Integer points close to a transcendental curve and correctly-rounded evaluation of a function

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Effective Aspects in Diophantine Approximation - March 28, 2023

(Binary) Floating Point (FP) Arithmetic

Given

$$\begin{cases} \text{a precision} & p \geqslant 1, \\ \text{a set of exponents} & E_{\mathsf{min}}, \cdots, E_{\mathsf{max}}. \end{cases}$$

A finite FP number x is represented by 2 integers:

- integer significand M, $2^{p-1} \leqslant |M| \leqslant 2^p - 1$,
- exponent E, $E_{\mathsf{min}} \leqslant E \leqslant E_{\mathsf{max}}$

such that

$$x = \frac{M}{2^{p-1}} \times 2^E.$$

IEEE Precisions

IEEE 754 standard (1984 then 2008).

See http://en.wikipedia.org/wiki/IEEE_floating_point

	precision p	min. exponent E_{min}	maximal exponent E_{max}
binary32 (single)	24	-126	127
binary64 (double)	53	-1022	1023
binary128 (quadruple)	113	-16382	16383

We have $x = \frac{M}{2^{p-1}} \times 2^E$ with $2^{p-1} \leqslant |M| \leqslant 2^p - 1$

and $E_{\mathsf{min}} \leqslant E \leqslant E_{\mathsf{max}}$.

Rounding modes

In the IEEE 754 standard, the user defines an *active rounding mode*.

In this talk, we use:

- round to nearest (default). If $x \in \mathbb{R}$, $\mathrm{RN}(x)$: the floating-point number closest to x. In case of a tie, value whose integral significand is even.

Breakpoint: a point where the rounding function changes.

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Breakpoint: a point where the rounding function changes.

Here, breakpoint $=$ the middle of two consecutive FP numbers.

Correct rounding

A correctly-rounded operation whose entries are FP numbers must return what we would get by infinitely precise operation followed by rounding.

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IEEE-754 (1985): Correct rounding for $+$, $-$, $\times$, $\div$, $\sqrt{}$ and some conversions.

IEEE-754 (2008): suggests correct rounding for some elementary functions ($\sqrt[n]{}$, $\sin$, $\cos$, $\arcsin$, $\arccos$, $\tan$, $\arctan$, $\exp$, $\log$, $\sinh$, $\cosh$...).

The Table Maker's Dilemma

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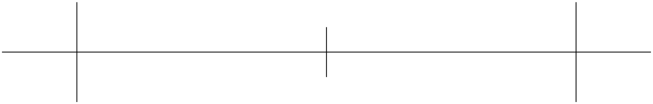

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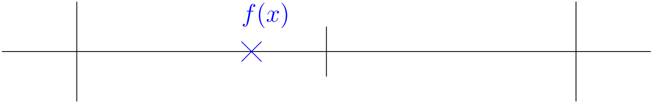

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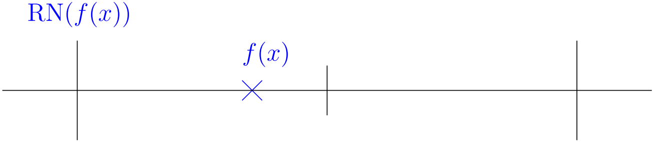

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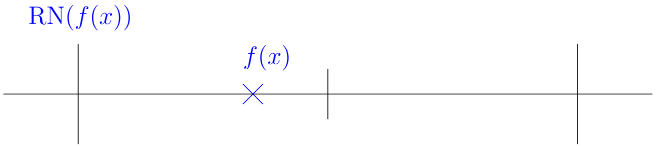

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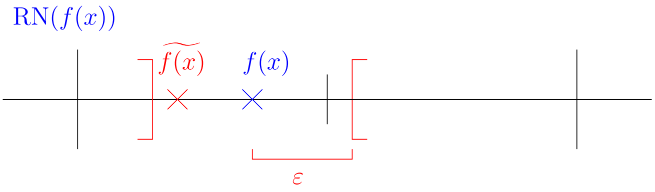

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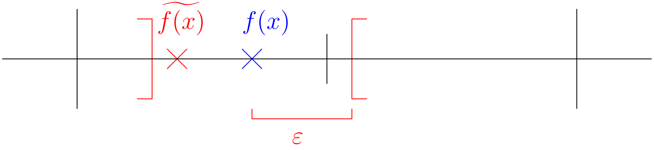

