Modular, Algebraic, and Gamma-Evaluations of Hypergeometric Series

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Joint with Frits Beukers.

Recall the Euler–Gauss hypergeometric function:

$$ {}_2F_1(a,b;c;z)=\sum_{n\ge0}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

for $|z|<1$ and suitably extended analytically to $\mathbb{C}$, with $(a)_n=a(a+1)\cdots(a+n-1)$, the Pochammer symbol.

Special evaluations:

- Gauss: if $\Re(c-a-b)>0$

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\;.$$

Simple integral manipulations.

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• Fricke: when τ is in the fundamental domain of the modular group (else multiply the RHS by a suitable $i^k(c\tau+d)$)

$${}_2F_1(1/12,5/12;1;1728/j(\tau))=E_4^{1/4}(\tau)\ .$$

Modular interpretation, differential equations.

• Beukers–Wolfart:

$${}_2F_1(1/12,5/12;1/2;1323/1331)=(3/4)11^{1/4}\ .$$

CM theory.

p-adic analogue for $p=7$ (note $7^2\mid 1323$):

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Goals

Ultimate goal: find the largest possible generalizations of all this. More precisely:

• Generalizing Gauss' 3-parameter evaluation at $z = 1$: Find all 2-parameter, 1-parameter, and even 0-parameter evaluations of a specific kind: conjecturally they can all be given in finitely many infinite families.

• Generalizing Fricke's E_4 evaluation: find all functional modular evaluations, in particular corresponding to hyperbolic arithmetic triangle groups.

• Generalizing Beukers–Wolfart's algebraic evaluation: find all algebraic evaluations of ${}_2F_1(a, b; c; z)$. Hopeless in general, but may be possible if we assume a, b, c, and also z rational. Note that all have p-adic analogs.

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Special Cases

If a, b, $c-a$, or $c-b$ is integral, ${}_2F_1(a,b;c;z)$ is degenerate in a suitable sense. For instance

$$\begin{aligned}{}_2F_1(a,-1;c;c/a)&={}_2F_1(a,c+1;c;c/(c-a))=0\\ {}_2F_1(a,2;c;(c-2)/(a-1))&=(a-1)(c-1)/(a-c+1)\\ {}_2F_1(1-a,b;b+2;b/(a+b))&=(b+1)(a/(a+b))^a\ .\end{aligned}$$

We exclude those.

Slightly more subtle: ${}_2F_1(a,b;c;z)$ may be an algebraic function of z. Easy criterion due to Schwarz. Examples:

$${}_2F_1(t,t+m/2;n/2;z),\ {}_2F_1(t,-t+p;n/2;z),\ {}_2F_1(t,t+m/2;2t+p;z)$$

with m and n odd integers and p integer. If we exclude a, b, $c-a$, $c-b$ integral as above, may be the only examples with a, b, and c linear in one parameter t. We also exclude those.

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Gamma Evaluations I

Well-known: there exist four two-parameter evaluations: one for $z=-1$, two for $z=1/2$, and one for $z=2$:

$$
\begin{aligned}
{}_2F_1(a,b;a-b+1;-1) &= \frac{1}{2}\frac{\Gamma(a/2)\Gamma(a-b+1)}{\Gamma(a)\Gamma(a/2-b+1)}\;,\\
{}_2F_1(a,b;(a+b+1)/2;1/2) &= \frac{\Gamma(1/2)\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}\;,\\
{}_2F_1(a,1-a;c;1/2) &= \frac{\Gamma(c/2)\Gamma((c+1)/2)}{\Gamma((c+a)/2)\Gamma((1+c-a)/2)}\;,\\
{}_2F_1(a,b/2;b;2) &= e^{\pi i a/2}\frac{\Gamma(1/2)\Gamma((b+1)/2)}{\Gamma((a+1)/2)\Gamma((b-a+1)/2)}\;.
\end{aligned}
$$

Gamma Evaluations II

Less well-known: these belong in fact to four infinite families, obtained non-trivially from the contiguity relations:

More precisely, they are evaluations as finite linear combinations of gamma quotients of:

$$
\begin{aligned}
&{}_2F_1(a,b;n+a-b;-1)\ ,\\
&{}_2F_1(a,b;n+(a+b+1)/2;1/2)\ ,\\
&{}_2F_1(a,n-a;c;1/2)\ ,\quad \text{and}\\
&{}_2F_1(a,b;n+2b;2)\ ,
\end{aligned}
$$

where $n \in \mathbb{Z}$.

