Polynomial system solving with the msolve library

https://msolve.lip6.fr

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Polynomial systems

Let $\mathbb{K}$, $\mathbb{K}'$ be fields with $\mathbb{K} \subset \mathbb{K}'$ and $\boldsymbol{f} = (f_1, \ldots, f_s)$ in $R = \mathbb{K}[x_1, \ldots, x_n]$

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"Solve" $f_1 = \cdots = f_s = 0$ over ${\mathbb{K}'}^n$ $\rightsquigarrow$ Solution set over ${\mathbb{K}'}^n$

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Typical settings.

☛ $\mathbb{K}$ is a finite field, $\mathbb{K}' = \mathbb{K}$ or $\mathbb{K}'$ is an algebraic closure of $\mathbb{K}$ (denoted by $\overline{K}$)

☛ $\mathbb{K} = \mathbb{Q}$ and $\mathbb{K}' = \mathbb{R}$ or $\mathbb{K}' = \mathbb{C}$

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- Non-linearity $\rightsquigarrow$ numerical issues

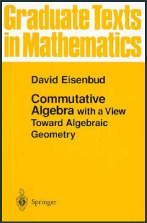
Algebra and geometry of polynomial system solving

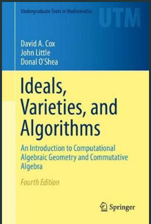
Algebraic representation → **Exact encoding** of the solution set

Equations → **Computing with solutions**

Zero test | **Sign determination** | **Dimension** | **Degree**

The algebra / geometry dictionary





☛ Non-linearity

⇝ vector spaces replaced by ideals

$$\langle f \rangle = \{ \sum_{i=1}^{s} q_i f_i, \, q_i \in R \}$$

☛ Need of normal forms (zero test)

⇝ computing "modulo" $I = \langle f \rangle$

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$\simeq$ 55 000 lines, license GPLv2+

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msolve's background: Gröbner bases

Let $\succ$ be an admissible monomial ordering $\rightsquigarrow \mathrm{lm}_{\succ}(f)$ for any $f \in R$

Gröbner bases $G \subset I$ finite such that $\langle \mathrm{lm}_{\succ}(G) \rangle = \langle \mathrm{lm}_{\succ}(I) \rangle$

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Lexicographic ordering $\rightsquigarrow$ eliminates variables

The elimination theorem

Projection $\leftrightarrow$ Elimination

$$\begin{cases} G_1 = I \cap \mathbb{K}[x_1] \\ \vdots \\ G_n = I \cap \mathbb{K}[x_1, \ldots, x_n] \end{cases}$$

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Description of finite solution set $V(I)$ in $\overline{\mathbb{K}}^n$

Ideals in shape position

Possible up to generic
linear change of coordinates

$$\begin{cases} w(x_1) = 0 \\ x_2 = w_2(x_1) \\ \vdots \\ x_n = w_n(x_1) \end{cases}$$

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$$\begin{cases} w(x_1) & = & 0 \\ x_2 & = & v_2(x_1)/w'(x_1) \\ & \vdots & \\ x_n & = & v_n(x_1)/w'(x_1) \end{cases}$$

Complexity issues

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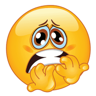

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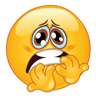

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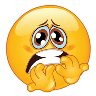

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- ☛ Is the worst case the "generic" one? **NO!**
- ☛ Better complexity through extra requirements ? **YES!**

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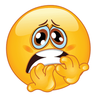

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Regular computations. **Bayer/Stillman/Lazard/Giusti, etc.**

$$E_d = \{\sum_{i=1}^{s} q_i f_i \mid q_i \in R, \deg(q_i f_i) \leq d\} \rightsquigarrow \text{finite dim. vector space}$$
$$B_{\succ,d} = \text{Basis of } E_d \text{ w.r.t. } \succ = \succ_{\text{graded}}$$
$$\langle \mathrm{lm}_{\succ}(B_{\succ,d}) \rangle = \langle \mathrm{lm}_{\succ}(I \cap R_{\leq d}) \rangle ?$$

Complexity

$$O\left(\left(\binom{n+\mathbb{D}_{\text{reg}}}{n}\right)^{\omega}\right) \text{ with } \mathbb{D}_{\text{reg}} = 1 + \sum_{i=1}^{s} (\deg(f_i) - 1)$$

Linearization technique and termination

$E_d = \{\sum_{i=1}^{s} q_i f_i \mid q_i \in R, \deg(q_i f_i) \leq d\} \rightsquigarrow$ finite dim. vector space

$B_d = \succ$-Basis of E_d with $\succ = \succ_{\text{graded}}$

$E_d \to B_{\succ,d}$ $\xrightarrow[\text{Gauss}]{\text{Generate}}$ $E_{d+1} \to B_{\succ,d+1}$ $\xrightarrow[\text{Gauss}]{\text{Generate}}$ $E_{d+2} \to B_{\succ,d+2}$

$\langle \mathrm{lm}_\succ(B_{\succ,\delta}) \rangle = \langle \mathrm{lm}_\succ(I) \rangle$?