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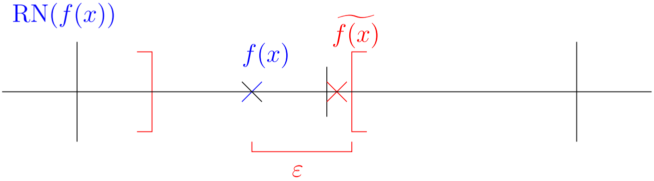

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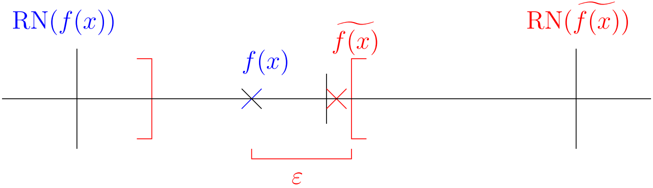

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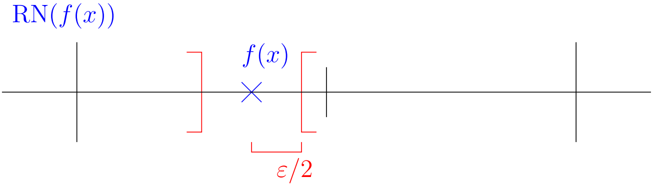

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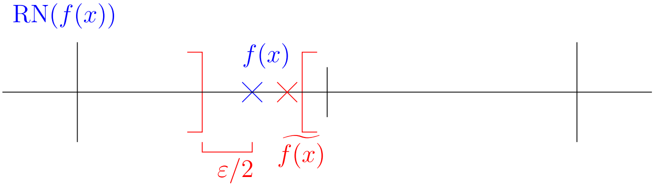

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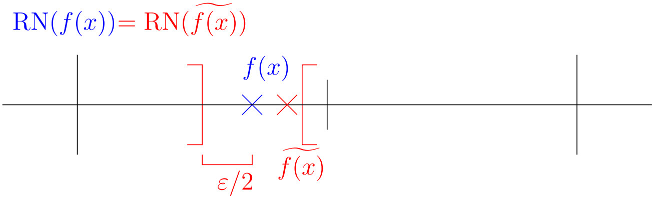

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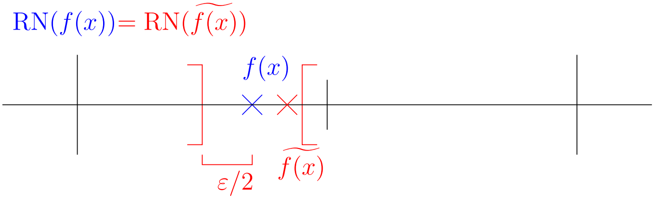

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Potential issues:

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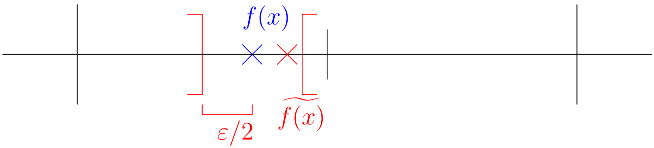

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Potential issues:

- What if $f(x)$ is a breakpoint?
- What about the number of subdivisions?
- ε should be uniform! And as large as possible!

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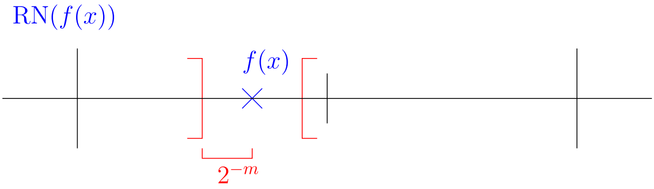

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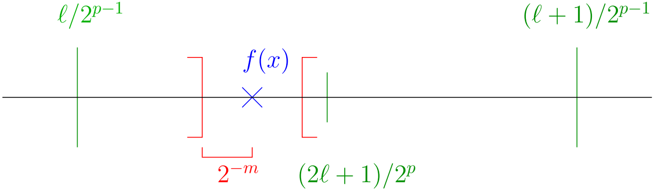

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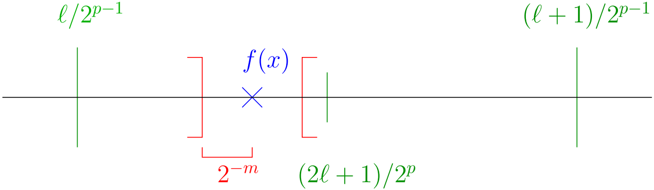

We want to find $m \in \mathbb{N}$ s.t.