Conjecturally, there are no other two-parameter evaluations.

Gamma Evaluations III

We now focus on one-parameter evaluations.
A pure Γ-expression $E(t)$ is of the type $E(t)=\phi(t)u^t\prod_{1\le i\le g}\Gamma(t+r_i)^{e_i}$ with $\phi(t)$ periodic of period 1 with finitely many Fourier coefficients (we can trivially replace $t+r_i$ by a_it+r_i, period 1 by another period, and generalize to more variables). A mixed Γ-expression is a finite linear combination of pure ones.

A Γ-evaluation is of the type ${}_2F_1(a(t),b(t);c(t);z)=E(t)$ with E a (pure or mixed) Γ-expression, and where we assume that $a(t)$, $b(t)$, and $c(t)$ are linear in t. Trivial equivalences, for instance $t\mapsto ut+v$, and notion of primitivity.

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Gamma Evaluations IV

Example:

$$ {}_2F_1(t,-2t+2;4/3;8/9)=\frac{1+e^{-2\pi it}}{2}(-1/3)^t\frac{\Gamma(t-1/2)}{\Gamma(t+1/6)}\ . $$

Systematic search using contiguity relations. Conjecture: up to equivalence and primitivity, there is only a finite number of pure Γ-evaluations, more precisely 120 with $z\in\mathbb{Q}$, 48 with $z\in\mathbb{Q}(\sqrt{5})$, 36 with $z\in\mathbb{Q}(\sqrt{3})$, 24 with $z\in\mathbb{Q}(\sqrt{2})$, 4 with $z\in\mathbb{Q}(\sqrt{-3})$, and no others, for a total of 232 pure Γ-evaluations.

In addition, one can show that all these 232 pure Γ-evaluations can be extended to 232 doubly infinite families of mixed Γ-evaluations (with m and $n\in\mathbb{Z}$ as above, such as ${}_2F_1(t,-2t+2+m;4/3+n;8/9)$). But...

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Gamma Evaluations V

There exist mixed Γ-evaluations which are not obtained by extending a pure one, for example

$$ {}_2F_1(3t,3t+1/2;4t+1;8/9)=108^t\left(\sqrt{2/3}\frac{\Gamma(t+1/4)\Gamma(t+3/4)}{\Gamma(t+1/3)\Gamma(t+2/3)}-\sqrt{2}\frac{\Gamma(t+1/4)\Gamma(t+3/4)}{\Gamma(t+1/6)\Gamma(t+5/6)}\right)\ . $$

We do not yet know how to classify those.

Some (exactly eight) Γ-evaluations do not involve gamma factors at all, and two are in fact periodic:

$$ {}_2F_1(2t,1/2;1-t;4)=(1+2e^{2\pi it})/3 $$

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In particular, in connection with our third subject, this gives eight infinite families of algebraic evaluations of ${}_2F_1(a,b;c;z)$ with all four variables rational.

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Hyperbolic Triangles

We now generalize Fricke's formula for $E_4^{1/4}$. Recall some hyperbolic geometry: $\overline{\mathcal{H}}$ completed upper half plane, $\Delta \subset \overline{\mathcal{H}}$ a hyperbolic triangle (edges geodesics) with angles $(\pi/p, \pi/q, \pi/r)$ with p, q, and r in $\mathbb{Z}_{\geq 2} \cup \infty$ with $1/p + 1/q + 1/r < 1$. It is cocompact if p, q, and r are finite. We simply write $\Delta = (p, q, r)$.