Buchberger's criterion

☛ Multivariate division $\leftrightarrow$ Gaussian elimination

F4 algorithm

Faugère'98

$G \leftarrow (f_1, \ldots, f_s)$

$\{(a_{i,j} g_i, b_{i,j} g_j) \mid \mathsf{lm}_\succ(a_{i,j} g_i) = \mathsf{lm}_\succ(b_{i,j} g_j) = \mathsf{lcm}(\mathsf{lm}_\succ(g_i), \mathsf{lm}_\succ(g_j))\}$

$\mathscr{P} \leftarrow \text{Pairs}(G, \succ)$

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Selection of the lcms of degree $d_{\min}$

$\mathscr{P}' \leftarrow \text{Select}(\mathscr{P})$ $\quad \mathscr{P} \leftarrow \mathscr{P} \setminus \mathscr{P}'$

$L \leftarrow \{af, bg \mid (af, bg) \in \mathscr{P}'\}$

$L' \leftarrow \text{SymbolicPreprocessing}(L, G)$

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Basis of $E_{d_{\min}}$

$L' \leftarrow \mathrm{SymbolicPreprocessing}(L, G)$

$H \leftarrow \mathrm{GaussianReduction}(\mathrm{Macaulay}(L', \succ))$

for $h \in H$, if $\mathrm{lm}_\succ(h) \notin \langle \mathrm{lm}_\succ(G) \rangle$
  $\mathscr{P} \leftarrow \mathscr{P} \cup \mathrm{Update}(G, h, \succ)$, $G \leftarrow G \cup \{h\}$

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$\mathscr{P} \neq \emptyset$?

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$g_j) = \mathrm{lcm}(\mathrm{lm}_\succ(g_i), \mathrm{lm}_\succ(g_j))\}$

More accurate "complexity estimate":

- ☛ Number of matrices + their sizes + their ranks
- ☛ Number of reducers and new elements in the GB

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☛ To **save memory**: one global hash table + one secondary local hash table

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Basis of $E_{d_{\min}}$

☛ Need of **fast** divisibility checks

☛ Use of divisor masks

$y(L', \succ))$

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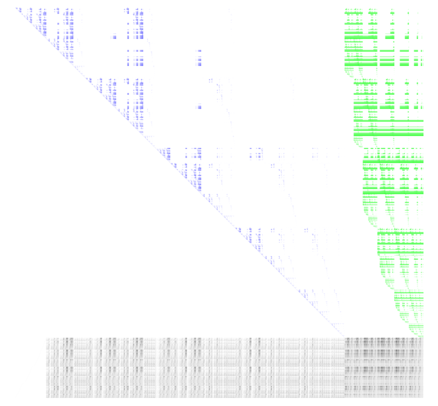
Matrices in F4



☛ rows stored in general in sparse format

☛ rows stored in sparse-dense hybrid format for denser matrices

☛ CPU intrinsics: AVX2 $\rightsquigarrow$ store eight 32-bit (unsigned) coefficients in one 256-bit `__m256i` type

☛ Probabilistic and deterministic reductions

☛ Implementation of a tracer for multi-modular computations

Traverso'88

F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

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./msolve -g 1 -f in.ms -o out.ms
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Examples	msolve F4 learn	msolve F4 tracer	(learn/tracer)	msolve prob	(prob / tracer)	maple	magma
Katsura-9	0.17	0.03	5.67	0.06	2	0.10	
Katsura-10	0.81	0.09	9	0.24	2.67	0.36	
Katsura-11	6.26	0.45	13.9	1.34	2.98	1.82	
Katsura-12	56.1	3.10	18.1	8.61	2.78	8.50	
Katsura-13	425	19	22.4	53	2.79	60.9	
Katsura-14	3336	128	26.1	318	2.5	393	
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Eco-12	11.6	1.1	10.54	2.25	2.05	2.97	
Eco-13	67.3	6.6	10.2	11.7	1.77	15.1	
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Eco-12	11.6	1.1	10.54	2.25	2.05	2.97	
Eco-13	67.3	6.6	10.2	11.7	1.77	15.1	
Eco-14	516	34.8	14.8	67	1.92	104.8	
Eco-15	3476	153	22.7	466.15	3	n.m.	
Eco-11	0.71			0.37			0.46
Eco-12	4.94			1.95			2.61
Eco-13	33.75			9.27			11.77

F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

```
./msolve -g 2 -f in.ms -o out.ms
./msolve -g 1 -f in.ms -o out.ms
```