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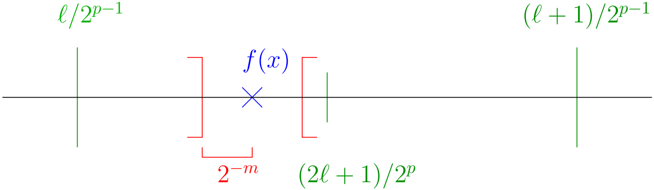

We want to find $m \in \mathbb{N}$, as small as possible, s.t. for all FP x:

- either there exists $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t. $f(x) = (2\ell+1)/2^p$,
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$$\text{for all } k \in [\![2^{p-1}, 2^p - 1]\!], \left| f(x) - \frac{2k+1}{2^p} \right| \geqslant 2^{-m}.$$

The Table Maker's Dilemma: Diophantine Problems

Assume, w.l.o.g., that x and $f(x) \in [1,2)$.

Q. (TMD) We want to determine $m \in \mathbb{N}$, as small as possible, s.t. for all $j \in [\![2^{p-1}, 2^p - 1]\!]$,

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The Table Maker's Dilemma: an Example

Consider the function 2^x and the binary64 FP number (base $2, p = 53$)

$$x = \frac{8520761231538509}{2^{62}}$$

We have

$$\begin{aligned} 2^{52+x} &= 4509371038706515.49999999999999999994026\cdots\text{(decimal)} \\ &= \underbrace{1\cdots}_{\text{53 bits}}.0\underbrace{1\cdots\cdots\cdots\cdots 1}_{\text{60 consecutive } 1's}0\cdots\text{(binary)}. \end{aligned}$$

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Heuristically, $m \sim 2p$.

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The Table Maker's Dilemma: First Challenge

A breakpoint is a point where the rounding function changes. In this talk, it is the middle of two consecutive FP numbers.

First challenge:

- Determine the set BP_f of all the FP numbers $x \in [1, 2)$ such that $f(x)$ is a breakpoint.

In other words, determine all $j, \ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t.

$$f\left(\frac{j}{2^{p-1}}\right) = \frac{2\ell + 1}{2^p}.$$

State of the Art

Transcendental elementary Functions $\sin$, $\cos$, $\arcsin$, $\arccos$, $\tan$, $\arctan$, $\exp$, $\log$, $\sinh$, $\cosh$. Hermite-Lindemann's theorem: $\alpha \neq 0$ algebraic $\Rightarrow$ e^{α} transcendental.

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What about the Euler Gamma function? For $\mathrm{Re}(z) > 0$,

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} \mathrm{d}t.$$

Very little is known:
$\Gamma(1/2), \Gamma(1/3), \Gamma(1/4), \Gamma(1/6), \Gamma(2/3), \Gamma(3/4), \Gamma(5/6)$ are transcendental and there are some partial irrationality results.

Our setting

Let $f : [1, 2) \mapsto [1, 2)$, f is transcendental and analytic in the neighborhood of $[1, 2)$.

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Let $g \in \mathcal{C}([a, b])$,

$$\|g\|_{\infty,[a,b]} := \max_{x\in[a,b]} |g(x)|.$$

Our Approach: Polynomial Interpolation and Lattice Basis Reduction

We want to find all $2^{p-1} \leqslant i, j \leqslant 2^p - 1$ s.t.

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We build a trap! We find a partition of $[1,2) = \cup_\ell I_\ell$ such that, for all ℓ, we can compute $P_{\ell,1}, P_{\ell,2} \in \mathbb{Z}[X,Y] \setminus \{0\}$ with

$$\left|P_{\ell,k}(2^{p-1}u, 2^p f(u))\right| < 1, \quad k = 1, 2,$$

for all $u \in I_\ell$.

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We eliminate (heuristic assumption!) one of the two variables and we get i and j, if they exist (Coppersmith; Boneh & Durfee; Stehlé).

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We build a trap! We find a partition of $[1,2) = \cup_\ell I_\ell$ such that, for all ℓ, we can compute $P_{\ell,1}, P_{\ell,2} \in \mathbb{Z}[X,Y] \setminus \{0\}$ with

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3. How do we compute these polynomials? Lattice basis reduction

Actually, the lattice reduction step makes it possible to refine the choice of the basis.

