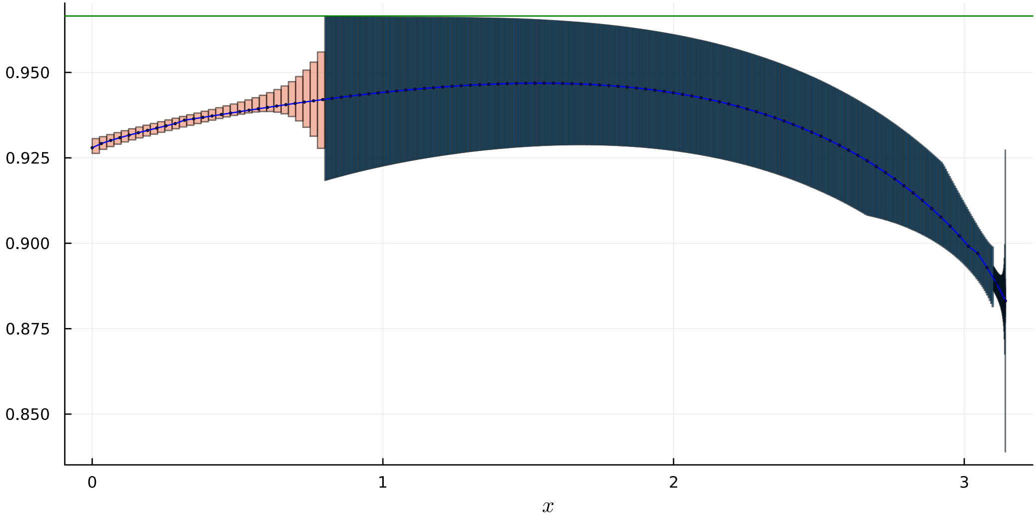
Case study: $\alpha = -0.33$ - δ_α

$$\delta_\alpha = \sup_{0<x<\pi} |F_\alpha(x)| \leq 8.590147533166941 \cdot 10^{-7}$$



Case study: $\alpha = -0.33$ - D_α

$$D_\alpha = \sup_{0<x<\pi} \mathcal{T}_\alpha(x) \leq 0.9665201427415013$$



Case study: $\alpha = -0.33$

$$\delta_\alpha \leq 8.590147533166941 \cdot 10^{-7} < 0.0003050051816611863 \leq \frac{(1 - D_\alpha)^2}{4n_\alpha}$$

Success!

Case study: $\alpha = -0.33$ - key tools

Evaluation of Clausen functions $\qquad \tilde{C}_{1.3}([1.2 \pm 0.001]) \subseteq [-3.97 \pm 4.95 \cdot 10^{-3}]$

Integration

$$\int_0^{\pi} |I_\alpha(x,y)| w_\alpha(y)\ dy$$

Asymptotic expansions

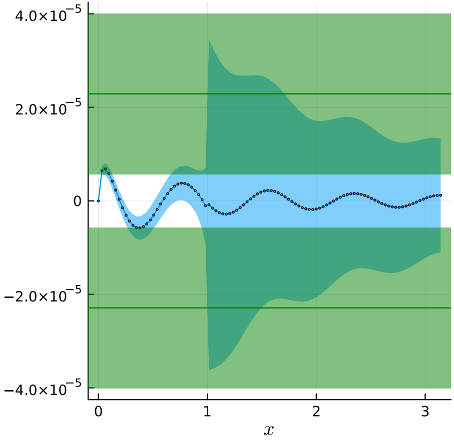
$$u_\alpha(x) = \sum_{j=0}^{N_{\alpha,0}} a_{\alpha,j}^0 |x|^{-\alpha + jp_\alpha} + \cdots$$

$[-1, 0)$

$$[-1, 0) = [-1, -1] \cup (-1, -0.9999) \cup [-0.9999, -0.0012] \cup (-0.0012, 0)$$

$[-0.9999, -0.0012]$

- Split into 72 034 subintervals
- For example $\alpha = [-0.3 \pm 3.4 \cdot 10^{-6}]$



Computational details

$[-0.9999, -0.0012]$ run on Dardel - NAISS computer cluser at KTH

	-1	$(-1, -0.9999)$	$[-0.9999, -0.0012]$	$(-0.0012, 0)$
n_α	5s	7s	500 core hours	$<$1s
$\delta_{-\alpha}$	15m	12m	4500 core hours	6s
$D_{-\alpha}$	90s	10m	6750 core hours	40s
LoC	2593	4056	5421	2821