Examples	msolve F4 learn	msolve F4 tracer	(learn/tracer)	msolve prob	(prob / tracer)	maple	magma
Katsura-9	0.17	0.03	5.67	0.06	2	0.10	
Katsura-10	0.81	0.09	9	0.24	2.67	0.36	
Katsura-11	6.26	0.45	13.9	1.34	2.98	1.82	
Katsura-12	56.1	3.10	18.1	8.61	2.78	8.50	
Katsura-13	425	19	22.4	53	2.79	60.9	
Katsura-14	3336	128	26.1	318	2.5	393	
Katsura-15	27960	738	27.96	2209	2.71	n.m.	
Katsura-16	259240	5548	46.7	12474	2.24	n.m.	
Katsura-11	3.60			1.15			1.63
Katsura-12	28.53			6.30			9.10
Katsura-13	246.37			39.43			57.77
Eco-10	0.28	0.05	5.6	0.1	2	0.14	
Eco-11	1.21	0.17	7.11	0.39	2.29	0.56	
Eco-12	11.6	1.1	10.54	2.25	2.05	2.97	
Eco-13	67.3	6.6	10.2	11.7	1.77	15.1	
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Eco-13	33.75			9.27			11.77
Henrion-6	0.22	0.07	3.14	0.11	1.57	0.17	
Henrion-7	27.5	6.5	4.23	9.55	1.47	12.8	

F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

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Henrion-7	27.5	6.5	4.23	9.55	1.47	12.8	
CP(3,6,2)	0.6	0.12	5	0.22	1.83	0.31	
CP(3,7,2)	8.18	1.23	6.65	1.97	1.6	2.78	
CP(3,8,2)	111.5	12.6	8.85	18.5	1.47	24.6	

F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

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Pol-Bill	190	-	-			348	291
SDK-Bill	150	-	-			268	4208

F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

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F4 timings

Gröbner bases for grevlex order computations modulo primes $< 2^{31}$

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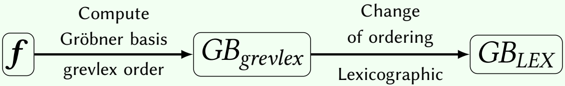
Pol-Bill	190	-	-			348	291
SDK-Bill	150	-	-			268	4208

Grevlex one block elimination orderings are also available

```
./msolve -e k -g 2 -f in.ms -o out.ms
./msolve -e k -g 1 -f in.ms -o out.ms
```

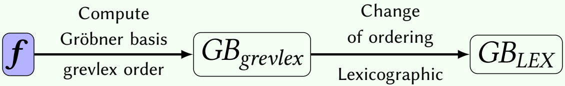
Describing solutions ⇝ Change of orders

The "usual" good way to do



Describing solutions $\rightsquigarrow$ Change of orders

The "usual" good way to do



Describing solutions $\rightsquigarrow$ Change of orders

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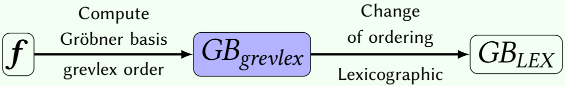

Describing solutions $\rightsquigarrow$ Change of orders

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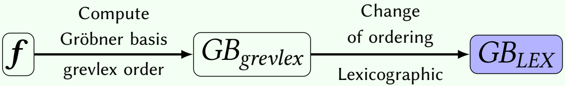

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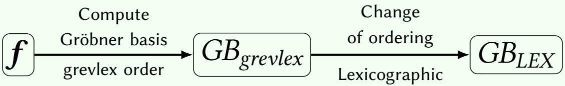

$\frac{\mathbb{K}[x_1,\ldots,x_n]}{I}$ is a finite dimensional vector space

Describing solutions $\rightsquigarrow$ Change of orders

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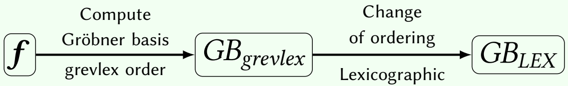

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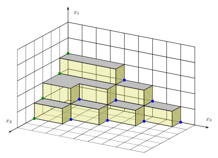

- Combinatorial structure of polynomial ideals
- Basis $\mathscr{B}$ of quotient ring $\frac{\mathbb{K}[x_1,\ldots,x_n]}{I}$
- Generic staircase **Moreno-Socías**
- $w(x_1) = 0, x_1 = w_2(x_1), \ldots, x_n = w_n(x_1)$

Describing solutions $\rightsquigarrow$ Change of orders

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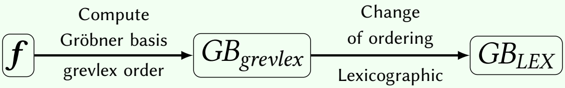

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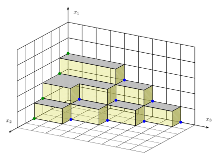