Basis in Use

Let $d \in \mathbb{N}$, if $X = 2^{p-1}x$ and $Y = 2^p f(x)$ the elements of the basis that we use are:

$$\begin{array}{cccccc}
1, & & & & & \\
X, & Y, & & & & \\
X^2, & XY, & Y^2, & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
X^{d-1}, & X^{d-2}Y, & X^{d-3}Y^2, & \cdots & Y^{d-1}, & \\
X^d, & X^{d-1}Y, & X^{d-2}Y^2, & \cdots & XY^{d-1}, & Y^d,
\end{array}$$

i.e., the basis of use is $\left((2^{p-1}x)^k (2^p f(x))^\ell\right)_{\substack{0 \leqslant \ell \leqslant d \\ 0 \leqslant k \leqslant d-\ell}}$.

Dimension $N = (d+1)(d+2)/2$.

Ensuring the Smallness of a Function: Interpolation at Chebyshev Nodes

Definition

Let $n \in \mathbb{N}$*, the Chebyshev nodes of the first kind of order* n *are the points* $\mu_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), k = 0, \ldots, n$*.*

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Let $p_n \in \mathbb{R}_n[X]$ that interpolates $\varphi \in \mathcal{C}([-1,1])$ at the $(\mu_k)_{k=0,..,n}$.

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The polynomial p_n is a quasi-optimal uniform approximation to φ:

$$\|\varphi - p_n\|_{\infty,[-1,1]} \leqslant 2\left(\frac{1}{\pi}\log(n+1)+1\right)\min_{Q\in\mathbb{R}_n[x]}\|\varphi - Q\|_{\infty,[-1,1]}.$$

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$$\underbrace{\|\varphi\|_{\infty,[-1,1]}}_{\text{small}} \leqslant \|p_n\|_{\infty,[-1,1]} + \|\varphi - p_n\|_{\infty,[-1,1]}$$

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Bounding the Interpolation Polynomial at Chebyshev Nodes

Definition

Let $n \in \mathbb{N}$, the Chebyshev nodes of the first kind of order n are the points $\mu_k = \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right), k = 0, \ldots, n$.

Let $P \in \mathbb{R}_n[X]$, we have

$$\max_{x \in [-1,1]} |P(x)| \leqslant \left(\frac{2}{\pi} \log(n+1) + 1\right) \max_{k=0,\ldots,n} |P(\mu_k)|.$$

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Let $\varphi \in \mathcal{C}([-1,1])$, let $p_n \in \mathbb{R}_n[X]$ that interpolates φ at the $(\mu_k)_{k=0,..,n}$, we have

$$\begin{aligned}\|p_n\|_\infty &\leqslant \left(\frac{2}{\pi}\log(n+1)+1\right)\max_{k=0,\dots,n}|p_n(\mu_k)| \\ &= \left(\frac{2}{\pi}\log(n+1)+1\right)\max_{k=0,\dots,n}|\varphi(\mu_k)|.\end{aligned}$$

Bounding the Remainder - Bernstein Ellipse

Let $\rho > 1$, let $\mathcal{E}_\rho := \left\{ \frac{\rho e^{i\theta} + \rho^{-1} e^{-i\theta}}{2}, \theta \in [0, 2\pi] \right\}$.

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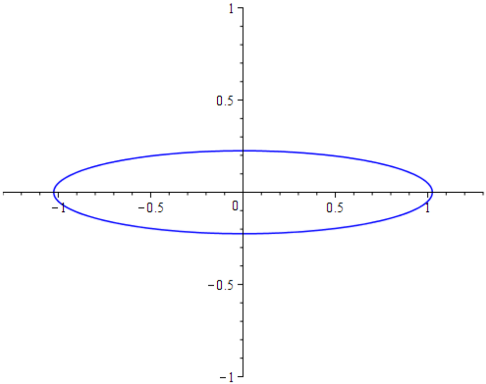

Bernstein ellipses for $\rho = 1.05$,

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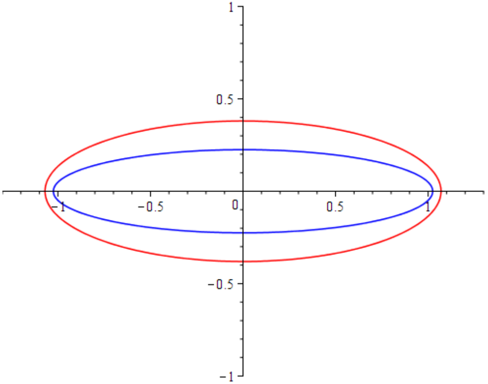