Given vertices τ_0, τ_1, τ_∞, the Schwarz reflection principle implies that there exists a unique meromorphic function J from $\overline{\mathcal{H}}$ to $\mathbb{P}^1(\mathbb{C})$ such that $J(\tau_0) = 0$, $J(\tau_1) = 1$, $J(\tau_\infty) = \infty$ and invariant under the group Γ of orientation-preserving maps generated by reflections along the sides of Δ: J is a Hauptmodul, and Γ a hyperbolic triangle group.

Note that given the vertices τ_0, τ_1, and τ_∞ we can give a completely explicit formula for $J^{-1}(z)$, the functional inverse, hence implicitly for $J(\tau)$, see below.

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The Hypergeometric Differential Equation I

Recall that a, b, c being fixed, $y(z) = {}_2F_1(a, b; c; z)$ is a solution of $(1 - z)y'' + (c - (a + b + 1)z)y' - aby = 0$, we denote it by $F_0(z)$. If $c \notin \mathbb{Z}$, a second independent solution is $F_1(z) = z^{1-c}{}_2F_1(a + 1 - c, b + 1 - c; 2 - c; z)$ (similarly if $c \in \mathbb{Z}$, but then a $\log(z)$ enters).

The crucial link with hyperbolic triangles (known since the 19th century, Schwarz theory) is to associate to a triangle (p, q, r) the hypergeometric functions with parameters a, b, and c such that

$$1 - c = 1/p, \quad c - a - b = 1/q, \quad \text{and} \quad b - a = 1/r$$

(more generally $\pm 1/p$, $\pm 1/q$, and $\pm 1/r$). Indeed:

Proposition (Schwarz) If we set $H(z) = F_1(z)/F_0(z)$, the image by H of the real line $\mathbb{R}$ is a hyperbolic triangle of type (p, q, r) in the disc model of the hyperbolic plane.

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Hypergeometric Differential Equation II

From this proposition, given the vertices τ_0 and τ_1 (τ_∞ is implicit since (p, q, r) is given) it is easy to give an explicit formula for $J^{-1}(z)$, the functional inverse of the Hauptmodul: if as above $H(z) = F_1(z)/F_0(z)$, then

$$J^{-1}(z) = \frac{\overline{\tau_0}H(z) - \delta\tau_0}{H(z) - \delta}\;, \quad \text{with} \quad \delta = H(1)\frac{\tau_1 - \overline{\tau_0}}{\tau_1 - \tau_0}\;.$$

The Main Theorem I

Let $J(\tau)$ be the above Hauptmodul for the triangle (p,q,r) and given vertices, and set

$$E(\tau)=J(\tau)^{-(1-1/p)}(1-J(\tau))^{-(1-1/q)}\frac{dJ(\tau)}{d\tau}\ .$$

The first main result is that E is a holomorphic modular form of weight 2 on the triangle group Γ, more correctly E^d is holomorphic of weight $2d$ for $d=\operatorname{lcm}(p,q)$, its zero set is the Γ-orbit of τ_∞, with known orders.

The second main result gives the link with hypergeometric functions. Recall $1-c=1/p$, $c-a-b=1/q$, $b-a=1/r$, the solutions $F_0(z)$ and $F_1(z)$ of the hypergeometric differential equation, and the vertices $(\tau_0,\tau_1,\tau_\infty)$ of the hyperbolic triangle Δ.

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The Main Theorem II

Theorem: locally around $\tau = \tau_0$ (we can be precise) we have

$F_0(J(\tau)) = E^{1/2}(\tau)$ if $\tau_0 = \infty$, and otherwise

$F_0(J(\tau)) = \gamma_0(\tau - \overline{\tau_0})E^{1/2}(\tau)$ with $\gamma_0 = E^{-1/2}(\tau_0)/(\tau_0 - \overline{\tau_0})$ and

$F_1(J(\tau)) = \gamma_1(\tau - \tau_0)E^{1/2}(\tau)$ with $\gamma_1 = E^{-1/2}(\tau_0)(J^{1-c})'(\tau_0)$:

$F_0(J(\tau))$ and $F_1(J(\tau))$ correspond to modular forms of weight 1.