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The generic staircase of grevlex Gröbner bases (Moreno-Socías)

For $m \in \mathscr{B}$, $mx_n \in \mathscr{B}$ or $mx_n \in \mathrm{LM}_{\succ_{grevlex}}(GB_{grevlex}) \rightarrow$ sparse matrix

Change of orders algorithms

Faugère/Lazard/Gianni/Mora $\rightsquigarrow$ FGLM algorithm

Complexity $O\left(D^3\right)$

relation reconstruction through linear algebra

Not implemented in `msolve`

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Faugère/Mou $\rightsquigarrow$ connection to Wiedemann's algorithm (sparsity)

Computation of minimal polynomial

Berlekamp-Massey $\rightsquigarrow$ parametrizations

Implemented in `msolve`

$t = \sharp\{m \mid mx_n \in \mathscr{B}\}$

Complexity $O\left(tD^2\right)$

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Complexity $O(tD^2)$

Berthomieu/Neiger/S.

Change of paradigm:

~~sparse~~ ☛ structured

Complexity $O(t^{\omega-1}D)$

Change of order timings

`msolve` implementation (prime fields, characteristic $< 2^{31}$)

☛ dedicated encoding of multiplication matrices

☛ AVX2 implementation

Examples	`msolve` FGLM	`maple` FGLM	ratio	`msolve` tracer	ratio (FGLM / tracer)
Katsura-10	0.11	0.15	1.36	0.09	1.2
Katsura-11	0.49	0.74	1.51	0.45	1.1
Katsura-12	3.96	5.4	1.36	3.10	1.28
Katsura-13	30.6	35.7	1.16	19	1.61
Katsura-14	210	271	1.29	128	1.64
Eco-11	0.07	0.12	1.71	0.17	0.41
Eco-12	0.34	0.85	2.5	1.07	0.31
Eco-13	2.12	6.7	3.16	6.6	0.32
Eco-14	25.9	69.1	2.67	34.8	0.74
Eco-15	146.3	n.m.		155.73	0.94
Henrion-6	0.11	0.11	1	0.07	1.57
Henrion-7	20.46	27.1	1.32	6.5	3.15
Noon-7	1.95	3.13	1.6	0.93	3.37
Noon-8	72.3	76.2	1.05	17.5	4.13

Change of order timings

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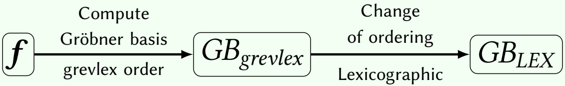
FGLM is **increasingly** dominant w.r.t. F4-tracer in msolve

Solving systems over the rational numbers

The “usual” good way to do



$$w(x_1) = 0, x_2 = w_2(x_1), \ldots, x_n = w_n(x_1) \quad \rightsquigarrow \quad w(x_1) = 0, x_2 = \frac{v_2(x_1)}{w'(x_1)}, \ldots, x_n = \frac{v_n(x_1)}{w'(x_1)}$$

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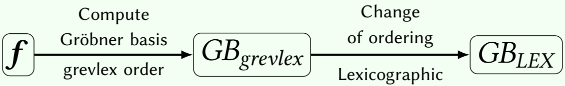

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- ☛ Multi-modular arithmetics
- ☛ Rational reconstruction
- ☛ Plenty of asymptotically optimal algorithms for univariate polynomials
- ☛ Dependency on the output bit size
- ☛ Probabilistic algorithm

Solving systems over the rational numbers

The "usual" good way to do

f —(Compute Gröbner basis, grevlex order)→ $GB_{grevlex}$ —(Change of ordering, Lexicographic)→ GB_{LEX}

$$w(x_1) = 0, x_2 = w_2(x_1), \ldots, x_n = w_n(x_1) \rightsquigarrow w(x_1) = 0, x_2 = \frac{v_2(x_1)}{w'(x_1)}, \ldots, x_n = \frac{v_n(x_1)}{w'(x_1)}$$

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Lift $GB_{grevlex}$ or lift GB_{lex}?

To lift or not to lift ($GB_{grevlex}$)?

`./msolve -g 2 -f in.ms -o out.ms` versus `./msolve -f in.ms -o out.ms`

Examples	`msolve` Grevlex	nprimes	`msolve` Param	nprimes	ratio (time)	ratio (nprimes)
Katsura-10	3.26	21	21.75	141	0.15	0.15
Katsura-11	29.1	34	179.67	307	0.16	0.11
Katsura-12	260	56	2 025.82	643	0.13	0.09
Katsura-13	1 326	81	47 539.59	1336	0.03	0.06
Katsura-14	12 101	108	738 259.30	2941	0.02	0.04

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Katsura-12	260	56	2 025.82	643	0.13	0.09
Katsura-13	1 326	81	47 539.59	1336	0.03	0.06
Katsura-14	12 101	108	738 259.30	2941	0.02	0.04
Eco-11	31.03	131	47.22	174	0.66	0.75
Eco-12	128.91	188	494.52	317	0.26	0.59
Eco-13	3 544.29	531	4 147.28	650	0.85	0.82
Eco-14	44 732.69	1284	67 361.61	1347	0.66	0.95

To lift or not to lift ($GB_{grevlex}$)?