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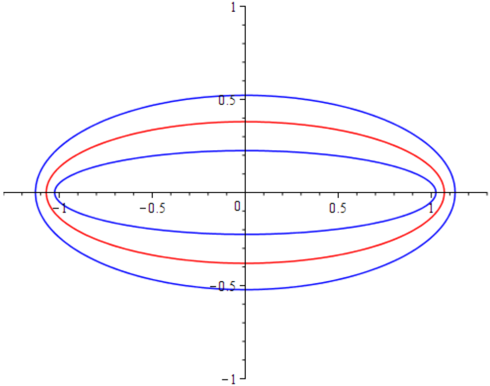

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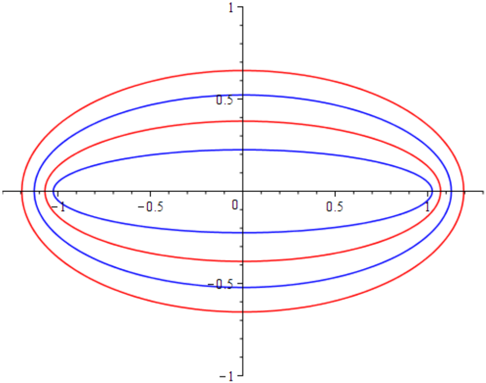

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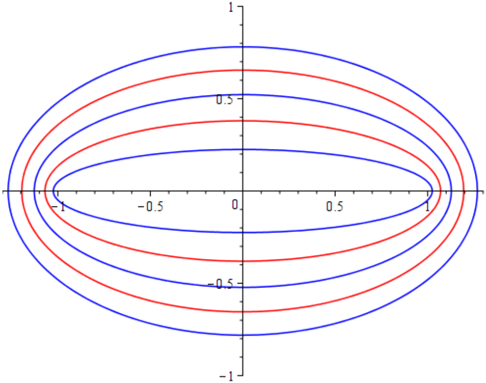

Bernstein ellipses for $\rho = 1.05, 1.25, 1.45, 1.65, 1.85$.

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$$\|\varphi - p_n\|_{\infty,[-1,1]} \leqslant \frac{4M_\rho(\varphi)}{\rho^n(\rho - 1)}.$$

where $M_\rho(\varphi) = \max_{z \in \mathcal{E}_\rho} |\varphi(z)|$.

Interpolation at Chebyshev Nodes and Uniform Approximation

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$$\begin{aligned}\|\varphi\|_\infty &\leqslant \|p_n\|_\infty + \|\varphi - p_n\|_\infty \\ &\leqslant \left(\frac{2}{\pi}\log(n+1)+1\right)\max_{k=0,\ldots,n}|\varphi(\mu_k)| + \frac{4M_\rho(\varphi)}{\rho^n(\rho-1)}.\end{aligned}$$

where $M_\rho(\varphi) = \max_{z\in\mathcal{E}_\rho}|\varphi(z)|$.

Interpolation at Chebyshev Nodes and Uniform Approximation: The case of $[a, b]$

Let $I = [a, b]$, one defines

- scaled Chebyshev nodes of the first kind of order n:
  $\mu_{k,[a,b]} = \frac{b-a}{2} \cos\left(\frac{(2k+1)\pi}{2(n+1)}\right) + \frac{a+b}{2}, k = 0, \ldots, n$,

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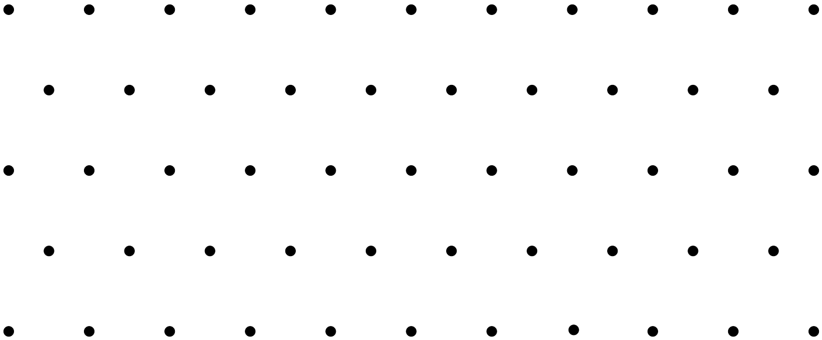
- a scaled Bernstein ellipse
  $\mathcal{E}_{\rho,a,b} = \left\{ \frac{b-a}{2} \frac{\rho e^{i\theta} + \rho^{-1} e^{-i\theta}}{2} + \frac{a+b}{2}, \theta \in [0, 2\pi] \right\}$.