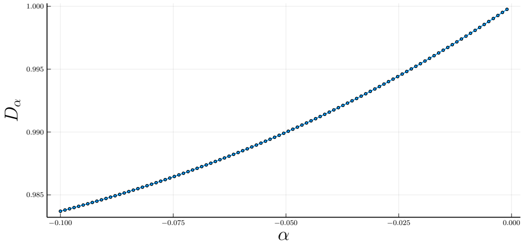
Difficulty 1 - removable singularities

- C_s and S_s for $s \in \mathbb{Z}_{\geq 0}$
- $\zeta(s,x) - \frac{1}{s-1}$ for $s = 1$
- $\frac{\sin x}{x}$

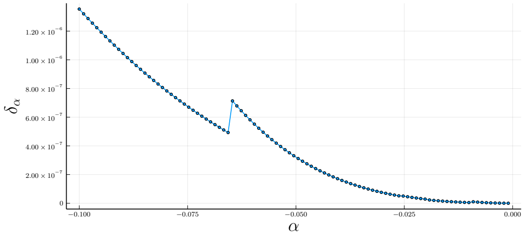
$$\frac{d^p}{dx^p}\frac{f(x)}{x^n} = \sum_{k=0}^{m} \frac{(k+p)!}{k!} f_{k+n+p}(0)x^k + \frac{(m+p+1)!}{(m+1)!} f_{m+n+p+1}(\xi)x^{m+1}, \quad \xi \in [0,x]$$

Difficulty 2 - near $\alpha = 0$

- $n_\alpha = \mathcal{O}(1)$
- $D_\alpha = 1 - \mathcal{O}(\alpha)$
- $\delta_\alpha = \mathcal{O}(\alpha^2)$

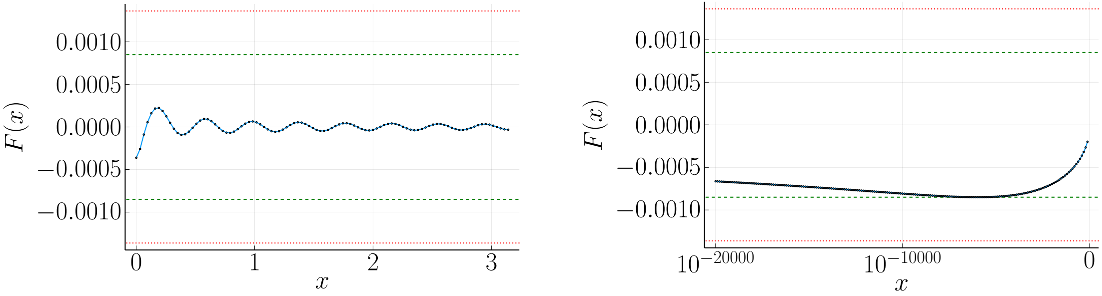


$$\delta_\alpha < \frac{(1 - D_\alpha)^2}{4 n_\alpha} \;\rightarrow\; \mathcal{O}(\alpha^2) < \mathcal{O}(\alpha^2)$$



Use Taylor models to control $\mathcal{O}$

Difficulty 3 - asymptotic defect for $\alpha = -1$

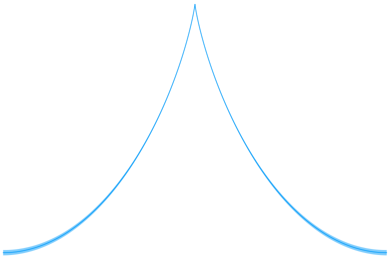


$$F_{-1}\left(\left[0, 10^{-10000000}\right]\right) \subseteq [0 \pm 4.27 \cdot 10^{-4}]$$

$$F_{-1}(x) \sim \left(\sqrt{\log(1/x)}\right)^{-1}$$

Code

Source code as well as notebooks containing all the proofs for Burgers-Hilbert ($\alpha = -1$) available at `https://github.com/Joel-Dahne/BurgersHilbertWave.jl`.



Special functions

- Gamma
- Zeta
- Polygamma
- Eta
- Incomplete beta function
- Hypergeometric 2F1 function
- Reciprocal gamma function
- Rising factorial
- Deflated zeta function
- Lerch Phi function
- Polylogarithm

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Thank you!