`./msolve -g 2 -f in.ms -o out.ms` versus `./msolve -f in.ms -o out.ms`

Examples	msolve Grevlex	nprimes	msolve Param	nprimes	ratio (time)	ratio (nprimes)
Katsura-10	3.26	21	21.75	141	0.15	0.15
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Random-8	103.87	5 120	30.84	1 172	3.35	4.36
Random-9	1 616.59	12 800	318.23	2 661	5.08	4.81
Random-10	24 612.11	31 744	3 520.40	5 915	6.99	5.37
Random-11	568 577.42	73 728	46 085.00	13 000	12.34	5.67

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Not so clear that there is in general interest to lift $GB_{grevlex}$

☛ msolve lifts GB_{lex}

Univariate real root isolation

Based on FLINT's univariate multiplication in `fmpz_poly`

Examples	♯ sols	msolve	maple		SLV		tdescartes	
		time	time	ratio	time	ratio	time	ratio
Katsura-10	120	3.1	4.8	1.5	3.8	1.2	20	6.5
Katsura-11	216	27	60	2.2	50.5	1.9	156	5.8
Katsura-12	326	207	656	3.2	555	2.7	2, 206	10.6
Katsura-13	582	2 220	16 852	7.6	13 651	6.1	22 945	10.3
Katsura-14	900	20 149	250 094	12.4	252 183	12.5	384 566	19.1
Katsura-15	1, 606	197 048	3 588 835	18.2	3 540 480	18.0	5 178 180	26.3
Katsura-16	2, 543	1 849 986	—	—	—	—	—	—
Katsura-17	4, 428	16 128 000	—	—	—	—	—	—

Real root isolation timings given in seconds

Warning: uses maple-v16

Timings for solving

Examples	DEG	msolve(trace)	msolve(prob)	speed-up	maple	speed-up	magma	speed-up
Katsura-9	256	4.89	7.49	1.53	104	21.27	2522	515
Katsura-10	512	43.7	70.5	1.61	1 278	29.24	82 540	1 888
Katsura-11	1024	424	814	1.92	7 812	18.4	-	
Katsura-12	2048	6 262	11 215	1.79	120 804	19.29	-	
Katsura-13	4096	89 390	148 372	1.66	-		-	
Katsura-14	8192	1 308 602	2 000 170	1.53	-		-	
Eco-10	256	12.5	21.2	1.69	26.3	2.1	6520	521.6
Eco-11	512	90.3	161	1.78	312	3.45	214 770	2378
Eco-12	1024	877	1 619	1.84	4 287	4.88	-	
Eco-13	2048	12 137	19 553	1.61	66 115	5.44	-	
Eco-14	4096	167 798	254 389	1.51	-		-	
Henrion-5	100	0.71	0.83	1.17	2.7	3.8	93	130.98
Henrion-6	720	138	157	1.13	1 470	10.65	-	
Henrion-7	5040	117 803	127 456	1.08	-		-	
CP(3,5,2)	288	18.1	19.2	1.06	249	13.75	-	
CP(3,6,2)	720	390	450	1.15	23 440	60	-	
CP(3,7,2)	1728	9 643	11 511	1.19	-		-	
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Noon-7	2173	4039	5 045	1.25	432	0.1	-	
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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	6520	521.6
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]70	2378
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On Noon examples we suffer from the bit size of our output parametrizations (which could be split in many small components)

Using Gröbner bases in geometry

Take C_1, C_2, C_3, C_4, C_5 in $\mathbb{Q}[x_1, x_2]$ of degree 2.
Compute $U \in \mathbb{Q}[x_1, x_2]$ such that
$V(U)$ is tangent to $V(C_i)$ for $1 \leq i \leq 5$.

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☛ Various modelings proposed, difficulty is to "force" U to be generic.
One suits better with numerical homotopy continuation

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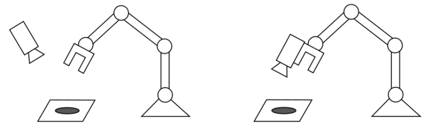
"New" alternative modeling which suits "well" to Gröbner bases

👍 `msolve` can solve one instance within $\simeq$ 2.5 hours (!)

👎 using 36 threads (memory consumption is ok but not tiny)...