Lattice Basis Reduction



An Approach based on Lattice Basis Reduction

Definition

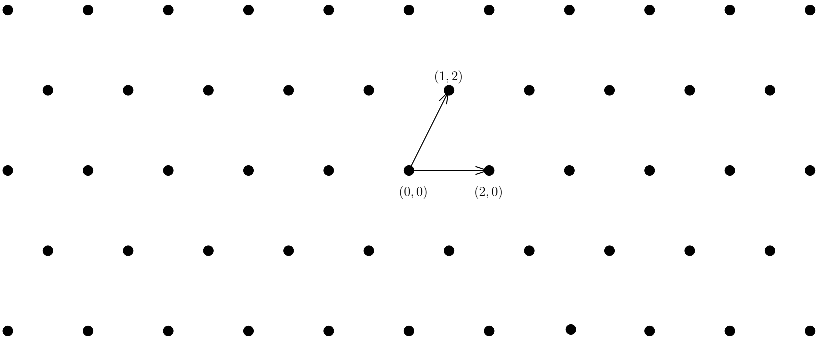
Let L be a nonempty subset of $\mathbb{R}^d$, L is a lattice iff there exists a set of vectors $b_1, \ldots, b_k$ $\mathbb{R}$-linearly independent such that

$$L = \mathbb{Z}.b_1 \oplus \cdots \oplus \mathbb{Z}.b_k.$$

$(b_1, \ldots, b_k)$ is a basis of the lattice L.

Examples. $\mathbb{Z}^d$, every subgroup of $\mathbb{Z}^d$.

Example: The Lattice $\mathbb{Z}(2,0) \oplus \mathbb{Z}(1,2)$



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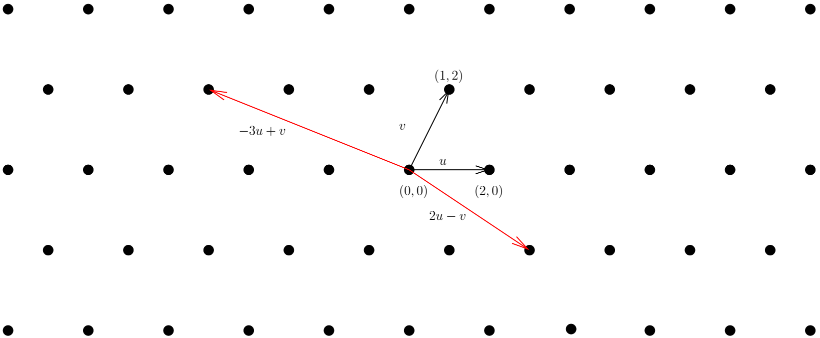

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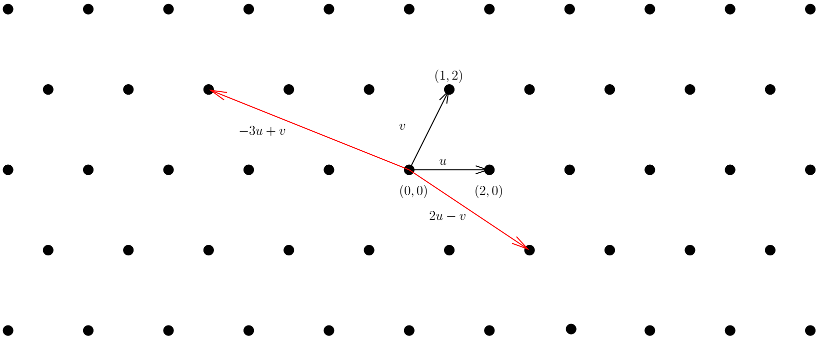

SVP (Shortest Vector Problem)

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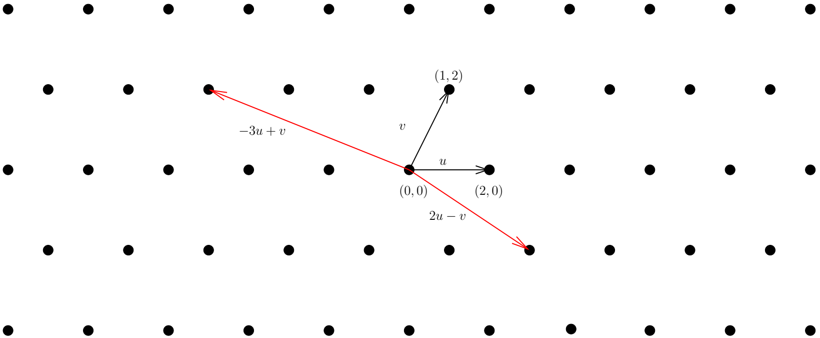

SVP (Shortest Vector Problem) is NP-hard.