Example: $(p, q, r) = (\infty, 2, 3)$, vertices $(\tau_0, \tau_1, \tau_\infty) = (\infty, i, -\overline{\rho})$ corresponds to the full modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, and we have $(a, b, c) = (1/12, 5/12, 1)$. Since $j(\infty) = \infty$, $j(i) = 1728$, and $j(-\overline{\rho}) = 0$, the corresponding Hauptmodul is

$$J(\tau) = 1728/j(\tau) = ((E_4^3 - E_6^2)/E_4^3)(\tau) .$$

An easy computation gives

$$E(\tau) = J(\tau)^{-1}(1 - J(\tau))^{-1/2} dJ(\tau)/d\tau = E_4^{1/2}(\tau) ,$$

so ${}_2F_1(1/12, 5/12; 1; 1728/j(\tau)) = E_4^{1/4}(\tau)$, giving Fricke.

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$$J(\tau)=1728/j(\tau)=((E_4^3-E_6^2)/E_4^3)(\tau)\ .$$

An easy computation gives

$$E(\tau)=J(\tau)^{-1}(1-J(\tau))^{-1/2}dJ(\tau)/d\tau=E_4^{1/2}(\tau)\ ,$$

so ${}_2F_1(1/12,5/12;1;1728/j(\tau))=E_4^{1/4}(\tau)$, giving Fricke.

The Main Theorem II

Theorem: locally around $\tau = \tau_0$ (we can be precise) we have

$F_0(J(\tau)) = E^{1/2}(\tau)$ if $\tau_0 = \infty$, and otherwise

$F_0(J(\tau)) = \gamma_0(\tau - \overline{\tau_0})E^{1/2}(\tau)$ with $\gamma_0 = E^{-1/2}(\tau_0)/(\tau_0 - \overline{\tau_0})$ and

$F_1(J(\tau)) = \gamma_1(\tau - \tau_0)E^{1/2}(\tau)$ with $\gamma_1 = E^{-1/2}(\tau_0)(J^{1-c})'(\tau_0)$:

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More interesting example: same triangle but vertices ordered differently $(p, q, r) = (2, 3, \infty)$, still the full modular group but now with $\tau_0 = i \neq \infty$, so we can apply the other cases of the theorem. Here the Hauptmodul is $J(\tau) = 1 - j(\tau)/1728$, and a similar computation gives

$$ {}_2F_1(1/12, 1/12; 1/2; 1 - j(\tau)/1728) = \frac{\tau + i}{2i} \frac{\eta^2(\tau)}{\eta^2(i)} \ , $$

with $\eta(\tau)$ Dedekind eta function of weight $1/2$.

Main point: if we choose $\tau \in \mathbb{Q}(i)$, CM theory tells us that the right-hand side is algebraic, and also that $J(\tau)$ is algebraic. Rational example: $\tau = 2i$, $J(\tau) = -1323/8$ gives the identity

$$ {}_2F_1(1/12, 1/12; 1/2; -1323/8) = 3/2^{7/4} \ , $$

equivalent via standard hypergeometric transformations to the famous example given above

$$ {}_2F_1(1/12, 5/12; 1/2; 1323/1331) = (3/4)11^{1/4} $$

The Main Theorem III

More interesting example: same triangle but vertices ordered differently $(p,q,r)=(2,3,\infty)$, still the full modular group but now with $\tau_0=i\neq\infty$, so we can apply the other cases of the theorem. Here the Hauptmodul is $J(\tau)=1-j(\tau)/1728$, and a similar computation gives

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Functional Hypergeometric Evaluations I

Using the main theorem, we have seen two functional (as opposed to numerical) hypergeometric evaluations expressing ${}_2F_1$ as a modular "form" of weight 1 (${}_3F_2$ would correspond to weight 2, and in particular give the 36 known rational Ramanujan-type formulas for $1/\pi$, see for instance an arXiv paper of mine). We want to generalize. Notion of arithmetic hyperbolic triangles, not difficult.