Vision-based control schemes in robotics



☛ eye-in-hand with configuration camera

☛ dynamic control observation

observation ⇝ desired position

Lyapunov theory ⇝ **critical points of a polynomial map**

local extrema ⇝ stability analysis

Briot/Chaumette/Colotti/Garcia-Fontan/Goldsztein/S.

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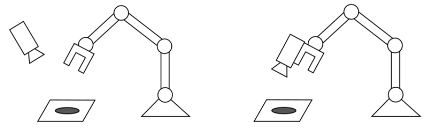

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sys2	24 days	1495 secs	1016/44	1016/44
sys3	27 days	1950 secs	1064/48	871/32
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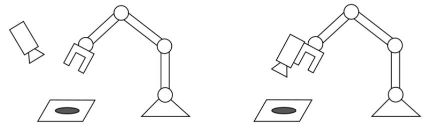

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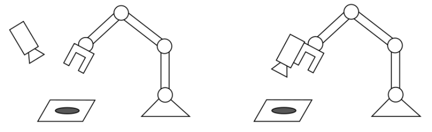

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System	msolve(×12)	HC.jl	msolve(×12)
sys1	15 days	1630 secs	172 secs
sys2	24 days	1495 secs	10243 secs
sys3	27 days	1950 secs	8035 secs
sys4	41 days	-	26h

- Symetries arise naturally in the formulation.
- Using GBs one can rewrite the polynomial system w.r.t. invariants.
- Last column reports on timings.

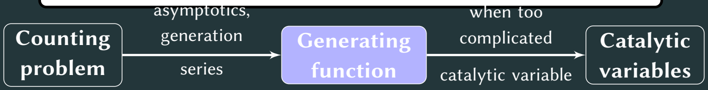
Briot/Chaumette/Colotti/Garcia-Fontan/Goldsztein/S.

Using Gröbner bases in combinatorics

The Flajolet-Sedgewick machinery for counting

Using Gröbner bases in combinatorics

The Flajolet-Sedgewick machinery for counting



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Counting problem —(asymptotics, generation / series)→ **Generating function** —(when too complicated / catalytic variable)→ **Catalytic variables**

Algebraicity result **Bousquet-Mélou/Jéhanne'06, Popescu'86**

Let $f \in \mathbb{Q}[u]$ and $Q \in \mathbb{Q}[x, y, t, u]$.
Let $\mathscr{F} \in \mathbb{Q}[u][[t]]$ be the unique solution to
$$\boxed{\mathscr{F} = f(u) + tQ(\mathscr{F}, \Delta(\mathscr{F}), \ldots, \Delta^{(k)}(\mathscr{F}), t, u)}$$
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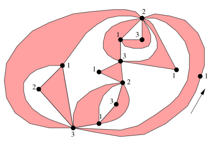

DDE → Polynomial systems

Elimination

msolve used

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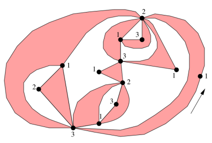

DDE → Polynomial systems

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`msolve` used

Bostan/Chyzak/Notarantonio/S.

What's next: syzygies and ideal-theoretic operations

A module approach $fg = gf \rightsquigarrow \mathrm{lt}(f)g = gf - \mathrm{tail}(f)g$

Compact representations of module of syzygies (F5) **Eder/Faugère**

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- Specializations of F5 in some structured setting
- Determinantal setting $\rightsquigarrow$ Crypto applications

Gopalakrishnan/Neiger/S.

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Ideal theoretic operations Nothing new since Bayer's PhD (!)

☛ F4 variant to compute saturation of ideals **Berthomieu/Eder/S.**

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☛ F4 variant to compute saturation of ideals **Berthomieu/Eder/S.**

☛ F5 variant for saturations + equidimensional decomposition

- Some reductions to 0 are unavoidable
- Exploit them $\rightsquigarrow$ decomposition of ideals

Eder/Lairez/Mohr/S.

The F4SAT algorithm

Berthomieu/Eder/S.

D_I: maximum degree reached to compute GB for I

$D_{\mathcal{J}}$: maximum degree reached to compute GB for $\mathcal{J}$

D_{rab}: maximum degree reached to compute GB for "rabinovitch" ideal

speedup1: Rabinowitsch / F4SAT (learn), speedup2: Rabinowitsch F4SAT (tracer)

speedup3: F4SAT / Maple (tracer)

system	D_I	$D_{\mathcal{J}}$	D_{rab}	learn	tracer	speedup1	speedup2	speedup3
d3-n6-p2	13	10	15	1.31	0.31	1.83	1.33	3.61
d3-n6-p3	16	13	18	43.7	1.84	3.25	9	19.24
d3-n6-p4	19	16	21	533	19.7	1.65	6.4	11.32
d4-n6-p3	24	20	27	31 101	596	1.4	10.6	14.8
d3-n7-p4	20	17	22	22 296	469	2.12	11.4	21.32
d3-n7-p5	23	20	25	126 006	2 881	1.62	7.96	11.67
d2-n8-p5	12	10	13	11.7	1.79	8.54	4.42	11.4
d3-n7-p3	17	14	19	1 263	32.4	2.89	12.53	30.37
d2-n9-p6	15	17	20	40 352	2 155	1.01	3.25	3.23
d2-n10-p5	13	14	15	66 845	2 141	0.9	10.8	32.1
steiner	19	19	24	115	67.2	5.34	2.28	3.57