Lenstra-Lenstra-Lovász Algorithm

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Factoring Polynomials with Rational Coefficients, A. K. Lenstra, H. W. Lenstra and L. Lovász, Math. Annalen **261**, 515-534, 1982.

The LLL algorithm gives an approximate solution to SVP in polynomial time.

Lenstra-Lenstra-Lovász Algorithm

Theorem

Let L a lattice of dimension k.
LLL provides a basis $(b_1, \ldots, b_k)$ made of "pretty" short vectors. We have $||b_1|| \leqslant 2^{(k-1)/2}\lambda_1(L)$ where $\lambda_1(L)$ denotes the norm of a shortest nonzero vector of L.
It terminates in at most $O(k^6 \ln^3 B)$ operations with $B \geqslant ||b_i||^2$ for all i.

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Let $(b_1, \ldots, b_k)$ be an LLL-reduced basis L then

$$||b_1|| \leqslant 2^{k/2} (\text{vol } L)^{1/k} \quad \text{and} \quad ||b_2|| \leqslant 2^{k/2} (\text{vol } L)^{\frac{1}{k-1}}.$$

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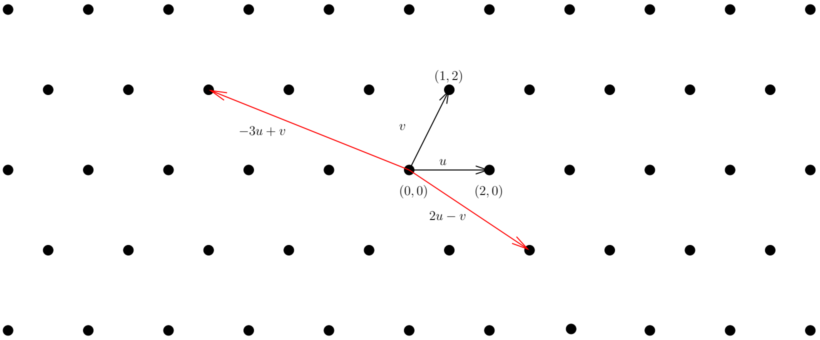

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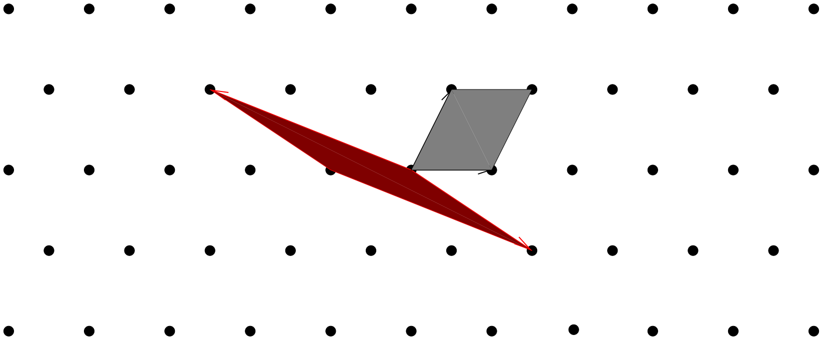

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How do we compute P_1 and P_2?

Let $d \in \mathbb{N}$, $P_1 = \sum_{0 \leqslant u+v \leqslant d} \alpha_{u,v} X^u Y^v$ and $P_2 = \sum_{0 \leqslant u+v \leqslant d} \beta_{u,v} X^u Y^v \in \mathbb{Z}[X, Y]$. We want to have

$$\left| P_k(2^{p-1}x, 2^p f(x)) \right| < 1, \qquad k = 1, 2,$$

for all $x \in I = [a, b]$.