Classified by Takeuchi in 1977: up to equivalence, exactly 85, with 9 non-compact, 76 compact. The 9 non-compact correspond to $(p, q, r) = (2, q, \infty)$, and (q, q, ∞) with $q = 3$, 4, or 6, plus (p, ∞, ∞) with $p = 2$, 3, and ∞, and the corresponding triangle groups are easy congruence subgroups of the modular group of levels 1, 2, 3, and 4.

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Explicit Hauptmoduln and modular forms, so explicit functional evaluations as above. Since several (a, b, c) can correspond to a given group (p, q, r), we find:

$$16, 16, 16, 8, 8, 8, 5, 5, 1$$

for a total of 83 functional evaluations for non-compact hyperbolic arithmetic triangles.

We can obtain additional functional evaluations using for instance derivatives. Example:

$$_2F_1(13/12, 5/12; 1; 1728/j(\tau)) = \frac{E_2(\tau)E_4^{5/4}(\tau)}{E_6(\tau)} .$$

Note that it is $E_2(\tau) - 3/(\pi\Im(\tau))$ which has nice CM properties.

Functional Hypergeometric Evaluations II

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Algebraic Evaluations I

However, gives numerical evaluations of two different kinds (assuming τ imaginary quadratic):
When $c < 1$ (equivalently $\tau_0 \neq \infty$), the right-hand side given by the theorem is of the form $f(\tau)/f(\tau_0)$, so is algebraic by CM theory as soon as $\tau \in \mathbb{Q}(\tau_0)$.

If $\tau \notin \mathbb{Q}(\tau_0)$, or if $c \geq 1$, we can prove using the Lerch, Chowla–Selberg formula that the RHS is algebraic times a product of $\Gamma(u_i)$ for $u_i \in \mathbb{Q}$, similar to Γ-evaluations above.

Thus, to obtain algebraic evaluations we need $c < 1$ and $\tau \in \mathbb{Q}(\tau_0)$.

Algebraic Evaluations II

Thanks to this, restricting to evaluations ${}_2F_1(a,b;c;z)$ with z rational, and using generalizations of the solution to the class number 1 problem and CM theory, we find a complete list of algebraic evaluations for non-compact arithmetic triangle groups coming from CM, generalizing Beukers–Wolfart's example, more precisely

$$9, 12, 26, 7, 13, 22, 2, 3, 0$$

evaluations corresponding to the nine groups, for a total of 94 algebraic numerical evaluations with z rational. We emphasize that here and in the sequel, all evaluations are understood with z rational (as well as a, b, and c), otherwise the classification would probably be hopeless.

Remarks:

• Several evaluations (12 out of the 94 above) are in fact rational, e.g., ${}_2F_1(1/4,1/2;3/4;80/81)=9/5$.

• If d is a common denominator of a, b, and c the main theorem implies that when $z\in\mathbb{Q}$ and $S={}_2F_1(a,b;c;z)$ is obtained as a CM value as above, then $S^d\in\mathbb{Q}(\tau)$ with $\tau=\sqrt{-1}$ or $\sqrt{-3}$.

• By a theorem of Wolfart and a deep transcendence result of Wüstholz, if $z=J(\tau)$ is algebraic (in particular rational) and $S={}_2F_1(a,b;c;z)$ is algebraic, then τ is imaginary quadratic, so we have indeed found all such evaluations in the non-compact case.

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p-adic Evaluations

In every case (including the ones we will see below) there are corresponding p-adic algebraic evaluations. Note that the corresponding number fields may be the same or very different, and that the p-adic value may vanish, while the complex one never does. Examples (in addition to the Beukers–Wolfart $11^{1/4}$ one where the number fields are the same):

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\begin{aligned}
{}_2F_1(1/6,1/3;1/2;25/37) &= (3/4)\sqrt{3}\\
{}_2F_1(1/6,1/3;1/2;25/37)_5 &= -(3/4)\sqrt{-1}\;,
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{}_2F_1(1/6,2/3;5/6;80/81) &= (3/5)\sqrt[6]{405}\\
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In the compact case, the main theorem also gives functional hypergeometric evaluations. However, the functions are automorphic functions on Shimura curves, completely explicit but less studied. Recall that the functional inverse $J^{-1}(z)$ is given by an explicit formula in terms of hypergeometric functions.