Equidimensional decomposition

name	nb. comp.	equidim	Maple	Oscar	Magma	Magma (prime dec.)	Bertini
8-3-config-Li	23	1.6	16 ×10	> 1h	> 1h	65 ×40	4 ×2.5
cyclic8	6	381	> 5h	> 5h	> 5h	> 5h	⚠ 126 ×0.3
dgp6	3	✋ 0.2	53	2.2	> 1h	1.2	75
Gonnet	3	✋ 0.2	2.1	2.8	> 1h	1.4	74
P4L1	6	0.3	2.4	1.8	0.7	1.5	⚠ 21
P4L3	8	0.3	3.3	10	¡0.1	1.5	11
KdV		> 4h	353	> 4h	> 4h	7109 ×20	> 4h
Leykin-1	13	✋ 2.6 ×1.9	4 ×3.2	641 ×468	> 1h	1.4	✖
C1	4	129	> 1h	> 1h	> 1h	> 1h	✖
C2	4	0.3	100	152	> 1h	2.0	✖
C3	13	10	55	7	0.3	1.5	✖
MontesS16	6	1.9 ×1.4	2.7 ×1.9	2.0 ×1.4	1.4	1.5 ×1.1	7 ×5
Ps(10)	2	1.7	> 1h	30 ×17	> 1h	6 ×3.3	9 ×5
Ps(12)	2	51	> 1h	> 1h	> 1h	2060 ×40	⚠ 38 ×0.7
Ps(6)	2	¡0.1	0.2	1.7	0.5	0.3	2.0
Ps(8)	2	¡0.1	4	1.7	1.2	0.8	6
Sing(10)	2	0.4	> 1h	> 1h	> 1h	> 1h	⚠ 495
Sing(4)	2	¡0.1	76	2.2	8	5	5
Sing(5)	2	¡0.1	> 1h	4	7	1636	⚠ 1.0
Sing(6)	2	¡0.1	> 1h	51	> 1h	> 1h	⚠ 8
Sing(7)	2	¡0.1	1704	399	> 1h	> 1h	54
Sing(8)	2	0.1	> 1h	995	> 1h	> 1h	139
Sing(9)	2	0.2	> 1h	> 1h	> 1h	> 1h	⚠ 271
sos(4,2)	2	¡0.1	16	2.2	1.3	1.2	1.0
sos(4,3)	2	¡0.1	694	2.6	3.4	6	3.0
sos(5,2)	2	¡0.1	> 1h	1.8	3.7	1.2	3.0
sos(5,3)	2	¡0.1	> 1h	> 1h	> 1h	149	15
sos(5,4)	2	0.5	> 1h	> 1h	> 1h	> 1h	21
sos(6,2)	2	¡0.1	> 1h	2.0	5	1.6	5
sos(6,3)	2	0.1	> 1h	> 1h	> 1h	> 1h	34
sos(6,4)	2	5	> 1h	> 1h	> 1h	> 1h	69 ×14
sos(6,5)	2	14	> 1h	> 1h	> 1h	> 1h	⚠ 40 ×2.9
steiner	2	870	> 12h	> 12h	> 12h	> 12h	✖

New change of order algorithm

Paradigm shift ~~sparse~~ → structured **Berthomieu/Neiger/S.**

$$\left[\begin{array}{cccc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-3 & -9 & -19 & -18 & 0 & -27 & -2 & -24
\end{array}\right]$$

New change of order algorithm

Paradigm shift ~~sparse~~ $\to$ structured **Berthomieu/Neiger/S.**

$$\left[\begin{array}{cccc|cc|cc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & -22 & -3 & -3 & -26 & -23 & 0 & -15\\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-17 & 0 & -3 & 0 & -15 & -28 & -19 & -5\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
-3 & -9 & -19 & -18 & 0 & -27 & -2 & -24
\end{array}\right] \rightsquigarrow \begin{bmatrix} x_3^4+3x_3^3+3x_3^2+22x_3 & 23x_3+26 & 15x_3\\ 3x_3^2+17 & x_3^2+28x_3+15 & 5x_3+19\\ 18x_3^3+19x_3^2+9x_3+3 & 27x_3 & x_3^2+24x_3+2\end{bmatrix} \in \mathbb{K}[x_3]^{t\times t}$$