How do we compute P_1 and P_2? The Lattice

Let $d \in \mathbb{N} \setminus \{0\}$ and $N = (d+1)(d+2)/2$. Let $(x_j)_{0 \leqslant j \leqslant N-1}$ denote Chebyshev nodes for the interval $I = [a, b]$.
We introduce, for $0 \leqslant k \leqslant d, 0 \leqslant \ell \leqslant d-k$,

$$f_{k,\ell}(x) = (2^{p-1}x)^{\ell}(2^p f(x))^k \text{ and } r_{k,\ell} = \frac{4M_\rho(f_{k,\ell})}{\rho^{N-1}(\rho-1)}.$$

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Our lattice: $\mathcal{L}$ generated by the rows of

$$\begin{pmatrix}
f_{0,0}(x_0) & \cdots & f_{0,0}(x_{N-1}) & r_{0,0} & 0 & \cdots & \cdots & 0 \\
f_{0,1}(x_0) & \cdots & f_{0,1}(x_{N-1}) & 0 & r_{0,1} & 0 & \cdots & 0 \\
\vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
f_{d-1,1}(x_0) & \cdots & f_{d-1,1}(x_{N-1}) & \vdots & \cdots & 0 & r_{d-1,1} & 0 \\
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\end{array}
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i.e.,

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Hence, the function g is "small" !

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LLL gives us two such short vectors, as long as the volume of the lattice is small.

How do we compute P_1 and P_2? Volume of the Lattice

Let $d \in \mathbb{N} \setminus \{0\}$ and $N = (d+1)(d+2)/2$. Let $(x_j)_{0 \leqslant j \leqslant N-1}$ denote Chebyshev nodes for the interval $I = [a, b]$.
We introduce, for $0 \leqslant k \leqslant d, 0 \leqslant \ell \leqslant d-k$,

$$f_{k,\ell}(x) = (2^{p-1}x)^{\ell}(2^p f(x))^k \text{ and } r_{k,\ell} = \frac{4M_\rho(f_{k,\ell})}{\rho^{N-1}(\rho-1)}.$$

Our lattice: $\mathcal{L}$ generated by the rows of

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Its volume $\mathrm{vol}(L)$: determinant of the matrix.

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$$\mathrm{vol}(\mathcal{L})^{1/N} \leqslant O(N) \frac{2^{2pd/3}}{\rho^{(N-1)/2}} \left| \frac{b-a}{2}\rho + \frac{b+a}{2} \right|^{d/3} M_{\rho,a,b}(f)^{d/3}$$

where $M_{\rho,a,b}(f) = \max_{z \in \mathcal{E}_{\rho,a,b}} |f(z)|$ and
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If $[a,b] = [1,2]$, 40 CPU minutes for $p = 53$ and 46 CPU days for $p = 113$.

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Our experiments were done in Sagemath[1] and heavily use the Arb[2] and fplll[3] libraries

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The Table Maker's Dilemma

A breakpoint is a point where the rounding function changes. In this talk, it is the middle of two consecutive FP numbers.

Two-step challenge:

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Holy Grail: $m \sim 2p$. True for $p = 53$ (V. Lefèvre et al).

Our results

Thanks to an extension of the presented ideas, we obtain for instance, for $p = 113$, for all $j, \ell \in [\![2^{p-1}, 2^p - 1]\!]$ and

$$\left|\exp\left(\frac{j}{2^{p-1}}\right) - \frac{2\ell+1}{2^p}\right| \geqslant \frac{1}{2^{12p}}$$

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Still, this work should hopefully help paving the way for correctly rounded elementary functions in IEEE binary128/quadruple precision.

Additional material

Some insight (Warning: Hand-waving!)...

Assume there exist $x \in [1, 2)$, $k \in \mathbb{N} \setminus \{0\}$ and $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ s.t.

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$$y = y_0.y_1 y_2 \cdots y_{p-1} \underbrace{01111111\cdots 11}_{k \text{ bits}} xxxxx \cdots$$

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Assessing the Heuristic: the Example of $\sin$

Here, $f = \sin$ over $[1, 2)$, $p = 16$.

k	Actual number of occurrences	Expected number of occurrences
1	16397	16384
2	8151	8192
3	4191	4096
4	2043	2048
5	1010	1024
6	463	512
7	255	256

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9	62	64
10	35	32
11	16	16
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13	6	4
14	0	2
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Here, the heuristic seems reasonable.

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$\rightarrow$ roughly,

$$\text{"}m_{opt} \sim 2p\text{"} \qquad \text{(Q)}.$$

Proving the Heuristic

NB, G. Hanrot and O. Robert (2017)

Let $f : [1, 2) \mapsto [1, 2)$, $f \in \mathcal{C}^2$, let $k \in \mathbb{N}$.

Determine the proportion of $j \in [\![2^{p-1}, 2^p - 1]\!]$ s.t. there exists $\ell \in [\![2^{p-1}, 2^p - 1]\!]$ with

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Proposition

For $\exp$ over $[1, 2)$, if $p \geqslant 24$, the heuristic is valid for $0 \leqslant k < p/3$.