We work backwards. Example: we find numerically that for the $(p,q,r)=(3,6,6)$ triangle, we have

$${}_2F_1(1/6,1/3;5/6;64/189)=(3/7)189^{1/6}\ .$$

Using suitable vertices for the triangle, we find that $J^{-1}(64/189)=\tau=(11\sqrt{-1}+6\sqrt{-2})/7$, which is thus the image in $\mathcal{H}$ of a CM point on the Shimura curve.

In all the evaluations found (849), the cross-ratio $[\tau,\tau_1;\tau_0,\overline{\tau_0}]$, which is independent of the choice of vertices, belongs to $\mathbb{Q}(\zeta_d)$, for instance in the above example it simply equals $2/3$.

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Algebraic Evaluations IV

We have made a thorough search for algebraic evaluations (excluding degenerate cases and specializations of general formulas) corresponding to the 76 compact arithmetic hyperbolic triangles (several weeks of CPU time), helped by powerful heuristics, for instance the conjecture that ${}_2F_1(a,b;c;z)^d \in \mathbb{Q}(\zeta_d)$ for d common denominator of (a,b,c), and the fact that both z and $1-z$ are smooth (only small prime factors), conjecture generalizing a theorem of Gross–Zagier on singular moduli. Also related to Belyi maps.

In addition to the 94 algebraic evaluations found in the non compact case, we have found 755 algebraic evaluations corresponding to compact arithmetic triangle groups. We believe that we have found more than 90% such evaluations. Note: all numerical, but work of J. Voight, Y. Yang, and others give algorithms to prove them all, which we have not done.

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Algebraic Evaluations V

Some extreme cases: For the $(2,3,8)$ triangle:
$x = {}_2F_1(17/48, 41/48; 7/8; 29884728384/34239431521)$ satisfies $x^{24} = a + b\sqrt{2}$ with a, b rational with 100 digit numerators and denominators. Note
$z = 29884728384/34239431521 = 2^6 3^4 7^8 31^{-2} 47^{-2} 127^{-2}$ and $1 - z = 23^3 71^3 31^{-2} 47^{-2} 127^{-2}$, both "smooth".

For the $(2,4,6)$ triangle:
$x = {}_2F_1(1/24, 7/24; 5/6; 3024000000/4097152081)$ satisfies $x^6 = 129536/117649$.

Evaluations found for 71 out of the 76 compact arithmetic groups. For instance none found for the $(2,3,7)$ triangle. In view of the example for $(2,3,8)$, height of z maybe very large.

As already mentioned, all of our 849 algebraic evaluations have p-adic analogues when the series converges (usually $v_p(z) > 0$). But using Dwork-type p-adic extensions, also for $v_p(z) = 0$, conjectural, due to F. Beukers.

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Algebraic Evaluations VI

We have also searched (much less thoroughly) for algebraic evaluations not corresponding to arithmetic triangle groups: we have found several for non-arithmetic triangle groups, for instance $x = {}_2F_1(3/14, 5/21; 20/21; -8)$ satisfies $x^{42} \in \mathbb{Q}(\zeta_{21})$ and of degree 252, and for non-triangle groups, for instance $x = {}_2F_1(1/10, 1/6; 3/5; 81)$ satisfies $x^{30} \in \mathbb{Q}(\zeta_{15})$ and of degree 240.

We have found 209 additional algebraic evaluations, but contrary to the case of arithmetic triangles, we do not believe that they represent most evaluations, but simply that our search was incomplete. Note that we use linear, quadratic, cubic, and quartic transformations of the hypergeometric function to obtain as many evaluations as we can.

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Thank you for your attention.