New change of order algorithm

Paradigm shift ~~sparse~~ → structured **Berthomieu/Neiger/S.**

$$\left[\begin{array}{cccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -9 & -19 & -18 & 0 & -27 & -2 & -24 \end{array}\right] \rightsquigarrow \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t\times t}$$

New change of order algorithm

Paradigm shift ~~sparse~~ → structured

Berthomieu/Neiger/S.

$$\left[\begin{array}{cccc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -22 & -3 & -3 & -26 & -23 & 0 & -15 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -17 & 0 & -3 & 0 & -15 & -28 & -19 & -5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -3 & -9 & -19 & -18 & 0 & -27 & -2 & -24 \end{array}\right] \rightsquigarrow \begin{bmatrix} x_3^4 + 3x_3^3 + 3x_3^2 + 22x_3 & 23x_3 + 26 & 15x_3 \\ 3x_3^2 + 17 & x_3^2 + 28x_3 + 15 & 5x_3 + 19 \\ 18x_3^3 + 19x_3^2 + 9x_3 + 3 & 27x_3 & x_3^2 + 24x_3 + 2 \end{bmatrix} \in \mathbb{K}[x_3]^{t \times t}$$

basis of $\mathbb{K}[x_3]$-module of $I \cap (\mathbb{K}[x_3] + x_2\mathbb{K}[x_3] + x_1\mathbb{K}[x_3])$

Hermite normal form $\rightsquigarrow$ lex Gröbner basis

Complexity: $O\left(t^{\omega-1}D\right)$

			Step 1: $\mathcal{G}_{\mathrm{drl}} \approx P$		Step 2: $\mathcal{G}_{\mathrm{lex}} \approx H$		
			msolve		msolve	NTL	PML
n, d	D	t	F_4	F_4-tr	Wied.	bl-Wied.	HNF
11, 2	2048	462	11.6	1.1	1.2	1.7	0.8
12, 2	4096	924	115.9	8.3	6.5	14.5	5.3
13, 2	8192	1716	970	62	103.6	110	34.8
14, 2	16384	3432	7921	460	1011	880	240
15, 2	32768	6435	61381	3193	7844	6691	1665
16, 2	65536	12870	482515	24523	58744	52709	11359
8, 3	6561	1107	122.6	12.8	23.6	44.7	15.1
9, 3	19683	3139	3552.7	361	1302	1163	314
10, 3	59049	8953	95052	8664	34844	29974	6709
6, 4	4096	580	9.9	2.2	4	8.8	3.5
7, 4	16384	2128	876	128	575	545	157
8, 4	65536	8092	57237	6977	36454	33452	7231

msolve's perspectives

1. ~~Lift Gröbner bases over the rationals~~ (done)
2. ~~Improve parallelism in hashing~~ (done)
3. Test more and stabilize new algorithms for ideal saturation (started, on-going, almost done)
4. Implement ideal decompositions (zero dimensional and positive dimensional case)
5. Better continuous integration (started, on-going, almost done)
6. Mix F5 and F4 $\rightsquigarrow$ F6 algorithm (started, on-going)
7. Implement new change of orderings algorithms (started, on-going)
8. Implement Hilbert series computations
9. Implement weighted orderings
10. Develop the `AlgebraicSolving.jl` package (basic solving)
11. Develop the `AlgebraicSolving.jl` package for semi-algebraic geometry
12. Use AVX512 + Apple M2 chip instructions
13. Use MPI to have `msolve` running on clusters
14. Write an interface to the tracer (in `AlgebraicSolving.jl`)
15. Write a C interface with a documented API
16. Integrate Hensel lifting techniques $\rightsquigarrow$ quadratic convergence when lifting rationals
17. Modular arithmetics with floating point arithmetics
18. Linear algebra improvements: matrices are not only sparse
    but structured $\rightsquigarrow$ matrix multiplication $\leftrightarrow$ Gaussian elimination
19. Use code generation techniques
20. Have a dedicated implementation for the boolean field and extension fields
21. Investigate the use of GPUs
22. Solve challenging applications
23. Continue to disseminate `msolve` in computer algebra systems
    (Oscar ✓, SageMath ✓, Macaulay2 ✗, Symbolics.jl ✗)
24. **Hunt bugs, write documentations**, etc, etc, etc, etc...

msolve's perspectives

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12. Use AVX
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14. Write ar

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Requests.

☛ Star us on `github.com`

☛ Register to mailing list

☛ Continue coding

☛ Join forces (resource sharing)

Recent trends in computer algebra

https://rtca2023.github.io/

- Fundamental Algorithms and Algorithmic Complexity (Sep. 25-29)
- Geometry of Polynomial System Solving, Optimization and Topology (Oct. 16-20)
- Computer Algebra for Functional Equations in Combinatorics and Physics (Dec. 4-8)